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BOUNDS FOR CONVEX FUNCTIONS OF ČEBYŠEV FUNCTIONAL VIA SONIN'S IDENTITY WITH APPLICATIONS

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Abstract. Some new bounds for the Čebyšev functional in terms of the Lebesgue norms $\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p}$ and the Δ-seminorms $\|f\|_p^{\Delta}:=\left(\int_a^b\int_a^b |f(t)-f(s)|^p\,dtds\right)^{\frac{1}{p}}$ are established. Applications for mid-point and trapezoid inequalities are provided as well.

1. Introduction

For two Lebesgue integrable functions $f,g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [7] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m,M,n,N such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [5], states that

$$|C(f,g)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

provided that f',g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,g:[a,b]\to\mathbb{R}$ are assumed to be absolutely continuous and $f',g'\in L_\infty[a,b]$ while $\|f'\|_\infty=ess\sup_{t\in[a,b]}|f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [12]:

$$|C(f,g)|\le\frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

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provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty [a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [9] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2 [a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [1] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|^p dt \right)^{\frac{1}{p}}, \tag{1.7}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess \sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|, \tag{1.8}$$

provided that $f \in L_p [a,b]$ and $g \in L_q [a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.7) we obtain

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \\ &\leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \end{aligned} \tag{1.9}$$

and if g satisfies (1.3), then

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \\ &\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \\ &\leq \frac{1}{2} (N-n) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt. \end{aligned} \tag{1.10}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [2].

For other recent results on the Grüss inequality, see [8], [10] and [13] and the references therein.

In this paper, some new bounds for the Čebyšev functional in terms of the Lebesgue norms $\left\| f - \frac{1}{b-a} \int_a^b f(t)\, dt \right\|_{[a,b],p}$ and the Δ-seminorms are established. Applications for mid-point and trapezoid inequalities are provided as well.

2. Some Results Via Sonin's Identity

The following result for convex functions of Čebyšev functional holds:

Theorem 1. *Let $f, g : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex on $\mathbb{R}$ then we have the inequality*

$$\Phi[C(f,g)] \leq \frac{1}{b-a} \inf_{\lambda\in\mathbb{R}} \int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)(g(x)-\lambda)\right] dx \tag{2.1}$$
$$\leq \frac{1}{(b-a)^2} \inf_{\lambda\in\mathbb{R}} \int_a^b \int_a^b \Phi[(f(x)-f(t))(g(x)-\lambda)]\,dtdx.$$

Proof. Start with Sonin's identity [11, p. 246]

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)(g(x)-\lambda)\,dx$$

that holds for any $\lambda \in \mathbb{R}$.

If we use Jensen's integral inequality we have

$$\begin{aligned}
\Phi[C(f,g)] &= \Phi\left[\frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)(g(x)-\lambda)\,dx\right] \\
&\leq \frac{1}{b-a}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)(g(x)-\lambda)\right] dx \\
&= \frac{1}{b-a}\int_a^b \Phi\left[\frac{1}{b-a}\int_a^b [(f(x)-f(t))(g(x)-\lambda)]\,dt\right] dx \\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \Phi[(f(x)-f(t))(g(x)-\lambda)]\,dtdx
\end{aligned}$$

for any $\lambda \in \mathbb{R}$.

Taking the infimum over $\lambda \in \mathbb{R}$ we deduce the desired inequalities (2.1). □

Remark 1. *If we write the inequality (2.1) for the convex function $\Phi(x) = |x|^p$, $p \geq 1$ then we get the inequality*

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \left\{\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right|^p |g(x)-\lambda|^p\,dx\right\}^{1/p} \tag{2.2}$$
$$\leq \frac{1}{(b-a)^{2/p}} \inf_{\lambda\in\mathbb{R}} \left\{\int_a^b\int_a^b |f(x)-f(t)|^p\,|g(x)-\lambda|^p\,dtdx\right\}^{1/p}.$$

Utilising Hölder's integral inequality we have

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p |g(x)-\lambda|^p\,dx$$

$$\leq \begin{cases} ess\sup_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p \\ \times \int_a^b |g(x)-\lambda|^p\,dx & \text{if } f\in L_\infty[a,b], g\in L_p[a,b]; \\ \\ \left(\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^{p\beta} dx\right)^{1/\beta} & \text{if } f\in L_{p\beta}[a,b], g\in L_{p\alpha}[a,b] \\ \times \left(\int_a^b |g(x)-\lambda|^{p\alpha}\,dx\right)^{1/\alpha} & \alpha>1, 1/\alpha+1/\beta=1; \\ \\ ess\sup_{x\in[a,b]} |g(x)-\lambda|^p \\ \times \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^p dxdx & \text{if } f\in L_p[a,b], g\in L_\infty[a,b]; \end{cases}$$

$$= \begin{cases} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty}^p \|g-\lambda\|_{[a,b],p}^p & f\in L_\infty[a,b], g\in L_p[a,b]; \\ \\ \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta}^p \|g-\lambda\|_{[a,b],p\alpha}^p & \text{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b], \alpha>1, \\ & 1/\alpha+1/\beta=1; \\ \\ \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p}^p \|g-\lambda\|_{[a,b],\infty}^p & \text{if } f\in L_p[a,b], \\ & g\in L_\infty[a,b]. \end{cases}$$

Utilising the first inequality in (2.2) we can state the following result:

Theorem 2. *Let $f,g:[a,b]\to\mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

$$|C(f,g)| \tag{2.3}$$

$$\leq \frac{1}{(b-a)^{1/p}} \times \begin{cases} \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty} & f\in L_\infty[a,b], \\ & g\in L_p[a,b]; \\ \\ \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta} & \textit{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b], \\ & \alpha>1, \\ & 1/\alpha+1/\beta=1; \\ \\ \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p} & \textit{if } f\in L_p[a,b], \\ & g\in L_\infty[a,b]. \end{cases}$$

We have the following particular cases of interest:

Corollary 1. *Let $f,g:[a,b]\to\mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \tag{2.4}$$
$$\times\begin{cases} \left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],p}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],\infty} & \begin{array}{c} f\in L_\infty[a,b],\\ g\in L_p[a,b]; \end{array}\\ \left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],p\alpha}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p\beta} & \begin{array}{c} \text{if } f\in L_{p\beta}[a,b],\\ g\in L_{p\alpha}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1; \end{array}\\ \left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],\infty}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} & \begin{array}{c} \text{if } f\in L_p[a,b],\\ g\in L_\infty[a,b]. \end{array} \end{cases}$$

If one function is bounded, then we can state the following result:

Corollary 2. *Assume that $f,g:[a,b]\to\mathbb{R}$ are Lebesgue measurable functions on $[a,b]$. If there exists the constant n, N such that $n\le g(t)\le N$ for a.e. $t\in[a,b]$, then*

$$|C(f,g)| \le \frac{1}{2}(N-n) \tag{2.5}$$
$$\times\begin{cases} \left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],\infty} & f\in L_\infty[a,b],\\ \frac{1}{(b-a)^{1/p\beta}}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p\beta} & \begin{array}{c} \text{if } f\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1; \end{array}\\ \frac{1}{(b-a)^{1/p}}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} & \text{if } f\in L_p[a,b]. \end{cases}$$

Proof. We observe that

$$\begin{aligned} \left\|g-\frac{n+N}{2}\right\|_{[a,b],p} &= \left(\int_a^b \left|g(t)-\frac{n+N}{2}\right|^p dt\right)^{1/p} \\ &\le \left(\int_a^b \left(\frac{N-n}{2}\right)^p dt\right)^{1/p} = \frac{N-n}{2}(b-a)^{1/p}, \end{aligned}$$

$$\begin{aligned} \left\|g-\frac{n+N}{2}\right\|_{[a,b],p\alpha} &= \left(\int_a^b \left|g(t)-\frac{n+N}{2}\right|^{p\alpha} dt\right)^{1/p\alpha} \\ &\le \frac{N-n}{2}(b-a)^{1/p\alpha} \end{aligned}$$

and

$$\left\|g-\frac{n+N}{2}\right\|_{[a,b],\infty} \le \frac{N-n}{2}.$$

Utilising (2.3) we deduce the desired result (2.5). □

When one function is of bounded variation, then we can state the following result:

Corollary 3. *If* $f:[a,b]\to\mathbb{R}$ *is Lebesgue integrable and* $g:[a,b]\to\mathbb{R}$ *is of bounded variation, then*

$$(2.6)\quad |C(f,g)| \le \frac{1}{2}\bigvee_a^b(g)\times\begin{cases}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],\infty} & f\in L_\infty[a,b],\\[2ex] \frac{1}{(b-a)^{1/p\beta}}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } f\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \frac{1}{(b-a)^{1/p}}\left\|f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} & \text{if } f\in L_p[a,b],\end{cases}$$

where $\bigvee_a^b(g)$ *is the total variation of the function* g *on the interval on* $[a,b]$.

Proof. Since $g:[a,b]\to\mathbb{R}$ is of bounded variation, then for any $t\in[a,b]$ we have

$$\begin{aligned}\left|g(t)-\frac{g(a)+g(b)}{2}\right| &= \left|\frac{g(t)-g(a)+g(t)-g(b)}{2}\right|\\ &\le \frac{1}{2}\left[|g(t)-g(a)|+|g(b)-g(t)|\right]\le\frac{1}{2}\bigvee_a^b(g).\end{aligned}$$

Then

$$\begin{aligned}\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p} &= \left(\int_a^b\left|g(t)-\frac{g(a)+g(b)}{2}\right|^p dt\right)^{1/p}\\ &\le\left(\int_a^b\left(\frac{1}{2}\bigvee_a^b(g)\right)^p dt\right)^{1/p}=\frac{1}{2}\bigvee_a^b(g)(b-a)^{1/p},\end{aligned}$$

$$\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p\alpha}\le\frac{1}{2}\bigvee_a^b(g)(b-a)^{1/p\alpha}$$

and

$$\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],\infty}\le\frac{1}{2}\bigvee_a^b(g).$$

Utilising (2.3) we deduce the desired result (2.6). □

For functions h that are *Lipschitzian in the middle point* with the constant $L_{\frac{a+b}{2}}$ and the exponent $q>0$, i.e., satisfying the condition

$$\left|h(t)-h\left(\frac{a+b}{2}\right)\right|\le L_{\frac{a+b}{2}}\left|t-\frac{a+b}{2}\right|^q$$

for any $t\in[a,b]$, we have the following result as well.

Corollary 4. *If $f:[a,b]\to\mathbb{R}$ is Lebesgue integrable and $g:[a,b]\to\mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q>0$, then*

$$|C(f,g)| \le L_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^q}{2^q(qp+1)^{1/p}} \left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],\infty} & f\in L_\infty[a,b], \\ \frac{(b-a)^{q-1/p\beta}}{2^q(qp\alpha+1)^{1/p\alpha}} \left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p\beta} & \text{if } f\in L_{p\beta}[a,b],\ \alpha>1,\ 1/\alpha+1/\beta=1; \\ \frac{(b-a)^{q-1/p}}{2^q} \left\| f-\frac{1}{b-a}\int_a^b f(t)\,dt\right\|_{[a,b],p} & \text{if } f\in L_p[a,b]. \end{cases} \tag{2.7}$$

Proof. We have

$$\begin{aligned} \left\| g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p} &= \left(\int_a^b \left| g(t)-g\left(\frac{a+b}{2}\right)\right|^p dt\right)^{1/p} \\ &\le \left(\int_a^b L^p_{\frac{a+b}{2}}\left|t-\frac{a+b}{2}\right|^{qp} dt\right)^{1/p} \\ &= L_{\frac{a+b}{2}}\left(\int_a^b \left|t-\frac{a+b}{2}\right|^{qp} dt\right)^{1/p}. \end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned} &\left(\int_a^b \left|t-\frac{a+b}{2}\right|^{qp} dt\right)^{1/p} \\ &= \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)^{qp} dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{qp} dt\right)^{1/p} \\ &= \left(2\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{qp} dt\right)^{1/p} = \left(2\,\frac{\left(t-\frac{a+b}{2}\right)^{qp+1}}{qp+1}\Bigg|_{\frac{a+b}{2}}^b\right)^{1/p} \\ &= \left(2\frac{\left(\frac{b-a}{2}\right)^{qp+1}}{qp+1}\right)^{1/p} = \left(\frac{(b-a)^{qp+1}}{2^{qp}(qp+1)}\right)^{1/p} = \frac{(b-a)^{q+1/p}}{2^q(qp+1)^{1/p}}. \end{aligned}$$

Then by (2.8) we have

$$\left\| g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p} \le L_{\frac{a+b}{2}}\frac{(b-a)^{q+1/p}}{2^q(qp+1)^{1/p}}.$$

Also

$$\left\| g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p\alpha} \le L_{\frac{a+b}{2}}\frac{(b-a)^{q+1/p\alpha}}{2^q(qp\alpha+1)^{1/p\alpha}}$$

and

$$\left\| g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \le L_{\frac{a+b}{2}}\frac{(b-a)^{q}}{2^q}.$$

Utilising the inequality (2.3)

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \times \begin{cases} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty} & f \in L_\infty[a,b], \\ \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } f \in L_{p\beta}[a,b], \\ \alpha > 1, \\ 1/\alpha + 1/\beta = 1;\end{array} \\ \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p} & \text{if } f \in L_p[a,b]. \end{cases}$$

$$\leq \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^{q+1/p}}{2^q(qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty} & f \in L_\infty[a,b], \\ \frac{(b-a)^{q+1/p\alpha}}{2^q(qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } f \in L_{p\beta}[a,b], \\ \alpha > 1, \\ 1/\alpha + 1/\beta = 1;\end{array} \\ \frac{(b-a)^q}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p} & \text{if } f \in L_p[a,b]. \end{cases}$$

$$= L_{\frac{a+b}{2}} \times \begin{cases} \frac{(b-a)^q}{2^q(qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty} & f \in L_\infty[a,b], \\ \frac{(b-a)^{q-1/p\beta}}{2^q(qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } f \in L_{p\beta}[a,b], \\ \alpha > 1, \\ 1/\alpha + 1/\beta = 1;\end{array} \\ \frac{(b-a)^{q-1/p}}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p} & \text{if } f \in L_p[a,b]. \end{cases}$$

and the inequality (2.7) is proved. □

Remark 2. *If the function g is Lipshitzian with the constant $L > 0$, then*

$$|C(f,g)| \leq L \times \begin{cases} \frac{b-a}{2(p+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],\infty} & f \in L_\infty[a,b], \\ \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p\beta} & \begin{array}{c}\textit{if } f \in L_{p\beta}[a,b], \\ \alpha > 1, \\ 1/\alpha + 1/\beta = 1;\end{array} \\ \frac{(b-a)^{1-1/p}}{2} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,dt \right\|_{[a,b],p} & \textit{if } f \in L_p[a,b]. \end{cases} \tag{2.9}$$

3. Δ-Seminorms and Related Inequalities

For $f \in L_p[a,b]$ $(p \in [1,\infty))$ we can define the functional (see [3] and [4])

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(t) - f(s)|^p \, dtds \right)^{\frac{1}{p}} \tag{3.1}$$

and for $f \in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\Delta} := ess \sup_{(t,s)\in[a,b]^2} |f(t) - f(s)|. \tag{3.2}$$

If we consider $f_\Delta : [a,b]^2 \to \mathbb{R}$,

$$f_\Delta(t,s) = f(t) - f(s),$$

then, obviously

$$\|f\|_p^{\Delta} = \|f_\Delta\|_p, \quad p \in [1,\infty],$$

where $\|\cdot\|_p$ are the usual Lebesgue p-norms on $[a,b]^2$.

Using the properties of the Lebesgue $p-$norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^{\Delta}$:

(i) $\|f\|_p^{\Delta} \geq 0$ for $f \in L_p[a,b]$ and $\|f\|_p^{\Delta} = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$;
(ii) $\|f+g\|_p^{\Delta} \leq \|f\|_p^{\Delta} + \|g\|_p^{\Delta}$ if $f,g \in L_p[a,b]$;
(iii) $\|\alpha f\|_p^{\Delta} = |\alpha| \, \|f\|_p^{\Delta}$.

We call $\|\cdot\|_p^{\Delta}$ as Δ-seminorms.

We note that if $p = 2$, then,

$$\begin{aligned} \|f\|_2^{\Delta} &= \left(\int_a^b \int_a^b (f(t) - f(s))^2 \, dtds \right)^{\frac{1}{2}} \\ &= \sqrt{2} \left[(b-a) \|f\|_2^2 - \left(\int_a^b f(t) \, dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

Using the inequalities (1.2), (1.4) and (1.6), we obtain the following estimate for $\|\cdot\|_2^{\Delta}$:

$$\|f\|_2^{\Delta} \leq \begin{cases} \frac{\sqrt{2}}{2}(M-m)(b-a) & \text{if} \quad m \leq f \leq M; \\ \\ \frac{\sqrt{2}}{2\sqrt{3}} \|f'\|_\infty (b-a)^2 & \text{if} \quad f' \in L_\infty[a,b]; \\ \\ \frac{\sqrt{2}}{\pi} \|f'\|_2 (b-a)^{\frac{3}{2}} & \text{if} \quad f' \in L_2[a,b], \end{cases} \tag{3.4}$$

since

$$\|f\|_2^{\Delta} = \sqrt{2}(b-a)[C(f,f)]^{\frac{1}{2}}.$$

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^{\Delta}$ in terms of $\|f'\|_p$.

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(*i*) *If* $p\in[1,\infty)$*, then we have the inequality*

$$\|f\|_p^{\Delta}\le\begin{cases}\frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\|f'\|_\alpha & \text{if } f'\in L_\alpha[a,b],\\ & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ (b-a)^{\frac{2}{p}}\|f'\|_1 & \text{if } f'\in L_1[a,b].\end{cases}\tag{3.5}$$

(*ii*) *If* $p=\infty$*, then we have the inequality*

$$\|f\|_\infty^{\Delta}\le\begin{cases}(b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\ (b-a)^{\frac{1}{\beta}}\|f'\|_\alpha & \text{if } f'\in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1;\\ \|f'\|_1.\end{cases}\tag{3.6}$$

The following result of Grüss type holds, see [4]:

Theorem 4. *Let* $f,g:[a,b]\to\mathbb{R}$ *be measurable on* $[a,b]$*. Then we have the inequality:*

$$|C(f,g)|\le\frac{1}{2(b-a)^2}\|f\|_p^{\Delta}\|g\|_q^{\Delta},\tag{3.7}$$

where $p=1$, $q=\infty$, *or* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *or* $q=1$ *and* $p=\infty$, *provided all integrals involved exist.*

The inequality is sharp in the sense that if we take $f(x)=g(x)=sgn(x-\alpha)$ *with* $\alpha=\frac{a+b}{2}$, *equality results.*

Making use of the double integral inequality

$$|C(f,g)|\le\frac{1}{(b-a)^{2/p}}\inf_{\lambda\in\mathbb{R}}\left\{\int_a^b\int_a^b|f(x)-f(t)|^p|g(x)-\lambda|^p\,dtdx\right\}^{1/p},\tag{3.8}$$

obtained in (2.2) we can state the following result as well:

Theorem 5. *Let* $f,g:[a,b]\to\mathbb{R}$ *be Lebesgue measurable functions on* $[a,b]$*. Then*

$$|C(f,g)|\le\begin{cases}\frac{1}{(b-a)^{1/p}}\inf_{\lambda\in\mathbb{R}}\|g-\lambda\|_{[a,b],p}\|f\|_\infty^{\Delta} & \text{if } f\in L_\infty[a,b],\\ & g\in L_p[a,b];\\ \frac{1}{(b-a)^{1/p+1/p\beta}}\inf_{\lambda\in\mathbb{R}}\|g-\lambda\|_{[a,b],p\alpha}\|f\|_{p\beta}^{\Delta} & \text{if } f\in L_{p\beta}[a,b],\\ & g\in L_{p\alpha}[a,b]\\ & \alpha>1,1/\alpha+1/\beta=1;\\ \frac{1}{(b-a)^{2/p}}\inf_{\lambda\in\mathbb{R}}\|g-\lambda\|_{[a,b],\infty}\|f\|_p^{\Delta} & \text{if } f\in L_p[a,b],\\ & g\in L_\infty[a,b].\end{cases}\tag{3.9}$$

Proof. Utilising Hölder's inequality for double integrals, we have

$$\int_a^b \int_a^b |f(x)-f(t)|^p |g(x)-\lambda|^p \, dtdx$$

$$\leq \begin{cases} ess\sup_{(x,y)\in[a,b]^2} |f(x)-f(t)|^p & \text{if } f\in L_\infty[a,b], \\ \times \int_a^b \int_a^b |g(x)-\lambda|^p \, dtdx & g\in L_p[a,b]; \\ \\ \left(\int_a^b \int_a^b |f(x)-f(t)|^{p\beta} \, dtdx\right)^{1/\beta} & \text{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b] \\ \times\left(\int_a^b \int_a^b |g(x)-\lambda|^{p\alpha} \, dtdx\right)^{1/\alpha} & \alpha>1, 1/\alpha+1/\beta=1; \\ \\ ess\sup_{x\in[a,b]} |g(x)-\lambda|^p & \text{if } f\in L_p[a,b], \\ \times \int_a^b \int_a^b |f(x)-f(t)|^p \, dtdx & g\in L_\infty[a,b]; \end{cases}$$

$$= \begin{cases} \left(\|f\|_\infty^\Delta\right)^p (b-a) \|g-\lambda\|_{[a,b],p}^p & \text{if } f\in L_\infty[a,b], \\ & g\in L_p[a,b]; \\ \\ \left(\|f\|_{p\beta}^\Delta\right)^p (b-a)^{1/\alpha} \|g-\lambda\|_{[a,b],p\alpha}^p & \text{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \\ \|g-\lambda\|_{[a,b],\infty}^p \left(\|f\|_p^\Delta\right)^p & \text{if } f\in L_p[a,b], \\ & g\in L_\infty[a,b]. \end{cases}$$

Then

$$|C(f,g)|^p \leq \frac{1}{(b-a)^2}$$

$$\times \begin{cases} \left(\|f\|_\infty^\Delta\right)^p (b-a) \|g-\lambda\|_{[a,b],p}^p & \text{if } f\in L_\infty[a,b], \\ & g\in L_p[a,b]; \\ \\ \left(\|f\|_{p\beta}^\Delta\right)^p (b-a)^{1/\alpha} \|g-\lambda\|_{[a,b],p\alpha}^p & \text{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \\ \|g-\lambda\|_{[a,b],\infty}^p \left(\|f\|_p^\Delta\right)^p & \text{if } f\in L_p[a,b], \\ & g\in L_\infty[a,b]. \end{cases}$$

$$= \begin{cases} \frac{1}{b-a} \left(\|f\|_\infty^\Delta\right)^p \|g-\lambda\|_{[a,b],p}^p & \text{if } f\in L_\infty[a,b], \\ & g\in L_p[a,b]; \\ \\ \frac{1}{(b-a)^{1+1/\beta}} \left(\|f\|_{p\beta}^\Delta\right)^p \|g-\lambda\|_{[a,b],p\alpha}^p & \text{if } f\in L_{p\beta}[a,b], \\ & g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \\ \frac{1}{(b-a)^2} \|g-\lambda\|_{[a,b],\infty}^p \left(\|f\|_p^\Delta\right)^p & \text{if } f\in L_p[a,b], \\ & g\in L_\infty[a,b]. \end{cases}$$

Taking the power $1/p$ and then the infimum over $\lambda\in\mathbb{R}$, we get the desired result (3.9). □

Some particular cases of interest are as follows:

Corollary 5. *Let $f,g:[a,b]\to\mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

$$|C(f,g)| \leq \begin{cases} \frac{1}{(b-a)^{1/p}}\left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],p}\|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\ g\in L_p[a,b]; \\ \frac{1}{(b-a)^{1/p+1/p\beta}}\left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],p\alpha}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \frac{1}{(b-a)^{2/p}}\left\|g-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|_{[a,b],\infty}\|f\|_p^\Delta & \text{if } f\in L_p[a,b],\ g\in L_\infty[a,b]. \end{cases} \tag{3.10}$$

The case when one function is bounded is as follows:

Corollary 6. *Assume that $f,g:[a,b]\to\mathbb{R}$ are Lebesgue integrable functions on $[a,b]$. If there exists the constant n,N such that $n\le g(t)\le N$ for a.e. $t\in[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{2}(N-n)\times\begin{cases} \|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\ g\in L_p[a,b]; \\ \frac{1}{(b-a)^{2/p\beta}}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \frac{1}{(b-a)^{2/p}}\|f\|_p^\Delta & \text{if } f\in L_p[a,b],\ g\in L_\infty[a,b]. \end{cases} \tag{3.11}$$

Proof. From (3.9) we have

$$|C(f,g)| \leq \begin{cases} \frac{1}{(b-a)^{1/p}}\left\|g-\frac{N+n}{2}\right\|_{[a,b],p}\|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\ g\in L_p[a,b]; \\ \frac{1}{(b-a)^{1/p+1/p\beta}}\left\|g-\frac{N+n}{2}\right\|_{[a,b],p\alpha}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b] \\ & \alpha>1, 1/\alpha+1/\beta=1; \\ \frac{1}{(b-a)^{2/p}}\left\|g-\frac{N+n}{2}\right\|_{[a,b],\infty}\|f\|_p^\Delta & \text{if } f\in L_p[a,b],\ g\in L_\infty[a,b]. \end{cases} \tag{3.12}$$

Since

$$\left\|g-\frac{n+N}{2}\right\|_{[a,b],p}\le\frac{N-n}{2}(b-a)^{1/p}$$

$$\left\|g-\frac{n+N}{2}\right\|_{[a,b],p\alpha}\le\frac{N-n}{2}(b-a)^{1/p\alpha}$$

and

$$\left\| g-\frac{n+N}{2}\right\|_{[a,b],\infty}\leq \frac{N-n}{2},$$

then by (3.12) we get (3.11). □

The case when one function is of bounded variation, is as follows:

Corollary 7. *If $f:[a,b]\to\mathbb{R}$ is Lebesgue integrable and $g:[a,b]\to\mathbb{R}$ is of bounded variation, then*

$$|C(f,g)| \leq \frac{1}{2}\bigvee_a^b (g)\times\begin{cases}\|f\|_{\infty}^{\Delta} & \text{if } f\in L_{\infty}[a,b],\ g\in L_p[a,b];\\[2ex] \frac{1}{(b-a)^{2/p\beta}}\|f\|_{p\beta}^{\Delta} & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b]\\ & \alpha>1, 1/\alpha+1/\beta=1;\\[2ex] \frac{1}{(b-a)^{2/p}}\|f\|_{p}^{\Delta} & \text{if } f\in L_{p}[a,b],\ g\in L_{\infty}[a,b].\end{cases} \tag{3.13}$$

Proof. From (3.9) we have

$$|C(f,g)| \leq \begin{cases}\frac{1}{(b-a)^{1/p}}\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p}\|f\|_{\infty}^{\Delta} & \text{if } f\in L_{\infty}[a,b],\ g\in L_p[a,b];\\[2ex] \frac{1}{(b-a)^{1/p+1/p\beta}}\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p\alpha}\|f\|_{p\beta}^{\Delta} & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b]\\ & \alpha>1, 1/\alpha+1/\beta=1;\\[2ex] \frac{1}{(b-a)^{2/p}}\left\|g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],\infty}\|f\|_{p}^{\Delta} & \text{if } f\in L_{p}[a,b],\ g\in L_{\infty}[a,b].\end{cases} \tag{3.14}$$

Since

$$\left\| g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p}\leq \frac{1}{2}\bigvee_a^b (g)(b-a)^{1/p},$$

$$\left\| g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],p\alpha}\leq \frac{1}{2}\bigvee_a^b (g)(b-a)^{1/p\alpha}$$

and

$$\left\| g-\frac{g(a)+g(b)}{2}\right\|_{[a,b],\infty}\leq \frac{1}{2}\bigvee_a^b (g),$$

then by (3.14) we get the desired result (3.13). □

Corollary 8. *If $f:[a,b]\to\mathbb{R}$ is Lebesgue integrable and $g:[a,b]\to\mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q>0$, then*

$$|C(f,g)| \le \frac{1}{2^q}L_{\frac{a+b}{2}}\times\begin{cases}\frac{(b-a)^q}{(qp+1)^{1/p}}\|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\ g\in L_p[a,b];\\[2ex] \frac{(b-a)^{q-2/p\beta}}{(qp\alpha+1)^{1/p\alpha}}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b]\\ & \alpha>1,1/\alpha+1/\beta=1;\\[2ex] (b-a)^{q-2/p}\|f\|_p^\Delta & \text{if } f\in L_p[a,b],\ g\in L_\infty[a,b].\end{cases} \tag{3.15}$$

Proof. From (3.9) we have

$$|C(f,g)| \le\begin{cases}\frac{1}{(b-a)^{1/p}}\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p}\|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\ g\in L_p[a,b];\\[2ex] \frac{1}{(b-a)^{1/p+1/p\beta}}\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p\alpha}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\ g\in L_{p\alpha}[a,b]\\ & \alpha>1,1/\alpha+1/\beta=1;\\[2ex] \frac{1}{(b-a)^{2/p}}\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty}\|f\|_p^\Delta & \text{if } f\in L_p[a,b],\ g\in L_\infty[a,b].\end{cases} \tag{3.16}$$

Since

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p}\le L_{\frac{a+b}{2}}\frac{(b-a)^{q+1/p}}{2^q(qp+1)^{1/p}},$$

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p\alpha}\le L_{\frac{a+b}{2}}\frac{(b-a)^{q+1/p\alpha}}{2^q(qp\alpha+1)^{1/p\alpha}}$$

and

$$\left\|g-g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty}\le L_{\frac{a+b}{2}}\frac{(b-a)^q}{2^q},$$

then from (3.16) we deduce the desired result (3.15). □

Remark 3. *If the function g is Lipshitzian with the constant $L>0$, then*

$$|C(f,g)| \le\frac{1}{2}L\times\begin{cases}\frac{b-a}{(p+1)^{1/p}}\|f\|_\infty^\Delta & \text{if } f\in L_\infty[a,b],\\[2ex] \frac{(b-a)^{1-2/p\beta}}{(p\alpha+1)^{1/p\alpha}}\|f\|_{p\beta}^\Delta & \text{if } f\in L_{p\beta}[a,b],\\ & \alpha>1,1/\alpha+1/\beta=1;\\[2ex] (b-a)^{1-2/p}\|f\|_p^\Delta & \text{if } f\in L_p[a,b].\end{cases} \tag{3.17}$$

4. Applications for Mid-point Inequalities

Consider the absolutely continuous function $h:[a,b]\to\mathbb{R}$. We have the following well known representation

$$h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt=\frac{1}{b-a}\int_a^b K(t)\,h'(t)\,dt,$$

where the kernel $K:[a,b]\to\mathbb{R}$ is defined by

$$K(t):=\begin{cases} t-a & \text{if } t\in\left[a,\frac{a+b}{2}\right];\\ t-b & \text{if } t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Since $\int_a^b K(t)\,dt=0$, then

$$\frac{1}{b-a}\int_a^b K(t)\,h'(t)\,dt=C(K,h').$$

Utilising the inequality (2.4) we have

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.1}$$
$$\leq\frac{1}{(b-a)^{1/p}}\times\begin{cases}\|K\|_{[a,b],p}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b],\\ \\ \|K\|_{[a,b],p\alpha}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } h'\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\ \\ \|K\|_{[a,b],\infty}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_p[a,b].\end{cases}$$

Observe that, for $q>0$ we have

$$\begin{aligned}\|K\|_{[a,b],q} &= \left[\int_a^b |K(t)|^q\,dt\right]^{1/q}\\ &= \left[\int_a^{\frac{a+b}{2}}(t-a)^q\,dt+\int_{\frac{a+b}{2}}^b (b-t)^q\,dt\right]^{1/q}\\ &= \left[\left.\frac{(t-a)^{q+1}}{q+1}\right|_a^{\frac{a+b}{2}}-\left.\frac{(b-t)^{q+1}}{q+1}\right|_{\frac{a+b}{2}}^b\right]^{1/q}\\ &= \left[\frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1}+\frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q}=\frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.\end{aligned}$$

Then

$$\|K\|_{[a,b],p}=\frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}},\quad \|K\|_{[a,b],p\alpha}=\frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$

We also have

$$\|K\|_{[a,b],\infty}=\frac{1}{2}(b-a).$$

Making use of (4.1) we get

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.2}$$
$$\leq\begin{cases}\frac{b-a}{2(p+1)^{1/p}}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b],\\[2ex] \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } h'\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \frac{1}{2}(b-a)^{1-1/p}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_p[a,b].\end{cases}$$

For $p=1$ we get the simpler inequalities

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.3}$$
$$\leq\begin{cases}\frac{1}{4}(b-a)\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b],\\[2ex] \frac{1}{2}(b-a)\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_1[a,b].\end{cases}$$

Utilising the inequality (2.5) we have

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.4}$$
$$\leq\frac{1}{2}(\Gamma-\gamma)\times\begin{cases}\|K\|_{[a,b],\infty}\\[2ex] \frac{1}{(b-a)^{1/p\beta}}\|K\|_{[a,b],p\beta} & \begin{array}{c}\alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p}\end{cases}$$

provided that $\gamma\leq h'(t)\leq\Gamma$ for a.e. $t\in[a,b]$.

Utilising the above calculations we then have:

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.5}$$
$$\leq\frac{1}{2}(\Gamma-\gamma)\times\begin{cases}\frac{1}{2}(b-a)\\[2ex] \frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} & \begin{array}{c}\alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \frac{b-a}{2(p+1)^{1/p}}\end{cases}$$

provided that $\gamma\leq h'(t)\leq\Gamma$ for a.e. $t\in[a,b]$.

In particular, for $p=1$ in the third inequality in (4.5) we have

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right|\leq\frac{1}{8}(\Gamma-\gamma)(b-a), \tag{4.6}$$

which is the best inequality one can get from (4.5).

If we use the inequality (2.6) and assume that h' is of bounded variation on $[a,b]$, then

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{4.7}$$
$$\leq\frac{1}{2}\bigvee_a^b(h')\times\begin{cases}\frac{1}{2}(b-a), & \\ \frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} & \alpha>1,\ 1/\alpha+1/\beta=1; \\ \frac{b-a}{2(p+1)^{1/p}}. & \end{cases}$$

From the last inequality in (4.7) for $p=1$ we get

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right|\leq\frac{1}{8}(b-a)\bigvee_a^b(h').$$

If we use the inequality (2.9) and assume that h' is Lipschitzian with the constant $U>0$ then

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right|\leq U\times\begin{cases}\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}, \\ \frac{1}{4}\frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}, \\ \frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}.\end{cases} \tag{4.8}$$

In particular, we get for $p=1$

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,dt\right|\leq\frac{1}{8}(b-a)^2U.$$

5. Applications for Trapezoid Inequalities

Consider the absolutely continuous function $h:[a,b]\to\mathbb{R}$. We have the following well known representation

$$\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt=\frac{1}{b-a}\int_a^b V(t)\,h'(t)\,dt$$

where the kernel $V:[a,b]\to\mathbb{R}$ is defined by

$$V(t):=t-\frac{a+b}{2}.$$

Since $\int_a^b V(t)\,dt=0$, then

$$\frac{1}{b-a}\int_a^b V(t)\,h'(t)\,dt=C(V,h').$$

Utilising the inequality (2.4) we have

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{5.1}$$
$$\le \frac{1}{(b-a)^{1/p}}\times\begin{cases}\|V\|_{[a,b],p}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b],\\[2ex] \|V\|_{[a,b],p\alpha}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } h'\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \|V\|_{[a,b],\infty}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_p[a,b].\end{cases}$$

Observe that, for $q>0$ we have

$$\begin{aligned}\|V\|_{[a,b],q} &= \left[\int_a^b |V(t)|^q\,dt\right]^{1/q}\\ &= \left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^q dt+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q dt\right]^{1/q}\\ &= \left[2\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q dt\right]^{1/q}\\ &= \left[\frac{2\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q}=\frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.\end{aligned}$$

Then

$$\|V\|_{[a,b],p}=\frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}},\quad \|V\|_{[a,b],p\alpha}=\frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$

We also have

$$\|V\|_{[a,b],\infty}=\frac{1}{2}(b-a).$$

Making use of (5.1) we get

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \tag{5.2}$$
$$\le\begin{cases}\frac{b-a}{2(p+1)^{1/p}}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b],\\[2ex] \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta} & \begin{array}{c}\text{if } h'\in L_{p\beta}[a,b],\\ \alpha>1,\\ 1/\alpha+1/\beta=1;\end{array}\\[2ex] \frac{1}{2}(b-a)^{1-1/p}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_p[a,b].\end{cases}$$

For $p = 1$ we get the simpler inequalities

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \begin{cases} \frac{1}{4}(b-a)\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty} & h'\in L_\infty[a,b], \\ \frac{1}{2}(b-a)\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p} & \text{if } h'\in L_1[a,b]. \end{cases} \tag{5.3}$$

Since the p-norms of the kernel V are the same as of K, then we can state the following results as well.

If $\gamma \leq h'(t) \leq \Gamma$ for a.e. $t \in [a,b]$, then we then have:

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \frac{1}{2}(\Gamma-\gamma)\times\begin{cases} \frac{1}{2}(b-a) & \\ \frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} & \alpha>1, \; 1/\alpha+1/\beta=1; \\ \frac{b-a}{2(p+1)^{1/p}}. & \end{cases} \tag{5.4}$$

In particular, for $p = 1$ in the third inequality in (5.4) we have

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \frac{1}{8}(\Gamma-\gamma)(b-a), \tag{5.5}$$

which is the best inequality one can get from (5.4).

If h' is of bounded variation on $[a,b]$, then

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \frac{1}{2}\bigvee_a^b(h')\times\begin{cases} \frac{1}{2}(b-a), & \\ \frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} & \alpha>1, \; 1/\alpha+1/\beta=1; \\ \frac{b-a}{2(p+1)^{1/p}}. & \end{cases} \tag{5.6}$$

From the last inequality in (4.7) for $p = 1$ we get

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,dt\right| \leq \frac{1}{8}(b-a)\bigvee_a^b(h'). \tag{5.7}$$

Assume that h' is Lipschitzian with the constant $U > 0$ then

$$\left| \frac{h(b)+h(a)}{2} - \frac{1}{b-a} \int_a^b h(t)\,dt \right| \leq U \times \begin{cases} \frac{1}{4} \frac{(b-a)^2}{(p+1)^{1/p}}, \\ \\ \frac{1}{4} \frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}, \\ \\ \frac{1}{4} \frac{(b-a)^2}{(p+1)^{1/p}}. \end{cases} \tag{5.8}$$

In particular, we get for $p = 1$

$$\left| \frac{h(b)+h(a)}{2} - \frac{1}{b-a} \int_a^b h(t)\,dt \right| \leq \frac{1}{8}(b-a)^2 U. \tag{5.9}$$

Some similar inequalities may be stated in terms of the Δ-seminorms. However the details are omitted.

6. Some Exponential Inequalities

We can state the following result:

Theorem 6. *Let $f, g : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi : \mathbb{R} \to \mathbb{R}$ is convex and monotonic nondecreasing on $\mathbb{R}$ then we have the inequality*

$$\Phi[C(f,g)] \leq \frac{1}{b-a} \inf_{\mu \in \mathbb{R}} \int_a^b \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] dx. \tag{6.1}$$

Proof. From Theorem 1 we have

$$\begin{aligned} &\Phi[C(f,g)] \\ &\leq \frac{1}{b-a} \int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,dt\right)\right] dx \end{aligned} \tag{6.2}$$

for any $\mu \in \mathbb{R}$.

Utilising the elementary inequality

$$\alpha\beta \leq \left(\frac{\alpha+\beta}{2}\right)^2$$

that holds for any $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} &\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\ &\leq \left(\frac{f(x)+g(x)}{2} - \mu\right)^2 \end{aligned} \tag{6.3}$$

for any $x \in [a,b]$.

Since $\Phi : \mathbb{R} \to \mathbb{R}$ is monotonic nondecreasing on $\mathbb{R}$ then

$$\begin{aligned} &\Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,dt\right)\right] \\ &\leq \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \end{aligned} \tag{6.4}$$

for any $x \in [a,b]$.

Integrating (6.4) over x in $[a,b]$ and taking the infimum over $\mu \in \mathbb{R}$, we deduce the desired result (6.1). $\square$

Remark 4. *Writing the inequality (6.1) for* $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = \exp x$ *we have*

$$\exp[C(f,g)] \leq \frac{1}{b-a} \inf_{\mu \in \mathbb{R}} \int_a^b \exp\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] dx. \tag{6.5}$$

This inequality can provide some exponential inequalitis as follows.

Assume that $f : [a,b] \to \mathbb{R}$ *is Lipschitzian with the constant* $L > 0$ *and* $g : [a,b] \to \mathbb{R}$ *is Lipschitzian with the constant* $K > 0$. *Then by taking*

$$\mu = \frac{f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)}{2}$$

we have

$$\left(\frac{f(x)+g(x)}{2} - \frac{f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)}{2}\right)^2 \leq \left(\frac{L+K}{2}\right)^2 \left(x - \frac{a+b}{2}\right)^2 \tag{6.6}$$

and by (6.5) we have

$$\exp[C(f,g)] \leq \frac{1}{b-a} \int_a^b \exp\left[\left(\frac{L+K}{2}\right)^2 \left(x - \frac{a+b}{2}\right)^2\right] dx. \tag{6.7}$$

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