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ON HARMONIC REPRESENTATION OF MEANS

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Abstract. We characterize continuous, symmetric and homogeneous means M that can be represented in the form

$$\frac{1}{M(x,y)} = \int_0^1 \frac{dt}{N\left(\frac{x+y}{2} - t\frac{x-y}{2}, \frac{x+y}{2} + t\frac{x-y}{2}\right)}.$$

New inequalities for means are derived from such representation.

1. Introduction, Definitions and notation

In paper [5] we investigated the representation of a symmetric, homogeneous mean $M : \mathbb{R}_+^2 \to \mathbb{R}$ of the form

$$M(x,y) = \frac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} \tag{1}$$

The main observation was that every symmetric, homogeneous mean admits such a representation. The mapping

$$M(x,y) \leftrightarrow f_M(z) = \frac{z}{M(1-z, 1+z)} \tag{2}$$

establishes one-to-one correspondence between the set of symmetric homogeneous means and the set of functions $f : (0,1) \to \mathbb{R}$ satisfying

$$\frac{z}{1+z} \le f(z) \le \frac{z}{1-z}, \tag{3}$$

called **Seiffert functions**, and the identity

$$M(x,y) = \frac{|x-y|}{2f_M\left(\frac{|x-y|}{x+y}\right)} \tag{4}$$

holds. Moreover, the formula (1) transforms Seiffert function into a symmetric, homogeneous mean.
Note that the outermost functions in (3) correspond to max and min means.

In this note we discuss the representation of means in the form

$$\frac{1}{M(x,y)} = \int_0^1 \frac{dt}{N\left(\frac{x+y}{2} + t\frac{x-y}{2}, \frac{x+y}{2} - t\frac{x-y}{2}\right)},$$

where N is also a homogeneous, symmetric mean.

We shall be using two facts from [5]

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Property 1. [5, Section 7] If f is a Seiffert mean, then for arbitrary $0 < t \le 1$ the function $f^{\{t\}}$ given by the formula $f^{\{t\}}(z) = \frac{f(tz)}{t}$ is also a Seiffert mean.

Lemma 1.1. *If f is a Seiffert function corresponding to the mean M, then $f^{\{t\}}$ is a Seiffert function for*

$$M^{\{t\}}(x,y) = M\left(\tfrac{x+y}{2} + t\tfrac{x-y}{2}, \tfrac{x+y}{2} - t\tfrac{x-y}{2}\right).$$

Proof. Let $z = \frac{|x-y|}{x+y}$. Then by (1) and (2) we have

$$\frac{|x-y|}{2f^{\{t\}}(z)} = \frac{t|x-y|}{2f(tz)} = \frac{t|x-y|M(1-tz, 1+tz)}{2tz}$$
$$= \frac{x+y}{2}M\left(1 - t\frac{|x-y|}{x+y}, 1 + t\frac{|x-y|}{x+y}\right) = M^{\{t\}}(x,y). \qquad \square$$

Following [5, Section 5], consider the integral operator on the set of continuous Seiffert functions, defined as

$$I(f)(z) = \int_0^z \frac{f(u)}{u}du. \tag{5}$$

Property 2. The operator I has the following properties:

- is monotone - if $f \le g$, then $I(f) \le I(g)$,
- preserves convexity - if f is convex, then so is $I(f)$ and for all $0 < z < 1$ the inequalities $z \le I(f)(z) \le f(z)$ hold, ([5, Theorem 5.1]),
- preserves concavity - if f is concave, then so is $I(f)$ and for all $0 < z < 1$ the inequalities $z \ge I(f)(z) \ge f(z)$ hold, ([5, Theorem 5.1]),
- $I(f)$ is a Seiffert function, ([5, Corollary 5.1]).

The next simple theorem characterizes the functions, which are of the form $I(f)$.

Theorem 1.1. *Let g be a real function defined on the interval $(0,1)$. The following conditions are equivalent*

- $\lim_{z\to 0} g(z) = 0$*, g is continuously differentiable, and for all* $0 < z < 1$

$$\frac{1}{1+z} \le g'(z) \le \frac{1}{1-z}, \tag{6}$$

- *there exist a continuous Seiffert function f such that $g = I(f)$.*

Proof. Multiplying (6) by z we see that $f(z) = zg'(z)$ is a continuous Seiffert function and clearly $I(f) = g$.
Conversely, if f is continuous, then $g = I(f)$ is differentiable. Since $\lim_{z\to 0} f(z)/z = 1$ we claim $\lim_{z\to 0} g(z) = 0$. Differentiating g we obtain $g'(z) = f(z)/z$, which yields (6) because f fulfills (3). $\square$

Now we are ready to formulate the main result of this note.

2. Harmonic representation of means

Definition 2.1. We say that a continuous mean N is a harmonic representation of mean M if

$$\frac{1}{M(x,y)} = \int_0^1 \frac{dt}{N^{\{t\}}(x,y)}.$$

Theorem 2.1. *A continuous mean M admits a harmonic representation if and only if its Seiffert function m can be represented as $I(n)$, where n is a continuous Seiffert function.*

Proof. Let N be the harmonic representation of M and let $z = \frac{|x-y|}{x+y}$. Denote by m and n the Seiffert functions of M and N respectively. Applying (1) and (2) we have

$$\begin{aligned}\frac{2}{|x-y|} I(n)(z) &= \frac{2}{|x-y|} \int_0^z \frac{n(u)}{u} du = \frac{2}{|x-y|} \int_0^1 n^{\{t\}}(z) dt \\ &= \int_0^1 \frac{dt}{N^{\{t\}}(x,y)} = \frac{1}{M(x,y)} = \frac{2}{|x-y|} m(z),\end{aligned}$$

which yields $m = I(n)$. Conversely, if $m = I(n)$ and N is a mean corresponding to n, then

$$\begin{aligned}\frac{1}{M(x,y)} &= \frac{2}{|x-y|} m(z) = \frac{2}{|x-y|} I(n)(z) = \frac{2}{|x-y|} \int_0^z \frac{n(u)}{u} du \\ &= \frac{2}{|x-y|} \int_0^1 n^{\{t\}}(z) dt = \int_0^1 \frac{dt}{N^{\{t\}}(x,y)}.\end{aligned}$$

□

From (3) we obtain by integration the inequalities

$$\log(1+z) \le I(f)(z) \le -\log(1-z), \tag{7}$$

which shows, that every mean admitting harmonic representation satisfies the inequalities

$$\frac{|x-y|}{2(\log A(x,y) - \log \min(x,y))} \le M(x,y) \le \frac{|x-y|}{2(\log \max(x,y) - \log A(x,y))}.$$

The inverse statement is not true. It is easy to construct a function satisfying (7) for which (6) fails.

3. Examples I

Example 3.1. The Seiffert function of the Seiffert mean $P(x,y) = \frac{|x-y|}{2 \arcsin z}$ is obviously arcsin. Let $g(z) = \frac{z}{\sqrt{1-z^2}}$. Then $\arcsin = I(g)$ and g is the Seiffert function of the geometric mean $G(x,y) = \sqrt{xy}$. Thus we obtain the identity

$$P(x,y) = \left(\int_0^1 \frac{dt}{G^{\{t\}}(x,y)} \right)^{-1}.$$

Example 3.2. The second Seiffert mean is given by $T(x,y) = \frac{|x-y|}{2 \arctan z}$. Let $C(x,y) = \frac{x^2+y^2}{x+y}$ be the contra-harmonic mean. Its Seiffert function is $c(z) = \frac{z}{1+z^2}$ and one can easily verify that $I(c) = \arctan$, so

$$T(x,y) = \left(\int_0^1 \frac{dt}{C^{\{t\}}(x,y)} \right)^{-1}.$$

Example 3.3. For the logarithmic mean $L(x,y)=\frac{x-y}{\log x-\log y}=\frac{|x-y|}{2\operatorname{artanh} z}$ we get

$$L(x,y)=\left(\int_0^1 \frac{dt}{H^{\{t\}}(x,y)}\right)^{-1},$$

where $H(x,y)=\frac{2xy}{x+y}$ denotes the harmonic mean.

Example 3.4. The Seiffert function of the root-mean square $R=\sqrt{\frac{x^2+y^2}{2}}$ is the function $r(z)=\frac{z}{\sqrt{1+z^2}}$, thus $I(r)(z)=\operatorname{arsinh} z$, which in turn is the Seiffert mean of the Neuman-Sándor mean $M(x,y)=\frac{|x-y|}{2\operatorname{arsinh} z}$, so

$$M(x,y)=\left(\int_0^1 \frac{dt}{R^{\{t\}}(x,y)}\right)^{-1},$$

In [5] we have shown that sin, tan, sinh and tanh are also Seiffert function. Let us check if their corresponding means admit harmonic representations. To do it we shall use Theorems 1.1 and 2.1

Example 3.5. For $g(z)=\sin z$ we want to show that g' satisfies (6). Obviously $\cos z<1<1/(1-z)$. To prove the other part observe that

$$(1+z)\cos z>(1+z)(1-z^2/2)>1+z(1-z/2)>1,$$

thus (6) holds, and one easily verifies that $z\cos z$ is the Seiffert function of the mean $M(x,y)=A(x,y)/\cos\frac{|x-y|}{x+y}$, which implies

$$\frac{x-y}{2\sin\frac{x-y}{x+y}}=\left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

Example 3.6. Now let $g(z)=\tan z$. We have

$$\frac{1}{1+z}<1<\frac{1}{\cos^2 z}=\frac{1}{(1+\sin z)(1-\sin z)}<\frac{1}{1-z},$$

so $z/\cos^2 z$ is the Seiffert function. It corresponds to the mean $M(x,y)=A(x,y)\cos^2\frac{|x-y|}{x+y}$ and

$$\frac{x-y}{2\tan\frac{x-y}{x+y}}=\left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

Example 3.7. With the hyperbolic sine the situation is simple. We have

$$1<\cosh z=\sum_{m=0}^{\infty}\frac{z^{2m}}{(2m)!}<\sum_{m=0}^{\infty} z^m=\frac{1}{1-z},$$

thus $z\cosh z$ is the Seiffert function, and its mean $M(x,y)=A(x,y)/\cosh\frac{|x-y|}{x+y}$ satisfies

$$\frac{x-y}{2\sinh\frac{x-y}{x+y}}=\left(\int_0^1 \frac{dt}{M^{\{t\}}(x,y)}\right)^{-1}.$$

Example 3.8. The last function is the hyperbolic tangent. Its derivative is $\cosh^{-2} z$ and $\cosh^{-2}(1)\approx 0.41997<\frac{1}{2}$, so the left inequality in (6) does not hold, and this yields the mean $\frac{x-y}{2\sinh\frac{x-y}{x+y}}$ does not have a harmonic representation.

We leave as a simple exercise the fact that there is no harmonic representation of the geometric mean.

4. The arithmetic-geometric mean

This section is devoted to the arithmetic-geometric mean given by the formula

$$AGM(x,y) = \left(\frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{x^2\cos^2\varphi + y^2\sin^2\varphi}}\right)^{-1}.$$

To find its Seiffert mean let us recall the famous result of Gauss [3]

$$AGM(1-z, 1+z) = \frac{\pi}{2K(z)}, \tag{8}$$

where K is the complete elliptic integral of the first kind

$$K(z) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-z^2\sin^2\varphi}} = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-z^2t^2}}. \tag{9}$$

Comparing (8) and (2) we see that $f_{AGM}(z) = \frac{2}{\pi} zK(z)$. We shall show that AGM admits the harmonic representation. By Theorem 1.1 it is enough to show that f'_{AGM}satisfies (6). To this end let us recall the power series expansion of K ([2, 900.00])

$$K(z) = \frac{\pi}{2}\left(1 + \sum_{m=1}^{\infty}\left[\frac{(2m-1)!!}{(2m)!!}\right]^2 z^{2m}\right). \tag{10}$$

We have

$$f'_{AGM}(z) = \frac{2}{\pi}\left(K(z) + z\frac{dK}{dz}\right) = 1 + \sum_{m=1}^{\infty}(2m+1)\left[\frac{(2m-1)!!}{(2m)!!}\right]^2 z^{2m} \tag{11}$$

Denoting the m^{th} coefficient in (11) by c_m we see that

$$\frac{c_{m+1}}{c_m} = \frac{2m+3}{2m+1}\left[\frac{(2m+1)!!((2m)!!}{(2m+2)!!(2m-1)!!}\right]^2 = \frac{(2m+1)(2m+3)}{(2m+2)^2} < 1,$$

and since $c_1 = 3/4$ we conclude that $c_m < 1$ for all ≥ 1. Thus $1 < f'_{AGM}(z) < 1 + z + z^2 + \cdots = 1/(1-z)$.

Theorem 1.1 implies that the arithmetic-geometric mean admits the harmonic representation. To derive its explicit form, recall that the derivative of K is given by $K'(z) = \frac{E(z)}{z(1-z^2)} - \frac{K(z)}{z}$(see. e.g. [2, 710.00]), thus

$$zf'_{AGM}(z) = \frac{2}{\pi}\left(zK(z) + z^2K'(z)\right) = \frac{2}{\pi}\frac{z}{1-z^2}E(z),$$

($E(z) = \int_0^{\pi/2}\sqrt{1-z^2\sin^2\varphi}\,d\varphi$ is the complete elliptic integral of the second kind). As $\frac{z}{1-z^2}$ is the Seiffert function of the harmonic mean we obtain the formula

$$\begin{aligned} V(x,y) =& \frac{\pi H(x,y)}{2E\left(\frac{|x-y|}{x+y}\right)} = \frac{\pi H(x,y)}{2E\left(\sqrt{1-\frac{G^2(x,y)}{A^2(x,y)}}\right)} \\ =& \frac{\pi G^2(x,y)}{2\int_0^{\pi/2}\sqrt{A^2(x,y)\cos^2\varphi + G^2(x,y)\sin^2\varphi}\,d\varphi}. \end{aligned}$$

This mean has a nice geometric interpretation: in the ellipsis with semi-axes $G(x,y)$ and $A(x,y)$ it represents the ratio of the area of inscribed disc to its semi-perimeter.

5. Hermite-Hadamard inequality for means

The Hermite-Hadamard inequality in its classic form says that if f is a convex function in an interval I, then for all $a, b \in I$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

A stronger inequality also holds

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right].$$

Suppose now that the mean N is the harmonic representation of M and its Seiffert function n is such that the function $n(u)/u$ is convex. Then, applying the Hermite-Hadamard inequality to (5) and taking into account that $\lim_{u\to 0} n(u)/u = 1$ we obtain

$$2n(z/2) \le I(n)(z) \le \frac{z+n(z)}{2}. \tag{12}$$

This yields (with help of (2)) the inequalities for means

$$H(A(x,y), N(x,y)) \le M(x,y) \le N\left(\frac{3x+y}{4}, \frac{x+3y}{4}\right). \tag{13}$$

The stronger version of the Hermite-Hadamard reads in this case:

$$I(n)(z) \le \frac{1}{2}\left[2n(z/2) + \frac{z+n(z)}{2}\right], \tag{14}$$

which yields

(15)

$$H(A(x,y), N^{\{1/2\}}(x,y), N^{\{1/2\}}(x,y), N(x,y)) \le M(x,y) \le N\left(\frac{3x+y}{4}, \frac{x+3y}{4}\right).$$

Obviously, if $n(u)/u$ is concave, the inequalities in (12)–(15) are reversed.

In the above we use the Hermite-Hadamard inequality with the left end fixed, so it may happen that (12) holds even if $n(u)/u$ is not convex. Of course, in such case an individual treatment would be required.

6. Examples II

Example 6.1. Let $N = G$. By Example 3.1 we know that $M = P$ is the first Seiffert mean. Since $n(u)/u = (1-u^2)^{-1/2}$ is convex and $G^{\{1/2\}} = \sqrt{3A^2+G^2}/2$, (13) and (14) yield

$$\frac{2AG}{A+G} \le 2\left(\frac{2}{\sqrt{3A^2+G^2}} + \frac{A+G}{2}\right)^{-1} \le P \le \frac{\sqrt{3A^2+G^2}}{2}.$$

Example 6.2. The Seiffert function c from Example 3.2 does not satisfy the convexity condition, but the reversed inequalities in (12) hold anyway, by the following lemma.

Lemma 6.1. *The inequalities*

$$\frac{4u}{4+u^2} > \arctan u > u\frac{2+u^2}{2+2u^2}$$

hold for $0 < u < 1$

Proof. Let $h(u) = \frac{4u}{4+u^2} - \arctan u$. As $h(0) = 0$ and $h'(u) = \frac{u^2(4-5u^2)}{(u^2+1)(u^2+4)^2}$ we see that h has local maximum at $u = 2/\sqrt{5}$ and since $h(1) > 0$ we conclude that $h(u) > 0$.

Let now $h(u) = \arctan u - u\frac{2+u^2}{2+2u^2}$. Then $h(0) = 0$ and $h'(u) = \frac{u^2(1-u^2)}{2(x^2+1)^2} > 0$, and the proof is complete. □

Thus for the contraharmonic mean and the second Seiffert mean we have

$$C^{\{1/2\}} = \frac{5A^2 - G^2}{4A} \le T \le H(A, C)$$

Example 6.3. The pair $(M, N) = (L, H)$ (see Example 3.3) gives the inequalities

$$\frac{2G^2A}{A^2 + G^2} \le \frac{4AG^2(3A^2 + G^2)}{3A^4 + 12A^2G^2 + G^4} \le L \le \frac{3A^2 + G^2}{4A}$$

Example 6.4. For the root-mean square and Neuman-Sándor means (Example 3.4) the convexity condition is not satisfied, but the following lemma shows that the reversed inequalities (12) are valid.

Lemma 6.2. *For* $0 < u < 1$ *the inequalities*

$$\frac{2u}{\sqrt{u^2 + 4}} \ge \operatorname{arsinh} u \ge \frac{u}{2} + \frac{u}{2\sqrt{u^2 + 1}}$$

hold.

Proof. To prove the left inequality it suffices to show that the function $h(u) = \operatorname{arsinh} u - \frac{2u}{\sqrt{u^2+4}}$ decreases, because $h(0) = 0$. Differentiating we obtain

$$h'(u) = \frac{(u^2 + 4)^{3/2} - 8(u^2 + 1)^{1/2}}{(u^2 + 4)^{3/2}(u^2 + 1)^{1/2}}. \tag{16}$$

Let p denote the numerator in (16). Then $p'(u) = u\left(3\sqrt{u^2 + 4)} - \frac{8}{\sqrt{u^2+1}}\right) := uq(u)$. The function q is a difference of an increasing and decreasing function, thus increases from $q(0) = -2$ to $q(1) = 3\sqrt{5} - 4\sqrt{2} > 0$, so we conclude that p has one local minimum in the interval $(0, 1)$. Since $p(0) = 0$ and $p(1) = \sqrt{125} - \sqrt{128} < 0$ we see that $p(u) < 0$ for all u, thus $h'(u) < 0$ and we are done.

For the right inequality the method is similar:

$$h(u) = \frac{u}{2} + \frac{u}{2\sqrt{u^2 + 1}} - \operatorname{arsinh} u, \quad h'(u) = \frac{(u^2 + 1)^{3/2} - (2u^2 + 1)}{2(u^2 + 1)^{3/2}}$$
$$p(u) = (u^2 + 1)^{3/2} - (2u^2 + 1), \quad p'(u) = u(3\sqrt{u^2 + 1} - 4) := uq(u).$$

As above, q increases from -1 to $3\sqrt{2} - 4$, so p has one local minimum, and since $p(0) = 0$ and $p(1) = \sqrt{8} - 3 < 0$ we conclude $h' < 0$. □

Thus for the Neuman-Sándor mean $M(x, y) = \frac{|x-y|}{2 \operatorname{arsinh} \frac{|x-y|}{x+y}}$ the inequality (13) in this case reads

$$R^{\{1/2\}} = \frac{\sqrt{5A^2 - G^2}}{2} \le M \le H(A, R).$$

Example 6.5. In Example 3.5 we consider the Seiffert functions $m(z) = \sin z$ and $n(z) = z\cos z$. Clearly $n(z)/z$ is concave and thus

$$\frac{x+y}{2\cos\frac{1}{2}\frac{|x-y|}{x+y}} \le \frac{|x-y|}{2\sin\frac{|x-y|}{x+y}} \le \frac{x+y}{1+\cos\frac{|x-y|}{x+y}}$$

Example 6.6. The function $\frac{1}{\cos^2 z}$ is convex, thus we can apply (12) to the functions from Example 3.6 to obtain

$$\frac{(x+y)\cos^2\frac{|x-y|}{x+y}}{1+\cos^2\frac{|x-y|}{x+y}} \le \frac{|x-y|}{2\tan\frac{|x-y|}{x+y}} \le A(x,y)\cos^2\frac{1}{2}\frac{|x-y|}{x+y}.$$

Example 6.7. In Example 3.7 the function cosh is convex, so we get

$$\frac{x+y}{1+\cosh\frac{|x-y|}{x+y}} \le \frac{|x-y|}{2\sinh\frac{|x-y|}{x+y}} \le \frac{x+y}{2\cosh\frac{1}{2}\frac{|x-y|}{x+y}}.$$

Example 6.8. In this example we deal with the AGM mean and its harmonic representation V described in Section 4. The Seiffert mean of V is $v(z) = \frac{2}{\pi}\frac{z}{1-z^2}E(z)$, so

$$\frac{v(z)}{z} = \frac{2}{\pi}\int_0^{\pi/2}\frac{\sqrt{1-z^2\sin^2\varphi}}{1-z^2}d\varphi. \tag{17}$$

We shall show that this function is convex. For $0 < a < 1$ let $h_a(u) = \frac{\sqrt{1-au^2}}{\sqrt{1-u^2}}$. Then

$$h_a'(u) = \frac{(1-a)u}{(1-au^2)^{1/2}(1-u^2)^{3/2}}.$$

Note the h_a' is nonnegative and increasing, since its numerator increases while denominator decreases. Thus h_a is positive, increasing and convex. The function $g(u) = 1/\sqrt{(1-u^2)}$ shares the same properties, so their product is convex [4, Theorem I.13C]. Since the integrands in (17) are convex, so is the left-hand side. Therefore by (13)

$$\frac{2AV}{A+V} \le AGM \le V^{\{1/2\}}.$$

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