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ANOTHER OSTROWSKI TYPE INEQUALITY VIA POMPEIU'S MEAN VALUE THEOREM

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Abstract. In this paper, a new Ostrowski type inequality via Pompeiu's mean value theorem is proved. Some applications for special means are also given.

1. Introduction

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

Following [6, p. 84 – 85], we will mention here a geometrical interpretation of Pompeiu's theorem.

The equation of the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$y = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

This line intersects the $y-$axis at the point $(0, y)$, where y is

$$\begin{aligned} y &= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(0 - x_1) \\ &= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2}. \end{aligned}$$

The equation of the tangent line at the point $(\xi, f(\xi))$ is

$$y = (x - \xi) f'(\xi) + f(\xi).$$

The tangent line intersects the $y-$axis at the point $(0, y)$, where

$$y = -\xi f'(\xi) + f(\xi).$$

Hence, the geometric meaning of Pompeiu's mean value theorem is that the tangent of the point $(\xi, f(\xi))$ intersects on the $y-$axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

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Theorem 2 (Ostrowski, 1938 [4])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)|\leq M<\infty$ for all $t\in(a,b)$. Then for any $x\in[a,b]$, we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a). \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 3 (Dragomir, 2005 [3])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\leq\frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty,\end{aligned} \tag{1.3}$$

where $\ell(t)=t$, $t\in[a,b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [7], E. C. Popa using a mean value theorem obtained a generalization of (1.3) as follows:

Theorem 4 (Popa, 2007 [7])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume that $\alpha\notin[a,b]$. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|\left(\frac{a+b}{2}-\alpha\right)f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)\,dt\right|\\&\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f-\ell_\alpha f'\|_\infty,\end{aligned} \tag{1.4}$$

where $\ell_\alpha(t)=t-\alpha$, $t\in[a,b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1\leq p\leq\infty$ which for $p=\infty$ give Dragomir's result.

Theorem 5 (Pečarić & Ungar, 2006 [5])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for $1\leq p,q\leq\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq PU(x,p)\|f-\ell f'\|_p, \tag{1.5}$$

for $x \in [a,b]$, where

$$PU(x,p) \quad : \quad = (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q} + \left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right].$$

In the cases $(p,q)=(1,\infty),(\infty,1)$ and $(2,2)$ the quantity $PU(x,p)$ has to be taken as the limit as $p\to 1,\infty$ and 2, respectively.

For other inequalities in terms of the p-norm of the quantity $f-\ell_\alpha f'$, where $\ell_\alpha(t)=t-\alpha$, $t\in[a,b]$ and $\alpha\notin[a,b]$ see [1] and [2].

In this paper, a new Ostrowski type inequality via Pompeiu's mean value theorem is proved. Applications for special means are also given.

2. Another Ostrowski Type Inequality Via Pompeiu's Result

The following new result holds.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $b>a>0$. Then for any $x\in[a,b]$, we have the inequality*

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \leq \frac{2}{b-a}\left\|f-\ell f'\right\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right), \tag{2.1}$$

where $\ell(t)=t$, $t\in[a,b]$.

The constant 2 is best possible in (2.1).

Proof. Applying Pompeiu's mean value theorem [6] (see also [8, p. 83]), for any $x,t\in[a,b]$, there is a ξ between x and t such that

$$tf(x)-xf(t)=[f(\xi)-\xi f'(\xi)](t-x)$$

giving

$$|tf(x)-xf(t)|\leq\sup_{\xi\in[a,b]}|f(\xi)-\xi f'(\xi)|\,|x-t|=\left\|f-\ell f'\right\|_\infty|x-t|$$

for any $t,x\in[a,b]$, or, by dividing with $x,t>0$, equivalently to

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|\leq\left\|f-\ell f'\right\|_\infty\left|\frac{1}{x}-\frac{1}{t}\right| \tag{2.2}$$

for any $t,x\in[a,b]$.

Integrating over $t\in[a,b]$, we get

$$\left|\frac{f(x)}{x}(b-a)-\int_a^b\frac{f(t)}{t}dt\right|\leq\int_a^b\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|dt \leq\left\|f-\ell f'\right\|_\infty\int_a^b\left|\frac{1}{x}-\frac{1}{t}\right|dt \tag{2.3}$$

and since

$$\begin{aligned}\int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| dt &= \left[\int_a^x \left(\frac{1}{t}-\frac{1}{x}\right) dt + \int_x^b \left(\frac{1}{x}-\frac{1}{t}\right) dt\right] \\ &= \left(\ln\frac{x}{a} - \frac{x-a}{x} + \frac{b-x}{x} - \ln\frac{b}{x}\right) \\ &= \left(\ln\frac{x^2}{ab} + \frac{a+b-2x}{x}\right) \\ &= 2\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right)\end{aligned}$$

for any $x \in [a,b]$, then we deduce from (2.3) the desired result (2.1).

Now, assume that (2.1) holds with a constant $k > 0$, i.e.,

$$\begin{aligned}&\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\ &\leq \frac{k}{b-a}\|f-\ell f'\|_\infty \left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right),\end{aligned} \tag{2.4}$$

for any $x \in [a,b]$.

Consider $f : [a,b] \to \mathbb{R}$, $f(t) = 1$. Then

$$\|f-\ell f'\|_\infty = 1, \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt = \frac{1}{b-a}\ln\frac{b}{a},$$

and by (2.4) we deduce

$$\left|\frac{1}{x} - \frac{1}{b-a}\ln\frac{b}{a}\right| \leq \frac{k}{b-a}\left(\ln\frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2}-x}{x}\right)$$

for any $x \in [a,b]$.

If we take in this inequality $x = a$, we get

$$\begin{aligned}\left|\frac{1}{a} - \frac{1}{b-a}\ln\frac{b}{a}\right| &\leq \frac{k}{b-a}\left(\ln\frac{a}{\sqrt{ab}} + \frac{b-a}{2a}\right) \\ &= \frac{k}{2(b-a)}\left(\ln\frac{a^2}{ab} + \frac{b-a}{a}\right) \\ &= \frac{k}{2(b-a)}\left(\ln\frac{a}{b} + \frac{b-a}{a}\right).\end{aligned} \tag{2.5}$$

In we multiply (2.5) with $2(b-a)$ we get

$$2\left|\frac{b-a}{a} - \ln\frac{b}{a}\right| \leq k\left(\frac{b-a}{a} - \ln\frac{b}{a}\right)$$

which implies that $k \geq 2$. □

The following interesting particular case holds.

Corollary 1. *With the assumptions in Theorem 6, we have*

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \leq \frac{2}{b-a}\|f-\ell f'\|_\infty \ln\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right). \tag{2.6}$$

Remark 1. *If we consider the function* $\psi : [a,b] \to \mathbb{R}$ *given by*

$$\psi(x) := \ln \frac{x}{\sqrt{ab}} + \frac{\frac{a+b}{2} - x}{x},$$

then we observe that

$$\psi'(x) = \frac{x - \frac{a+b}{2}}{x^2},$$

which shows that

$$\inf_{x \in [a,b]} \psi(x) = \psi\left(\frac{a+b}{2}\right) = \ln\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right),$$

meaning that the inequality (2.6) is the best possible one can get from (2.1).

Remark 2. *We can state from (2.1) the following inequality as well:*

$$\left|\frac{f\left(\sqrt{ab}\right)}{\sqrt{ab}} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt\right| \le \frac{2}{b-a}\|f - \ell f'\|_\infty \left(\frac{\frac{a+b}{2} - \sqrt{ab}}{\sqrt{ab}}\right). \tag{2.7}$$

Corollary 2. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $b > a > 0$. *Then we have the inequality*

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{a+b}{2}\int_a^b \frac{f(t)}{t}dt\right| \\ &\le \|f - \ell f'\|_\infty \left[\left(\frac{b^2 + a^2}{2}\right)\frac{\ln b - \ln a}{b-a} - 1\right]. \end{aligned} \tag{2.8}$$

Proof. Utilizing (2.1) we have

$$\begin{aligned} &\left|\int_a^b f(x)\,dx - \frac{a+b}{2}\int_a^b \frac{f(t)}{t}dt\right| \\ &\le \int_a^b \left|f(x) - \frac{x}{b-a}\int_a^b \frac{f(t)}{t}dt\right| dx \\ &\le \frac{2}{b-a}\|f - \ell f'\|_\infty \left(\int_a^b x \ln\frac{x}{\sqrt{ab}}dx + \int_a^b \left(\frac{a+b}{2} - x\right)dx\right) \\ &= \frac{2}{b-a}\|f - \ell f'\|_\infty \int_a^b x \ln\frac{x}{\sqrt{ab}}dx. \end{aligned} \tag{2.9}$$

Since

$$\begin{aligned}\int_a^b x\ln\frac{x}{\sqrt{ab}}dx &= \frac{x^2}{2}\ln\frac{x}{\sqrt{ab}}\bigg|_a^b - \frac{1}{2}\int_a^b xdx \\ &= \frac{b^2}{2}\ln\frac{b}{\sqrt{ab}} - \frac{a^2}{2}\ln\frac{a}{\sqrt{ab}} - \frac{1}{2}(b-a) \\ &= \frac{b^2}{2}\ln\sqrt{\frac{b}{a}} - \frac{a^2}{2}\ln\sqrt{\frac{a}{b}} - \frac{1}{2}(b-a) \\ &= \left(\frac{b^2+a^2}{2}\right)\ln\sqrt{\frac{b}{a}} - \frac{1}{2}(b-a) \\ &= \left(\frac{b^2+a^2}{4}\right)\ln\frac{b}{a} - \frac{1}{2}(b-a),\end{aligned}$$

then

$$\frac{2}{b-a}\int_a^b x\ln\frac{x}{\sqrt{ab}}dx = \left(\frac{b^2+a^2}{2}\right)\frac{\ln b - \ln a}{b-a} - 1$$

and by (2.9) we deduce the desired result (2.8). □

3. The Weighted Case

We consider now the weighted integral case.

Theorem 7. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ with $b>a>0$. If $w:[a,b]\to\mathbb{R}$ is nonnegative integrable on $[a,b]$, then for each $x\in[a,b]$, we have the inequality:*

$$\begin{aligned}&\left|\frac{f(x)}{x}\int_a^b w(t)\,dt - \int_a^b \frac{f(t)}{t}w(t)\,dt\right| \\ &\quad\le \|f-\ell f'\|_\infty\left[\int_a^x \frac{w(t)\,dt}{t} - \int_x^b \frac{w(t)}{t}dt \right.\\ &\qquad\left. + \frac{1}{x}\left(\int_x^b w(t)\,dt - \int_a^x w(t)\,dt\right)\right].\end{aligned} \tag{3.1}$$

Proof. Using the inequality (2.2), we have

$$\begin{aligned}
&\left|\frac{f(x)}{x}\int_a^b w(t)\,dt-\int_a^b \frac{f(t)}{t}w(t)\,dt\right| \\
&\leq \int_a^b \left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| w(t)\,dt \\
&\leq \|f-\ell f'\|_\infty \int_a^b \left|\frac{1}{x}-\frac{1}{t}\right| w(t)\,dt \\
&= \|f-\ell f'\|_\infty \left[\int_a^x \left(\frac{1}{t}-\frac{1}{x}\right) w(t)\,dt+\int_x^b \left(\frac{1}{x}-\frac{1}{t}\right) w(t)\,dt\right] \\
&= \|f-\ell f'\|_\infty \left[\int_a^x \frac{w(t)\,dt}{t}-\frac{1}{x}\int_a^x w(t)\,dt\right. \\
&\left.+\frac{1}{x}\int_x^b w(t)\,dt-\int_x^b \frac{w(t)}{t}dt\right] \\
&= \|f-\ell f'\|_\infty \left[\int_a^x \frac{w(t)\,dt}{t}-\int_x^b \frac{w(t)}{t}dt\right. \\
&\left.+\frac{1}{x}\left(\int_x^b w(t)\,dt-\int_a^x w(t)\,dt\right)\right],
\end{aligned}$$

from where we get the desired inequality (3.1). □

Remark 3. *If we take in (3.1) $w(t)=t$, then we have the following estimate for the integral mean*

$$\begin{aligned}
&\left|\frac{b^2-a^2}{2}\cdot\frac{f(x)}{x}-\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{x}\|f-\ell f'\|_\infty \left[\left(x-\frac{a+b}{2}\right)^2+\left(\frac{b-a}{2}\right)^2\right]
\end{aligned}$$

for $x\in(a,b)$, that is equivalent to (2.1) for $0<a<b$.

If we take in (3.1) $w(t)=t^2$, then we have

$$\begin{aligned}
&\left|\frac{b^3-a^3}{3}\cdot\frac{f(x)}{x}-\int_a^b f(t)\,t\,dt\right| \\
&\leq \|f-\ell f'\|_\infty \left[\frac{2x^3-3x\left(a^2+b^2\right)+2\left(b^3+a^3\right)}{6x}\right]
\end{aligned} \tag{3.2}$$

for $x\in(a,b)$.

4. Applications for Special Means

In the following we will use the following inequality obtained in Corollary 1,

$$\left|\frac{f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}}-\frac{1}{b-a}\int_a^b \frac{f(t)}{t}dt\right| \leq \frac{2}{b-a}\|f-\ell f'\|_\infty \ln\left(\frac{\frac{a+b}{2}}{\sqrt{ab}}\right) \tag{4.1}$$

provided $0<a<b$.

(1) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(t)=t^p,\ p\in\mathbb{R}\backslash\{-1,0\}.$ Then

$$f\left(\frac{a+b}{2}\right)=A^p,\ \frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt=L_{p-1}^{p-1},\ p\in\mathbb{R}\backslash\{0,1\},$$

$$\|f-\ell f'\|_{[a,b],\infty}=\begin{cases}(1-p)\,a^p & \text{if } p\in(-\infty,0)\backslash\{-1\},\\ \\ |1-p|\,b^p & \text{if } p\in(0,1)\cup(1,\infty),\end{cases}$$

where

$$L_p=L_p(a,b):=\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\ p\in R\backslash\{-1,0\},\ a,b>0,\ a\neq b$$

is the *p-Logarithmic mean* and $A=A(a,b):=\frac{a+b}{2}$, $a,b\geq 0$ is the *arithmetic mean.*

Consequently, by (4.1) we deduce

$$\left|A^{p-1}-L_{p-1}^{p-1}\right|\leq\frac{2}{b-a}\ln\left(\frac{A}{G}\right)\times\begin{cases}(1-p)\,a^p & \text{if } p\in(-\infty,0)\backslash\{-1\},\\ \\ |1-p|\,b^p & \text{if } p\in(0,1)\cup(1,\infty).\end{cases}\tag{4.2}$$

(2) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(t)=\frac{1}{t}$. Then

$$f\left(\frac{a+b}{2}\right)=\frac{1}{A},\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt=\frac{1}{G^2},\|f-\ell f'\|_{[a,b],\infty}=\frac{2}{a},$$

where

$$G(a,b):=\sqrt{ab},\ a,b>0,$$

is the *Geometric mean.*

Consequently, by (4.1) we deduce

$$0\leq A^2-G^2\leq\frac{4}{(b-a)\,a}A^2G^2\ln\left(\frac{A}{G}\right).\tag{4.3}$$

(3) Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f(t)=\ln t$. Then

$$f\left(\frac{a+b}{2}\right)=\ln A,\ \frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt=\frac{\ln^2 b-\ln^2 a}{2(b-a)},$$

$$\|f-\ell f'\|_{[a,b],\infty}=\max\left\{\left|\ln\left(\frac{a}{e}\right)\right|,\left|\ln\left(\frac{b}{e}\right)\right|\right\},$$

where

$$I=I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\ a,b>0,\ a\neq b$$

is the *Identric mean.*

Consequently, by (4.1) we deduce

$$\left|\frac{\ln A}{A}-\frac{\ln^2 b-\ln^2 a}{2(b-a)}\right|\leq\frac{2}{b-a}\max\left\{\left|\ln\left(\frac{a}{e}\right)\right|,\left|\ln\left(\frac{b}{e}\right)\right|\right\}\ln\left(\frac{A}{G}\right).\tag{4.4}$$

If we use the *Logarithmic mean*, i.e.

$$L = \frac{b-a}{\ln b - \ln a},\ a,b>0,\ a\neq b,$$

then (4.4) can be written as

$$\left|\frac{\ln A}{A} - \frac{\ln G}{L}\right| \leq \frac{2}{b-a}\max\left\{\left|\ln\left(\frac{a}{e}\right)\right|, \left|\ln\left(\frac{b}{e}\right)\right|\right\}\ln\left(\frac{A}{G}\right). \tag{4.5}$$

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