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A Functional Generalization of Trapezoid Inequality

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Abstract. We show in this paper amongst other that, if $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then

$$\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)$$
$$\leq (\geq)\ \frac{x-a}{(b-a)^2}\int_a^b \Phi[f(a)-f(t)]\,dt+\frac{b-x}{(b-a)^2}\int_a^b \Phi[f(b)-f(t)]\,dt$$

for any $x\in[a,b]$.

Natural applications for power and exponential functions are provided as well. Bounds for the Lebesgue p-norms are also given.

1. Introduction

Inequalities providing upper bounds for the quantity

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|,\qquad x\in[a,b] \tag{1.1}$$

are known in the literature as *generalized trapezoid inequalities*,

It has been shown in [**3**], see also [**1**] that

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{1.2}$$

for any $x\in[a,b]$, provided that f is of bounded variation on $[a,b]$.

In particular, we have the *trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f). \tag{1.3}$$

The constant $\frac{1}{2}$ is the best possible.

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If f is absolutely continuous on $[a,b]$, then (see [**2**, p. 93])

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.4}$$
$$\leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1/q}\|f'\|_{[a,b],p} & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_{[a,b],1} \end{cases}$$

for any $x\in[a,b]$.

We used here the Lebesgue norms

$$\|g\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|g(t)|$$

and

$$\|g\|_{[a,b],p}:=\left(\int_a^b|g(t)|^p\,dt\right)^{1/p} \text{ for } p\geq 1.$$

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.5}$$
$$\leq \begin{cases} \frac{1}{4}(b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_p & \text{if } f'\in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|f'\|_1 \end{cases}$$

The constants $\frac{1}{4}$, $\frac{1}{2(q+1)^{1/q}}$ and $\frac{1}{2}$ are the best possible.

Finally, for convex functions $f:[a,b]\to\mathbb{R}$, we have [**5**]

$$\begin{aligned} &\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] \\ &\leq (b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt \\ &\leq \frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_-(a)\right] \end{aligned} \tag{1.6}$$

for any $x\in(a,b)$, provided that $f'_-(b)$ and $f'_+(a)$ are finite. As above, the second inequality also holds for $x=a$ and $x=b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (1.6).

In particular, we have

$$\begin{aligned}(1.7)\qquad &\frac{1}{8}(b-a)^2\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\ &\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &\leq \frac{1}{8}(b-a)\left[f'_-(b)-f'_-(a)\right].\end{aligned}$$

The constant $\frac{1}{8}$ is best possible in both inequalities.

For other recent results on the trapezoid inequality, see [**4**], [**6**], [**7**], [**8**] and [**9**].

2. Generalized Trapezoid Inequalities

The following result holds:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequalities*

$$\begin{aligned}(2.1)\qquad &\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\\ &\leq(\geq)\frac{x-a}{(b-a)^2}\int_a^b\Phi[f(a)-f(t)]\,dt+\frac{b-x}{(b-a)^2}\int_a^b\Phi[f(b)-f(t)]\,dt\end{aligned}$$

for any $x\in[a,b]$.

Proof. We have

$$\begin{aligned}(2.2)\qquad &\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^b\left[\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-f(t)\right]dt\\ &=\frac{1}{b-a}\int_a^b\frac{(x-a)[f(a)-f(t)]+(b-x)[f(b)-f(t)]}{b-a}dt,\end{aligned}$$

for any $x\in[a,b]$.

Using Jensen's inequality for the convex function $\Phi:\mathbb{R}\to\mathbb{R}$ we have

$$\begin{aligned}(2.3)\qquad &\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\\ &=\Phi\left(\frac{1}{b-a}\int_a^b\frac{(x-a)[f(a)-f(t)]+(b-x)[f(b)-f(t)]}{b-a}dt\right)\\ &\leq\frac{1}{b-a}\int_a^b\Phi\left(\frac{(x-a)[f(a)-f(t)]+(b-x)[f(b)-f(t)]}{b-a}\right)dt\end{aligned}$$

for any $x\in[a,b]$.

By the convexity of Φ we also have

$$\begin{aligned}(2.4)\qquad &\Phi\left(\frac{(x-a)[f(a)-f(t)]+(b-x)[f(b)-f(t)]}{b-a}\right)\\ &\leq\frac{x-a}{b-a}\Phi[f(a)-f(t)]+\frac{b-x}{b-a}\Phi[f(b)-f(t)]\end{aligned}$$

for any $x,t\in[a,b]$.

Integrating (2.4) over $t\in[a,b]$ we get

$$\frac{1}{b-a}\int_a^b \Phi\left(\frac{(x-a)[f(a)-f(t)]+(b-x)[f(b)-f(t)]}{b-a}\right)dt \tag{2.5}$$
$$\le \frac{x-a}{(b-a)^2}\int_a^b \Phi[f(a)-f(t)]\,dt+\frac{b-x}{(b-a)^2}\int_a^b \Phi[f(b)-f(t)]\,dt$$

for any $x\in[a,b]$.

Utilising (2.3) and (2.5) we deduce the desired result (2.1). □

Remark 1. *If we write the inequality (2.1) for the convex function $\Phi:\mathbb{R}\to[0,\infty)$, $\Phi(x)=|x|^p$, $p\ge 1$ we have*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p \tag{2.6}$$
$$\le \frac{x-a}{(b-a)^2}\int_a^b |f(t)-f(a)|^p\,dt+\frac{b-x}{(b-a)^2}\int_a^b |f(b)-f(t)|^p\,dt$$

for any $x\in[a,b]$.

If we assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then

$$|f(t)-f(a)|\le\bigvee_a^t(f)\ \text{ and }\ |f(b)-f(t)|\le\bigvee_t^b(f)$$

and by (2.6) we get

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p \tag{2.7}$$
$$\le \frac{x-a}{(b-a)^2}\int_a^b\left(\bigvee_a^t(f)\right)^p dt+\frac{b-x}{(b-a)^2}\int_a^b\left(\bigvee_t^b(f)\right)^p dt$$
$$\le\begin{cases}\frac{1}{b-a}\int_a^b\left[\left(\bigvee_a^t(f)\right)^p+\left(\bigvee_t^b(f)\right)^p\right]dt\\ \qquad\times\left(\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right);\\ \\ \frac{1}{b-a}\max\left\{\int_a^b\left(\bigvee_a^t(f)\right)^p dt,\int_a^b\left(\bigvee_t^b(f)\right)^p dt\right\}.\end{cases}$$

For $p=1$ we have from (2.6) that

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.8}$$
$$\le \frac{x-a}{(b-a)^2}\int_a^b |f(t)-f(a)|\,dt+\frac{b-x}{(b-a)^2}\int_a^b |f(b)-f(t)|\,dt$$

for any $x \in [a,b]$, *while from (2.7) we have*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.9}$$
$$\leq \frac{x-a}{(b-a)^2}\int_a^b\left(\bigvee_a^t(f)\right)dt+\frac{b-x}{(b-a)^2}\int_a^b\left(\bigvee_t^b(f)\right)dt$$
$$\leq\begin{cases}\left(\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right)\bigvee_a^b(f);\\ \frac{1}{b-a}\left(\frac{1}{2}\bigvee_a^b(f)(b-a)+\frac{1}{2}\left|\int_a^b\left[\bigvee_a^t(f)-\bigvee_t^b(f)\right]dt\right|\right);\end{cases}$$

for any $x \in [a,b]$

REMARK 2. *If there exists* $L_a, L_b > 0$ *and* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha p+1, \beta p+1 > 0$ *and such that*

$$|f(t)-f(a)| \leq L_a(t-a)^\alpha \text{ for any } t\in(a,b]$$

and

$$|f(b)-f(t)| \leq L_b(b-t)^\beta \text{ for any } t\in[a,b)$$

then by (2.6)

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p \tag{2.10}$$
$$\leq \frac{L_a}{(\alpha p+1)}(x-a)(b-a)^{\alpha p-1}+\frac{L_b}{(\beta p+1)}(b-x)(b-a)^{\beta p-1}$$

for any $x \in [a,b]$.

If $f:[a,b]\to\mathbb{R}$ *is Lipschitzian with the constant* $L>0$, *then from (2.10) we have*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{L^{1/p}}{(p+1)^{1/p}}(b-a) \tag{2.11}$$

for any $x \in [a,b]$.

COROLLARY 1. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *If* $\Phi:\mathbb{R}\to\mathbb{R}$ *is convex (concave) on* $\mathbb{R}$ *then we have the inequalities*

$$\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \tag{2.12}$$
$$\leq \frac{x-a}{(b-a)^2}\int_a^b\left(\frac{1}{t-a}\int_a^t\Phi\left[(a-t)f'(s)\right]ds\right)dt$$
$$+\frac{b-x}{(b-a)^2}\int_a^b\left(\frac{1}{b-t}\int_t^b\Phi\left[(b-t)f'(s)\right]ds\right)dt$$

for any $x \in [a,b]$.

PROOF. By Jensen's integral inequality we have

$$\Phi\left[f(a)-f(t)\right]=\Phi\left[-\int_a^t f'(s)\,ds\right]=\Phi\left[\frac{1}{t-a}\int_a^t\left[(a-t)f'(s)\right]ds\right]\leq\frac{1}{t-a}\int_a^t\Phi\left[(a-t)f'(s)\right]ds \tag{2.13}$$

for any $t\in(a,b]$ and

$$\Phi\left[f(b)-f(t)\right]=\Phi\left[\int_t^b f'(s)\,ds\right]=\Phi\left[\frac{1}{b-t}\int_t^b\left[(b-t)f'(s)\right]ds\right]\leq\frac{1}{b-t}\int_t^b\Phi\left[(b-t)f'(s)\right]ds \tag{2.14}$$

for any $t\in[a,b)$.

Integrating the inequalities (2.13) and (2.14) over t we get

$$\int_a^b\Phi\left[f(a)-f(t)\right]dt\leq\int_a^b\left(\frac{1}{t-a}\int_a^t\Phi\left[(a-t)f'(s)\right]ds\right)dt \tag{2.15}$$

and

$$\int_a^b\Phi\left[f(b)-f(t)\right]dt\leq\int_a^b\left(\frac{1}{b-t}\int_t^b\Phi\left[(b-t)f'(s)\right]ds\right)dt. \tag{2.16}$$

By making use of (2.1), (2.15) and (2.16) we get the desired result (2.12). □

REMARK 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we write the inequality (2.12) for the convex function $\Phi(x)=|x|^p,\ p\geq 1$ we have*

$$\begin{aligned}&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p\\&\leq\frac{x-a}{(b-a)^2}\int_a^b\left((t-a)^{p-1}\int_a^t|f'(s)|^p\,ds\right)dt\\&+\frac{b-x}{(b-a)^2}\int_a^b\left((b-t)^{p-1}\int_t^b|f'(s)|^p\,ds\right)dt\\&:=K(x)\end{aligned} \tag{2.17}$$

for any $x\in[a,b]$, provided $f'\in L_p[a,b]$.

We also have the bounds

$$K(x) \leq \begin{cases} \frac{1}{b-a}\int_a^b \left[(t-a)^{p-1}\int_a^t |f'(s)|^p\,ds + (b-t)^{p-1}\int_t^b |f'(s)|^p\,ds\right] dt \\ \qquad \times\left(\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right) \\ \\ \frac{1}{b-a}\max\left\{\int_a^b\left((t-a)^{p-1}\int_a^t |f'(s)|^p\,ds\right), \right. \\ \left. \int_a^b\left((b-t)^{p-1}\int_t^b |f'(s)|^p\,ds\right)dt\right\} \end{cases} \tag{2.18}$$

for any $x\in[a,b]$.

Since

$$\int_a^t |f'(s)|^p\,ds \leq \|f'\|_{[a,b],p}^p \;\; \text{and} \;\; \int_t^b |f'(s)|^p\,ds \leq \|f'\|_{[a,b],p}^p, \;\; t\in[a,b],$$

then we have

$$K(x) \leq \frac{(b-a)^{p-1}}{p}\|f'\|_{[a,b],p}^p,$$

which is giving the simpler inequality

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^{1-1/p}}{p^{1/p}}\|f'\|_{[a,b],p} \tag{2.19}$$

for any $x\in[a,b]$.

If we assume that $f'\in L_{[a,b],\infty}$ *then*

$$\int_a^t |f'(s)|^p\,ds \leq (t-a)\|f'\|_{[a,b],\infty}^p$$

and

$$\int_t^b |f'(s)|^p\,ds \leq (b-t)\|f'\|_{[a,b],\infty}^p.$$

From the definition of $K(x)$ *we have*

$$K(x) \leq \frac{(b-a)^p}{p+1}\|f'\|_{[a,b],\infty}^p,$$

which is giving the simpler inequality

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)}{(p+1)^{1/p}}\|f'\|_{[a,b],\infty} \tag{2.20}$$

for any $x\in[a,b]$.

We also have the following inequality for absolutely continuous functions:

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *If* $\Phi:\mathbb{R}\to\mathbb{R}$ *is convex (concave) on* $\mathbb{R}$ *then we have the inequalities*

$$\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b \Phi\left((t-x)f'(t)\right)dt. \tag{2.21}$$

PROOF. Integrating by parts we have the equality, see also [**1**]

$$\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b (t-x)f'(t)\,dt \tag{2.22}$$

for any $x\in[a,b]$.

Using Jensen's integral inequality we have

$$\begin{aligned}&\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\\ &=\Phi\left(\frac{1}{b-a}\int_a^b (t-x)f'(t)\,dt\right)\\ &\le \frac{1}{b-a}\int_a^b \Phi\left((t-x)f'(t)\right)dt\end{aligned}$$

for any $x\in[a,b]$, and the result is proved. □

REMARK 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we write the inequality (2.21) for the convex function $\Phi(x)=|x|^p$, $p\ge 1$ we have*

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^p \le \frac{1}{b-a}\int_a^b |t-x|^p\,|f'(t)|^p\,dt := T(x) \tag{2.23}$$

for any $x\in[a,b]$.

Utilising Hölder's integral inequality we have

$$T(x) \le \begin{cases} \frac{1}{p+1}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right] \times (b-a)^p\,\|f'\|_{[a,b],\infty}^p & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(\alpha p+1)^{1/\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha p+1}+\left(\frac{b-x}{b-a}\right)^{\alpha p+1}\right]^{1/\alpha} \times (b-a)^p\,\|f'\|_{[a,b],p\beta}^p & \text{if } f'\in L_{p\beta}[a,b],\\ & \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^p \|f'\|_{[a,b],p}^p \end{cases} \tag{2.24}$$

and then

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.25}$$

$$\leq\begin{cases}\frac{1}{(p+1)^{1/p}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{1/p} & \text{if } f'\in L_{\infty}[a,b]; \\ \times(b-a)\|f'\|_{[a,b],\infty} & \\ \frac{1}{(\alpha p+1)^{1/p\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha p+1}+\left(\frac{b-x}{b-a}\right)^{\alpha p+1}\right]^{1/p\alpha} & \text{if } f'\in L_{p\beta}[a,b], \\ \times(b-a)\|f'\|_{[a,b],p\beta} & \\ & \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_{[a,b],p} & \end{cases}$$

for any $x\in[a,b]$.

3. Trapezoid Inequalities

Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$ then by taking $x=\frac{a+b}{2}$ in by (2.1) we have

$$\Phi\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \tag{3.1}$$
$$\leq(\geq)\frac{1}{(b-a)}\int_a^b\frac{\Phi[f(a)-f(t)]+\Phi[f(b)-f(t)]}{2}dt.$$

We can refine this inequality as follows:

Proposition 1. *Let* $f:[a,b]\to\mathbb{R}$ *be a Lebesgue integrable function on* $[a,b]$. *If* $\Phi:\mathbb{R}\to\mathbb{R}$ *is convex (concave) on* $\mathbb{R}$*, then*

$$\Phi\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \tag{3.2}$$
$$\leq(\geq)\frac{1}{b-a}\int_a^b\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx$$
$$\leq(\geq)\frac{1}{b-a}\int_a^b\frac{\Phi[f(a)-f(t)]+\Phi[f(b)-f(t)]}{2}dt.$$

Proof. Integrating the inequality (2.1) over $x\in[a,b]$, we get

$$\frac{1}{b-a}\int_a^b\Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \tag{3.3}$$
$$\leq(\geq)\frac{1}{b-a}\int_a^b\left[\frac{x-a}{(b-a)^2}\int_a^b\Phi[f(a)-f(t)]\,dt\right.$$
$$\left.+\frac{b-x}{(b-a)^2}\int_a^b\Phi[f(b)-f(t)]\,dt\right]dx$$
$$=\frac{1}{(b-a)}\int_a^b\frac{\Phi[f(a)-f(t)]+\Phi[f(b)-f(t)]}{2}dt.$$

By Jensen's integral inequality we also have

$$\Phi\left(\frac{1}{b-a}\int_a^b\left[\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]dx\right) \tag{3.4}$$
$$\leq\frac{1}{b-a}\int_a^b\Phi\left(\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx$$

and since

$$\frac{1}{b-a}\int_a^b\left[\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]dx$$
$$=\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

then by (3.3) and (3.4) we get (3.2). □

We denote by $\mathbf{e}:[a,b]\to\mathbb{R}$ the identity function, i.e., $\mathbf{e}(x)=x$. If we write the inequality (3.2) for the convex function $\Phi(x)=|x|^p$, $p\geq 1$, then we have the p-norm inequalities

$$(b-a)^{1/p}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.5}$$
$$\leq\left\|\frac{(\mathbf{e}-a)\,f(a)+(b-\mathbf{e})\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}$$
$$\leq\frac{1}{2^{1/p}}\left[\|f(a)-f\|_{[a,b],p}^p+\|f(b)-f\|_{[a,b],p}^p\right]^{1/p}$$

where f is a Lebesgue integrable function.

Proposition 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $\Phi:\mathbb{R}\to\mathbb{R}$ is convex (concave) on $\mathbb{R}$, then we have the inequalities*

$$\Phi\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \tag{3.6}$$
$$\leq(\geq)\,\frac{1}{b-a}\int_a^b\Phi\left(\frac{(x-a)\,f(a)+(b-x)\,f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx$$
$$\leq(\geq)\,\frac{1}{2(b-a)}\int_a^b\left(\frac{1}{t-a}\int_a^t\Phi\left[(a-t)\,f'(s)\right]ds\right.$$
$$\left.+\frac{1}{b-t}\int_t^b\Phi\left[(b-t)\,f'(s)\right]ds\right)dt.$$

The proof follows (2.12) and the Jensen inequality.

If we write the inequality (3.6) for the convex function $\Phi(x) = |x|^p, p \geq 1$, then we have the p-norm inequalities

$$
\begin{aligned}
(3.7)\quad & (b-a)^{1/p}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \left\|\frac{(\mathbf{e}-a)f(a)+(b-\mathbf{e})f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} \\
&\leq \frac{1}{2^{1/p}}\left[\int_a^b\left((t-a)^{p-1}\int_a^t |f'(s)|^p\,ds+(b-t)^{p-1}\int_t^b |f'(s)|^p\,ds\right)dt\right]^{1/p} \\
&\leq \frac{1}{p^{1/p}}(b-a)\|f'\|_{[a,b],p}.
\end{aligned}
$$

Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then from (2.21) for $x = \frac{a+b}{2}$ we have the inequality

$$
\begin{aligned}
(3.8)\quad & \Phi\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq \frac{1}{b-a}\int_a^b \Phi\left(\left(t-\frac{a+b}{2}\right)f'(t)\right)dt.
\end{aligned}
$$

Utilising Jensen's inequality and (2.21) we also have:

Proposition 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex (concave) on $\mathbb{R}$ then we have the inequalities*

$$
\begin{aligned}
(3.9)\quad & \Phi\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq (\geq)\,\frac{1}{b-a}\int_a^b \Phi\left(\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)dx \\
&\leq (\geq)\,\frac{1}{(b-a)^2}\int_a^b\int_a^b \Phi((t-x)f'(t))\,dt\,dx.
\end{aligned}
$$

If we take the convex function $\Phi(x)=|x|^p, p\geq 1$, then

$$\begin{aligned}
&\int_a^b\int_a^b \Phi((t-x)f'(t))\,dtdx \\
&=\int_a^b\int_a^b |t-x|^p |f'(t)|^p\,dtdx \\
&=\int_a^b\left(\int_a^b |t-x|^p\,dx\right)|f'(t)|^p\,dt \\
&=\int_a^b\left(\int_a^t (t-x)^p\,dx+\int_t^b (x-t)^p\,dx\right)|f'(t)|^p\,dt \\
&=\int_a^b\left(\frac{(t-a)^{p+1}+(b-t)^{p+1}}{p+1}\right)|f'(t)|^p\,dt \\
&=\frac{1}{p+1}\int_a^b\left((t-a)^{p+1}+(b-t)^{p+1}\right)|f'(t)|^p\,dt \\
&=\frac{1}{p+1}\left\|\left[(\mathbf{e}-a)^{1+1/p}+(b-\mathbf{e})^{1+1/p}\right]f'\right\|_{[a,b],p}^p
\end{aligned} \tag{3.10}$$

and then from (3.10) we have the p-norm inequalities

$$\begin{aligned}
&(b-a)^{1/p}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq\left\|\frac{(\mathbf{e}-a)f(a)+(b-\mathbf{e})f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} \\
&\leq\frac{1}{(p+1)^{1/p}(b-a)^{1/p}}\left\|\left[(\mathbf{e}-a)^{1+1/p}+(b-\mathbf{e})^{1+1/p}\right]f'\right\|_{[a,b],p}.
\end{aligned} \tag{3.11}$$

4. Inequalities for the Exponential

Let $g:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function on $[a,b]$. If we take $\Phi:\mathbb{R}\to(0,\infty)$, $\Phi(x)=\exp x$ in the inequality (2.1) then we have the inequality

$$\begin{aligned}
&\exp\left(\frac{(x-a)g(a)+(b-x)g(b)}{b-a}-\frac{1}{b-a}\int_a^b g(t)\,dt\right) \\
&\leq\frac{x-a}{(b-a)^2}\int_a^b \exp[g(a)-g(t)]\,dt+\frac{b-x}{(b-a)^2}\int_a^b \exp[g(b)-g(t)]\,dt \\
&=\left[\frac{x-a}{(b-a)^2}\exp[g(a)]+\frac{b-x}{(b-a)^2}\exp[g(b)]\right]\int_a^b \exp[-g(t)]\,dt
\end{aligned}$$

for any $x \in [a,b]$, which is equivalent with

$$\frac{\exp\left[\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right]}{\exp\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right]} \leq \left[\frac{x-a}{(b-a)^2}\exp[g(a)] + \frac{b-x}{(b-a)^2}\exp[g(b)]\right]\int_a^b \exp[-g(t)]\,dt \tag{4.1}$$

for any $x \in [a,b]$.

Proposition 4. *Let $f : [a,b] \to (0,\infty)$ be a Lebesgue integrable function on $[a,b]$. Then*

$$\frac{[f(a)]^{\frac{x-a}{b-a}}[g(b)]^{\frac{b-x}{b-a}}}{\exp\left[\frac{1}{b-a}\int_a^b \ln f(t)\,dt\right]} \leq \left[\left(\frac{x-a}{b-a}\right)f(a) + \left(\frac{b-x}{b-a}\right)f(b)\right]\frac{1}{b-a}\int_a^b \frac{dt}{f(t)} \tag{4.2}$$

for any $x \in [a,b]$.

The proof follows by (4.1) on taking $g = \ln f$.

Let $g : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we take $\Phi : \mathbb{R} \to (0,\infty)$, $\Phi(x) = \exp x$ in the inequality (2.1) then we have the inequality

$$\begin{aligned}\frac{\exp\left[\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right]}{\exp\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right]} &\leq \frac{x-a}{(b-a)^2}\int_a^b\left(\frac{1}{t-a}\int_a^t \exp[(a-t)g'(s)]\,ds\right)dt \\ &+ \frac{b-x}{(b-a)^2}\int_a^b\left(\frac{1}{b-t}\int_t^b \exp[(b-t)g'(s)]\,ds\right)dt.\end{aligned} \tag{4.3}$$

Moreover, if we assume that there exists the constants γ and Γ such that

$$\gamma \leq g'(s) \leq \Gamma \text{ for almost every } s \in [a,b] \tag{4.4}$$

then

$$\begin{aligned}\int_a^t \exp[(a-t)g'(s)]\,ds &\leq \int_a^t \exp[(a-t)\gamma]\,ds \\ &= (t-a)\exp[(a-t)\gamma]\end{aligned}$$

and

$$\begin{aligned}\int_t^b \exp[(b-t)g'(s)]\,ds &\leq \int_t^b \exp[(b-t)\Gamma]\,ds \\ &= (b-t)\exp[(b-t)\Gamma].\end{aligned}$$

Therefore

$$\int_a^b \left(\frac{1}{t-a}\int_a^t \exp\left[(a-t)\,g'(s)\right]ds\right)dt \le \int_a^b \exp\left[(a-t)\,\gamma\right]dt = \frac{1}{\gamma} - \frac{\exp\left[-(b-a)\,\gamma\right]}{\gamma}$$

and

$$\int_a^b \left(\frac{1}{b-t}\int_t^b \exp\left[(b-t)\,g'(s)\right]ds\right)dt \le \int_a^b \exp\left[(b-t)\,\Gamma\right]dt = -\frac{1}{\Gamma} + \frac{\exp\left[(b-a)\,\Gamma\right]}{\Gamma}.$$

We can state the following proposition:

Proposition 5. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we assume that there exists the constants γ and Γ such that (4.4) holds true, then*

$$\frac{\exp\left[\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right]}{\exp\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right]} \le \frac{b-x}{(b-a)^2}\left(\frac{\exp\left[(b-a)\,\Gamma\right]}{\Gamma} - \frac{1}{\Gamma}\right) + \frac{x-a}{(b-a)^2}\left(\frac{1}{\gamma} - \frac{\exp\left[-(b-a)\,\gamma\right]}{\gamma}\right) \tag{4.5}$$

for any $x\in[a,b]$.

Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we take $\Phi:\mathbb{R}\to(0,\infty)$, $\Phi(x)=\exp x$ in the inequality (2.21) then we have the inequality

$$\frac{\exp\left[\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right]}{\exp\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right]} \le \frac{1}{b-a}\int_a^b \exp\left((t-x)\,g'(t)\right)dt. \tag{4.6}$$

If we assume that there exists the constants γ and Γ such that (4.4) holds true, then

$$\begin{aligned}
&\int_a^b \exp\left((t-x)\,g'(t)\right)dt \\
&= \int_a^x \exp\left((t-x)\,g'(t)\right)dt + \int_x^b \exp\left((t-x)\,g'(t)\right)dt \\
&\le \int_a^x \exp\left((t-x)\,\gamma\right)dt + \int_x^b \exp\left((t-x)\,\Gamma\right)dt \\
&= \frac{1}{\gamma} - \frac{\exp\left(-\gamma\,(x-a)\right)}{\gamma} + \frac{\exp\left(\Gamma\,(b-x)\right)}{\Gamma} - \frac{1}{\Gamma}.
\end{aligned}$$

We can state then the following result:

Proposition 6. *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If we assume that there exists the constants γ and Γ such that (4.4) holds* *true, then*

$$\frac{\exp\left[\frac{(x-a)g(a)+(b-x)g(b)}{b-a}\right]}{\exp\left[\frac{1}{b-a}\int_a^b g(t)\,dt\right]} \leq \frac{\exp(\Gamma(b-x))}{\Gamma}-\frac{1}{\Gamma}+\frac{1}{\gamma}-\frac{\exp(-\gamma(x-a))}{\gamma}, \tag{4.7}$$

for any $x \in [a, b]$.

The interested reader may state some similar inequalities by using the convex function $\Phi : \mathbb{R} \to (0, \infty)$, $\Phi(x) = \cosh(x) := \frac{e^x + e^{-x}}{2}$. The details are omitted.

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