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INEQUALITIES OF POMPEIU'S TYPE FOR ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS TO OSTROWSKI'S INEQUALITY

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Abstract. In this paper, some new Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

1. Introduction

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1 (Pompeiu, 1946 [6])**.** *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 2 (Ostrowski, 1938 [4])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)| \leq M < \infty$ for all $t \in (a,b)$. Then for any $x \in [a,b]$, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a). \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 3 (Dragomir, 2005 [3])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x \in [a,b]$, we have*

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the inequality

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty, \tag{1.3}$$

where $\ell(t)=t,\ t\in[a,b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller constant.*

In [7], E. C. Popa using a mean value theorem obtained a generalization of (1.3) as follows:

Theorem 4 (Popa, 2007 [7])**.** *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b). *Assume that* $\alpha\notin[a,b]$. *Then for any* $x\in[a,b]$, *we have the inequality*

$$\left|\left(\frac{a+b}{2}-\alpha\right)f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f-\ell_\alpha f'\|_\infty, \tag{1.4}$$

where $\ell_\alpha(t)=t-\alpha,\ t\in[a,b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1\le p\le\infty$ which for $p=\infty$ give Dragomir's result.

Theorem 5 (Pečarić & Ungar, 2006 [5])**.** *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $1\le p,q\le\infty$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le PU(x,p)\|f-\ell f'\|_p, \tag{1.5}$$

for $x\in[a,b]$, *where*

$$\begin{aligned} PU(x,p) \;:&= (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right. \\ &\left.+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{1/q}\right]. \end{aligned}$$

In the cases $(p,q)=(1,\infty),(\infty,1)$ *and* $(2,2)$ *the quantity* $PU(x,p)$ *has to be taken as the limit as* $p\to1,\infty$ *and* 2, *respectively.*

For other inequalities in terms of the p-norm of the quantity $f-\ell_\alpha f'$, where $\ell_\alpha(t)=t-\alpha$, $t\in[a,b]$ and $\alpha\notin[a,b]$ see [1] and [2].

In this paper, some new Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

2. Pompeiu's Type Inequalities

The following inequality is useful to derive some Ostrowski type inequalities.

Corollary 1 (Pompeiu's Inequality)**.** *With the assumptions of Theorem 1 and if* $\|f-\ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ *where* $\ell(t) = t$, $t \in [a,b]$, *then*

$$|tf(x) - xf(t)| \le \|f - \ell f'\|_\infty |x-t| \tag{2.1}$$

for any $t, x \in [a,b]$.

The inequality (2.1) was stated by the author in [3].

We can generalize the above inequality (2.1) for the larger class of functions that are absolutely continuous and complex-valued as well as for other norms of the difference $f - \ell f'$.

Theorem 6. *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b > a > 0$. *Then for any* $t, x \in [a,b]$ *we have*

$$|tf(x) - xf(t)| \le \begin{cases} \|f-\ell f'\|_\infty |x-t| & \text{if } f-\ell f' \in L_\infty[a,b], \\ \frac{1}{(2q-1)^{1/q}} \|f-\ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \text{if } f-\ell f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f-\ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases} \tag{2.2}$$

or, equivalently

$$\left|\frac{f(x)}{x} - \frac{f(t)}{t}\right| \le \begin{cases} \|f-\ell f'\|_\infty \left|\frac{1}{t} - \frac{1}{x}\right| & \text{if } f-\ell f' \in L_\infty[a,b], \\ \frac{1}{(2q-1)^{1/q}} \|f-\ell f'\|_p \left|\frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}}\right|^{1/q} & \text{if } f-\ell f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f-\ell f'\|_1 \frac{1}{\min\{t^2,x^2\}}. \end{cases} \tag{2.3}$$

Proof. If f is absolutely continuous, then f/ℓ is absolutely continuous on the interval $[a,b]$ that does not containing 0 and

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \frac{f(x)}{x} - \frac{f(t)}{t}$$

for any $t, x \in [a,b]$ with $x \ne t$.

Since

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \int_t^x \frac{f'(s)s - f(s)}{s^2} ds$$

then we get the following identity

$$tf(x) - xf(t) = xt \int_t^x \frac{f'(s)s - f(s)}{s^2} ds \tag{2.4}$$

for any $t, x \in [a,b]$.

We notice that the equality (2.4) was proved for the smaller class of differentiable real valued functions and in a different manner in [5].

Taking the modulus in (2.4) we have

$$|tf(x)-xf(t)| = \left|xt\int_t^x \frac{f'(s)s-f(s)}{s^2}ds\right| \leq xt\left|\int_t^x\left|\frac{f'(s)s-f(s)}{s^2}\right|ds\right| := I \tag{2.5}$$

and utilizing Hölder's integral inequality we deduce

$$I \leq xt\begin{cases}\sup_{s\in[t,x]([x,t])}|f'(s)s-f(s)|\left|\int_t^x\frac{1}{s^2}ds\right|, & \\ \left|\int_t^x|f'(s)s-f(s)|^p ds\right|^{1/p}\left|\int_t^x\frac{1}{s^{2q}}ds\right|^{1/q} & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left|\int_t^x|f'(s)s-f(s)|ds\right|\sup_{s\in[t,x]([x,t])}\left\{\frac{1}{s^2}\right\}, & \end{cases} \tag{2.6}$$

$$= \begin{cases}\|f-\ell f'\|_\infty |x-t|, & \\ \frac{1}{(2q-1)^{1/q}}\|f-\ell f'\|_p\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}, & p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \|f-\ell f'\|_1\frac{\max\{t,x\}}{\min\{t,x\}}, & \end{cases}$$

and the inequality (2.3) is proved. □

Remark 1. *The first inequality in (2.2) also holds in the same form for* $0>b>a$.

Remark 2. *If we take in (2.2)* $x=A=A(a,b):=\frac{a+b}{2}$ *(the arithmetic mean) and* $t=G=G(a,b):=\sqrt{ab}$ *(the geometric mean) then we get the simple inequality for functions of means:*

$$|Gf(A)-Af(G)| \leq \begin{cases}\|f-\ell f'\|_\infty (A-G) & \text{if } f-\ell f'\in L_\infty[a,b], \\ \frac{1}{(2q-1)^{1/q}}\|f-\ell f'\|_p\frac{\left(A^{2q-1}-G^{2q-1}\right)^{1/q}}{A^{1/p}G^{1/p}} & \text{if } f-\ell f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|f-\ell f'\|_1\frac{A}{G}. & \end{cases} \tag{2.7}$$

3. Evaluating the Integral Mean

The following new result holds.

Theorem 7. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $x \in [a, b]$ we have*

(3.1)

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \begin{cases} \frac{b-a}{x}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty & \text{if } f-\ell f'\in L_\infty[a,b], \\ \\ \frac{1}{(2q-1)^{1/q}x(b-a)^{1/q}}\|f-\ell f'\|_p\left[B_q(a,b;x)\right]^{1/q} & \begin{array}{c}\text{if } f-\ell f'\in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1,\end{array} \\ \\ \frac{1}{b-a}\|f-\ell f'\|_1\left(\ln\frac{x}{a}+\frac{b^2-x^2}{2x^2}\right), \end{cases},$$

where

$$B_q(a,b;x)=\begin{cases}\frac{x^q}{2-q}\left(2x^{q-2}-a^{q-2}-b^{q-2}\right) & \\ +\frac{1}{x^{q-1}(q+1)}\left(b^{q+1}+a^{q+1}-2x^{q+1}\right), & q\neq 2 \\ \\ x^2\ln\frac{x^2}{ab}+\frac{b^3+a^3-2x^3}{3x}, & q=2.\end{cases} \tag{3.2}$$

Proof. The first inequality can be proved in an identical way to the case of differentiable functions from [3] by utilizing the first inequality in (2.2).

Utilising the second inequality in (2.2) we have

$$\begin{aligned}\left|\frac{a+b}{2}\cdot f(x)-\frac{x}{b-a}\int_a^b f(t)\,dt\right| &\leq \frac{1}{b-a}\int_a^b |tf(x)-xf(t)|\,dt \\ &\leq \frac{1}{(2q-1)^{1/q}(b-a)}\|f-\ell f'\|_p\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}dt.\end{aligned} \tag{3.3}$$

Utilising Hölder's integral inequality we have

$$\begin{aligned}\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}dt &\leq \left(\int_a^b dt\right)^{1/p}\left(\int_a^b\left[\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}\right]^q dt\right)^{1/q} \\ &= (b-a)^{1/p}\left(\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|dt\right)^{1/q}.\end{aligned} \tag{3.4}$$

For $q \neq 2$ we have

$$
\begin{aligned}
&\int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dt \\
&= \int_a^x \left(\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right) dt + \int_x^b \left(\frac{t^q}{x^{q-1}} - \frac{x^q}{t^{q-1}} \right) dt \\
&= x^q \int_a^x \frac{dt}{t^{q-1}} - \frac{1}{x^{q-1}} \int_a^x t^q dt + \frac{1}{x^{q-1}} \int_x^b t^q dt - x^q \int_x^b \frac{1}{t^{q-1}} dt \\
&= \frac{x^q}{2-q} \left(\frac{1}{x^{2-q}} - \frac{1}{a^{2-q}} \right) - \frac{1}{x^{q-1}(q+1)} \left(x^{q+1} - a^{q+1} \right) \\
&+ \frac{1}{x^{q-1}(q+1)} \left(b^{q+1} - x^{q+1} \right) - \frac{x^q}{2-q} \left(\frac{1}{b^{2-q}} - \frac{1}{x^{2-q}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^q}{2-q} \left(\frac{1}{x^{2-q}} - \frac{1}{a^{2-q}} - \frac{1}{b^{2-q}} + \frac{1}{x^{2-q}} \right) \\
&+ \frac{1}{x^{q-1}(q+1)} \left(b^{q+1} - x^{q+1} - x^{q+1} + a^{q+1} \right) \\
&= \frac{x^q}{2-q} \left(2x^{q-2} - a^{q-2} - b^{q-2} \right) + \frac{1}{x^{q-1}(q+1)} \left(b^{q+1} + a^{q+1} - 2x^{q+1} \right) \\
&= B_q(a, b; x).
\end{aligned}
$$

For $q = 2$ we have

$$
\begin{aligned}
&\int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt \\
&= \int_a^x \left(\frac{x^2}{t} - \frac{t^2}{x} \right) dt + \int_x^b \left(\frac{t^2}{x} - \frac{x^2}{t} \right) dt \\
&= x^2 \int_a^x \frac{dt}{t} - \frac{1}{x} \int_a^x t^2 dt + \frac{1}{x} \int_x^b t^2 dt - x^2 \int_x^b \frac{1}{t} dt \\
&= x^2 \ln \frac{x}{a} - \frac{1}{x} \frac{x^3 - a^3}{3} + \frac{1}{x} \frac{b^3 - x^3}{3} - x^2 \ln \frac{b}{x} \\
&= x^2 \ln \frac{x^2}{ab} + \frac{1}{x} \frac{b^3 + a^3 - 2x^3}{3} = B_2(a, b; x).
\end{aligned}
$$

Utilizing (3.3) and (3.4) we get the second inequality in (3.1).

Utilising the third inequality in (2.2) we have

$$
\begin{aligned}
(3.5) \qquad \left| \frac{a+b}{2} \cdot f(x) - \frac{x}{b-a} \int_a^b f(t)\, dt \right| &\le \frac{1}{b-a} \int_a^b |t f(x) - x f(t)|\, dt \\
&\le \frac{1}{b-a} \|f - \ell f'\|_1 \int_a^b \frac{\max\{t, x\}}{\min\{t, x\}} dt.
\end{aligned}
$$

Since

$$
\int_a^b \frac{\max\{t, x\}}{\min\{t, x\}} dt = \int_a^x \frac{x}{t} dt + \int_x^b \frac{t}{x} dt = x \ln \frac{x}{a} + \frac{1}{x} \frac{b^2 - x^2}{2},
$$

then by (3.5) we have

$$\left|\frac{a+b}{2}\cdot f(x)-\frac{x}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{b-a}\int_a^b |tf(x)-xf(t)|\,dt$$
$$\le \frac{1}{b-a}\|f-\ell f'\|_1\left[x\ln\frac{x}{a}+\frac{1}{x}\frac{b^2-x^2}{2}\right],$$

and the last part of (3.1) is thus proved. □

Remark 3. *If we take in (3.1)* $x=A=A(a,b):=\frac{a+b}{2}$*, then we get*

$$\left|f(A)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.6}$$
$$\le\begin{cases}\frac{b-a}{4A}\|f-\ell f'\|_\infty & \text{if } f-\ell f' \in L_\infty[a,b],\\[2ex] \frac{1}{(2q-1)^{1/q}A(b-a)^{1/q}}\|f-\ell f'\|_p\left[B_q(a,b;A)\right]^{1/q} & \text{if } f-\ell f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{b-a}\|f-\ell f'\|_1\left[\ln\frac{A}{a}+\frac{1}{2}(b-a)\left(\frac{a+3b}{4}\right)A\right],\end{cases},$$

where

$$B_q(a,b;A)=\begin{cases}\frac{2A^q}{2-q}\left(A^{q-2}-A\left(a^{q-2},b^{q-2}\right)\right)\\ +\frac{2}{(q+1)A^{q-1}}\left(A\left(b^{q+1},a^{q+1}\right)-A^{q+1}\right), & q\ne 2\\[2ex] 2A^2\ln\frac{A}{G}+\frac{1}{2}(b-a)^2, & q=2.\end{cases}$$

4. A Related Result

The following new result also holds.

Theorem 8. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b>a>0$*. Then for any* $x\in[a,b]$ *we have*

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \tag{4.1}$$
$$\le\begin{cases}\frac{2}{b-a}\|f-\ell f'\|_\infty\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right) & \text{if } f-\ell f'\in L_\infty[a,b]\\[2ex] \frac{1}{(2q-1)^{1/q}(b-a)^{1/q}}\|f-\ell f'\|_p\left(C_q(a,b;x)\right)^{1/q} & \text{if } f-\ell f'\in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1\\[2ex] \frac{1}{b-a}\|f-\ell f'\|_1\frac{x^2+ab-2ax}{x^2a},\end{cases},$$

where

$$C_q(a,b;x)=\frac{1}{x^{2q-1}}(b+a-2x)+\frac{a^{2-2q}+b^{2-2q}-2x^{2-2q}}{2(q-1)},q>1. \tag{4.2}$$

Proof. From the first inequality in (3.2) we have

$$\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \leq \frac{1}{b-a}\int_a^b\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|dt \tag{4.3}$$
$$\leq \|f-\ell f'\|_\infty \frac{1}{b-a}\int_a^b\left|\frac{1}{t}-\frac{1}{x}\right|dt.$$

Since

$$\begin{aligned}\int_a^b\left|\frac{1}{x}-\frac{1}{t}\right|dt &= \left[\int_a^x\left(\frac{1}{t}-\frac{1}{x}\right)dt+\int_x^b\left(\frac{1}{x}-\frac{1}{t}\right)dt\right]\\ &= \left(\ln\frac{x}{a}-\frac{x-a}{x}+\frac{b-x}{x}-\ln\frac{b}{x}\right)\\ &= \left(\ln\frac{x^2}{ab}+\frac{a+b-2x}{x}\right)\\ &= 2\left(\ln\frac{x}{\sqrt{ab}}+\frac{\frac{a+b}{2}-x}{x}\right)\end{aligned}$$

for any $x\in[a,b]$, then we deduce from (4.3) the first inequality in (4.1).

From the second inequality in (3.2) we have

$$\begin{aligned}&\left|\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b\frac{f(t)}{t}dt\right| \\ &\leq \frac{1}{b-a}\int_a^b\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|dt\\ &\leq \frac{1}{(2q-1)^{1/q}(b-a)}\|f-\ell f'\|_p\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}dt.\end{aligned} \tag{4.4}$$

Utilising Hölder's integral inequality we have

$$\begin{aligned}&\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}dt\\ &\leq\left(\int_a^b dt\right)^{1/p}\left(\int_a^b\left[\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|^{1/q}\right]^q dt\right)^{1/q}\\ &=(b-a)^{1/p}\left(\int_a^b\left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right|dt\right)^{1/q}.\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}
&\int_a^b \left|\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right| dt \\
&= \int_a^x \left(\frac{1}{t^{2q-1}}-\frac{1}{x^{2q-1}}\right) dt + \int_x^b \left(\frac{1}{x^{2q-1}}-\frac{1}{t^{2q-1}}\right) dt \\
&= \frac{x^{2-2q}-a^{2-2q}}{2-2q} - \frac{1}{x^{2q-1}}(x-a) + \frac{1}{x^{2q-1}}(b-x) - \frac{b^{2-2q}-x^{2-2q}}{2-2q} \\
&= \frac{1}{x^{2q-1}}(b+a-2x) + \frac{2x^{2-2q}-a^{2-2q}-b^{2-2q}}{2-2q} \\
&= \frac{1}{x^{2q-1}}(b+a-2x) + \frac{a^{2-2q}+b^{2-2q}-2x^{2-2q}}{2(q-1)} = C_q(a,b;x)
\end{aligned}$$

then by (4.4) and (4.5) we get

$$\begin{aligned}
&\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\
&\leq \frac{1}{(2q-1)^{1/q}(b-a)} \|f-\ell f'\|_p (b-a)^{1/p} (C_q(a,b;x))^{1/q}
\end{aligned}$$

and the second inequality in (4.1) is proved.

From the third inequality in (3.2) we have

$$\begin{aligned}
\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| &\leq \frac{1}{b-a}\int_a^b \left|\frac{f(x)}{x}-\frac{f(t)}{t}\right| dt \\
&\leq \frac{1}{b-a}\|f-\ell f'\|_1 \int_a^b \frac{1}{\min\{t^2,x^2\}} dt.
\end{aligned} \tag{4.6}$$

Since

$$\begin{aligned}
\int_a^b \frac{1}{\min\{t^2,x^2\}} dt &= \int_a^x \frac{dt}{t^2} + \int_x^b \frac{dt}{x^2} = \frac{x-a}{xa} + \frac{b-x}{x^2} \\
&= \frac{x^2+ab-2ax}{x^2 a},
\end{aligned}$$

then by (4.6) we deduce the last part of (4.1). □

Remark 4. *If we take in (4.1) $x = A = A(a,b) := \frac{a+b}{2}$, then we get*

$$\begin{aligned}
&\left|\frac{f(A)}{A} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t} dt\right| \\
&\leq \begin{cases}
\frac{2}{b-a}\|f-\ell f'\|_\infty \ln\left(\frac{A}{G}\right) & \text{if } f-\ell f' \in L_\infty[a,b], \\[2ex]
\frac{1}{(2q-1)^{1/q}(b-a)^{1/q}}\|f-\ell f'\|_p (C_q(a,b;A))^{1/q} & \begin{array}{c}\text{if } f-\ell f' \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1,\end{array} \\[2ex]
\frac{1}{2}\|f-\ell f'\|_1 \frac{A+a}{A^2 a},
\end{cases}
\end{aligned} \tag{4.7}$$

where

$$C_q(a,b;A) = \frac{A\left(a^{2-2q}, b^{2-2q}\right) - A^{2-2q}}{q-1}, q > 1.$$

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