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TRAPEZOID TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE WITH APPLICATIONS FOR UNITARY OPERATORS IN HILBERT SPACES

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Abstract. Some trapezoid type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of integrators of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided.

1. Introduction

A simple way to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ is by using the *trapezoidal rule*

$$\frac{f(a)+f(b)}{2}\cdot[u(b)-u(a)] \tag{1.1}$$

under different assumptions for the *integrand* f and the *integrator* u for which the above integral exists.

A priory error bounds, namely, upper bounds for the quantity

$$\left|\int_a^b f(t)\,du(t)-\frac{f(a)+f(b)}{2}\cdot[u(b)-u(a)]\right|$$

are known for various pairs (f,u) for which the integral $\int_a^b f(t)\,du(t)$ exists. We present here some simple ones.

Theorem 1 (Dragomir, 2001, [11])**.** *Let $f:[a,b]\to\mathbb{C}$ be a $p-H$-Hölder type function, that is, it satisfies the condition*

$$|f(x)-f(y)|\le H|x-y|^p \text{ for all } x,y\in[a,b], \tag{1.2}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$. Then we have the inequality:

$$\left|\frac{f(a)+f(b)}{2}\cdot[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\le\frac{1}{2^p}H(b-a)^p\bigvee_a^b(u). \tag{1.3}$$

The constant $C=1$ on the right hand side of (1.3) cannot be replaced by a smaller quantity.

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In the case when u is monotonic nondecreasing, we have the following result as well:

Theorem 2 (Dragomir, 2011, [14])**.** *Let $f:[a,b]\to\mathbb{C}$ be a $p-H$-Hölder type mapping where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{R}$ a monotonic nondecreasing function on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}(1.4)\qquad &\left|\frac{f(a)+f(b)}{2}\cdot[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\\ &\leq\frac{1}{2}H\left\{(b-a)^p[u(b)-u(a)]-p\int_a^b\left[\frac{(b-t)^{1-p}-(t-a)^{1-p}}{(b-t)^{1-p}(t-a)^{1-p}}\right]u(t)\,dt\right\}\\ &\leq\frac{1}{2^p}H(b-a)^p[u(b)-u(a)].\end{aligned}$$

The inequalities in (1.4) are sharp.

The case when both the integrand and the integrator are of bounded variation is as follows:

Theorem 3 (Dragomir, 2011, [14])**.** *Let $f,\ u:[a,b]\to\mathbb{C}$ be of bounded variation on $[a,b]$. If one of them is continuous on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and we have the inequality*

$$(1.5)\qquad\left|\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\leq\frac{1}{2}\bigvee_a^b(f)\bigvee_a^b(u).$$

The constant $\frac{1}{2}$ is best possible in (1.5).

For other results of this type see [14] where applications for functions of selfadjoint operators on complex Hilbert spaces are given as well.

For other inequalities for Riemann-Stieltjes integral, see [1]-[5], [6]-[10], [12]-[18] and [20].

Motivated by the above facts, we consider in the present paper the problem of approximating the Riemann-Stieltjes integral $\int_a^b f(e^{is})\,du(s)$ by the trapezoidal rule

$$\frac{f(e^{ib})+f(e^{ia})}{2}[u(b)-u(a)]$$

for continuous complex valued function $f:\mathcal{C}(0,1)\to\mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation. We denote the *error functional* by

$$(1.6)\qquad T_{\mathcal{C}}(f,u;a,b):=\frac{f(e^{ib})+f(e^{ia})}{2}[u(b)-u(a)]-\int_a^b f(e^{is})\,du(s).$$

The Riemann-Stieltjes integral $\int_0^{2\pi} f(e^{is})\,du(s)$ is related with functions of unitary operators U defined on complex Hilbert spaces as follows.

We recall here some basic facts on unitary operators and spectral families that will be used in the sequel.

We say that the bounded linear operator $U:H\to H$ on the Hilbert space H is *unitary* iff $U^*=U^{-1}$. Simple examples of unitary operators are for instance the exponential operators $\exp(iA)$ where A is a selfadjoint bounded linear operator on H.

It is well known that (see for instance [19, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{1.7}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{1.8}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{1.9}$$

for any $x, y \in H$.

From the above properties it follows that the function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[0, 2\pi]$ for any $x \in H$.

Examples of such functions of unitary operators are

$$\exp(U) = \int_0^{2\pi} \exp\left(e^{i\lambda}\right) dE_\lambda$$

and

$$U^n = \int_0^{2\pi} e^{in\lambda} dE_\lambda$$

for n an integer.

We can also define the *trigonometric functions* for a unitary operator U by

$$\sin(U) = \int_0^{2\pi} \sin\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cos(U) = \int_0^{2\pi} \cos\left(e^{i\lambda}\right) dE_\lambda$$

and the *hyperbolic functions* by

$$\sinh(U) = \int_0^{2\pi} \sinh\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cosh(U) = \int_0^{2\pi} \cosh\left(e^{i\lambda}\right) dE_\lambda$$

where

$$\sinh(z) := \frac{1}{2}\left[\exp z - \exp(-z)\right] \text{ and } \cosh(z) := \frac{1}{2}\left[\exp z + \exp(-z)\right], z \in \mathbb{C}.$$

2. Inequalities for the Riemann-Stieltjes Integral

We have the following result.

Theorem 4. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H |z - w|^r \tag{2.1}$$

for any $w, z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given.

If $[a,b] \subseteq [0, 2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then

$$|T_{\mathcal{C}}(f, u; a, b)| \le 2^{r-1} H \max_{s\in[a,b]} B_r(a,b;s) \bigvee_a^b (u) \le \frac{1}{2^r} H (b-a)^r \bigvee_a^b (u) \tag{2.2}$$

for any $t \in [a,b]$, where the bound $B_r(a,b;s)$ is given by

$$\begin{aligned} B_r(a,b;s) &:= \sin^r\left(\frac{b-s}{2}\right) + \sin^r\left(\frac{s-a}{2}\right) \\ &\le \frac{1}{2^r}\left[(b-s)^r + (s-a)^r\right]. \end{aligned} \tag{2.3}$$

Moreover, if $f : \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then

$$|T_{\mathcal{C}}(f, u; a, b)| \le 2K \sin\left(\frac{b-a}{4}\right) \bigvee_a^b (u) \le \frac{1}{2} K (b-a) \bigvee_a^b (u). \tag{2.4}$$

The constant 2 in the first inequality in (2.4) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We have the equality

$$T_{\mathcal{C}}(f, u; a, b) = \int_a^b \left[\frac{f(e^{ib}) + f(e^{ia})}{2} - f(e^{is})\right] du(s). \tag{2.5}$$

It is known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\, dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)\, dv(t)\right| \le \max_{t\in[c,d]} |p(t)| \bigvee_c^d (v). \tag{2.6}$$

Taking the modulus in the equality (2.5) and utilizing the property (2.6) we deduce

$$
\begin{aligned}
(2.7) \qquad \left|T_{\mathcal{C}}(f,u;a,b)\right| &\leq \left|\int_a^b \left[\frac{f\left(e^{ib}\right)+f\left(e^{ia}\right)}{2}-f\left(e^{is}\right)\right]du(s)\right| \\
&\leq \max_{s\in[a,b]}\left|\frac{f\left(e^{ib}\right)+f\left(e^{ia}\right)}{2}-f\left(e^{is}\right)\right|\bigvee_a^b(u) \\
&\leq \frac{1}{2}\max_{s\in[a,b]}\left[\left|f\left(e^{ib}\right)-f\left(e^{is}\right)\right|+\left|f\left(e^{is}\right)-f\left(e^{ia}\right)\right|\right]\bigvee_a^b(u) \\
&\leq \frac{1}{2}H\max_{s\in[a,b]}\left[\left|e^{ib}-e^{is}\right|^r+\left|e^{is}-e^{ia}\right|^r\right]\bigvee_a^b(u).
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|e^{is}-e^{it}\right|^2 &= \left|e^{is}\right|^2-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^2 \\
&= 2-2\cos(s-t)=4\sin^2\left(\frac{s-t}{2}\right)
\end{aligned}
$$

for any $t,s\in\mathbb{R}$, then

$$
(2.8) \qquad \left|e^{is}-e^{it}\right|^r = 2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r
$$

for any $t,s\in\mathbb{R}$.

For $[a,b]\subseteq[0,2\pi]$ we have

$$
\left|e^{ib}-e^{is}\right|^r = 2^r\sin^r\left(\frac{b-s}{2}\right)
$$

and

$$
\left|e^{is}-e^{ia}\right|^r = 2^r\sin^r\left(\frac{s-a}{2}\right)
$$

for any $s\in[a,b]$.

Utilising the inequality (2.7) we deduce the first inequality in (2.2).

By the elementary inequality $\sin x\leq x$ for $x\in[0,\pi]$ we have the inequality (2.3).

Consider the function $\varphi:[a,b]\to\mathbb{R}$, $\varphi(s)=(b-s)^r+(s-a)^r$. We have

$$
\varphi'(s)=r(s-a)^{r-1}-r(b-s)^{r-1}=r\frac{(b-s)^{1-r}-(s-a)^{1-r}}{(b-s)^{1-r}(s-a)^{1-r}}
$$

and

$$
\varphi''(s)=r(r-1)\left[(s-a)^{r-2}+(b-s)^{r-2}\right]
$$

for any $s\in(a,b)$. We observe that $\varphi'(s)=0$ iff $s=\frac{a+b}{2}$, $\varphi'(s)>0$ for $s\in\left(a,\frac{a+b}{2}\right)$ and $\varphi'(s)<0$ for $s\in\left(\frac{a+b}{2},b\right)$ which shows that the function φ is strictly increasing on $\left(a,\frac{a+b}{2}\right)$ and strictly decreasing on $\left(\frac{a+b}{2},b\right)$. Since $\varphi''(s)<0$ for any $s\in(a,b)$, the functions φ is strictly concave on $[a,b]$. We have the bounds

$$
\max_{s\in[a,b]}\varphi(s)=\varphi\left(\frac{a+b}{2}\right)=2^{1-r}(b-a)^r
$$

and

$$\min_{s\in[a,b]}\varphi(s)=\varphi(a)=\varphi(b)=(b-a)^r.$$

This proves the last part of (2.2).

For $r=1$ we have

$$\begin{aligned}B_1(a,b;s)&:=\sin\left(\frac{b-s}{2}\right)+\sin\left(\frac{s-a}{2}\right)\\&=2\sin\left(\frac{b-a}{4}\right)\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\end{aligned}$$

which implies that

$$\begin{aligned}\max_{s\in[a,b]}B_1(a,b;s)&=2\sin\left(\frac{b-a}{4}\right)\max_{s\in[a,b]}\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)\\&=2\sin\left(\frac{b-a}{4}\right)\le\frac{b-a}{2}\end{aligned}$$

which proves the desired result (2.4).

Now, for the best constant, assume that there is a $D>0$ such that

$$\begin{aligned}&\left|\frac{f(e^{ib})+f(e^{ia})}{2}[u(b)-u(a)]-\int_a^b f(e^{is})\,du(s)\right| \qquad (2.9)\\&\le DK\sin\left(\frac{b-a}{4}\right)\bigvee_a^b(u)\end{aligned}$$

for an interval $[a,b]\subseteq[0,2\pi]$ a K-Lipschitzian function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and a function of bounded variation $u:[a,b]\to\mathbb{C}$.

If we take $[a,b]=[0,2\pi]$, $f(z)=z$ then $K=1$ and the inequality (2.9) becomes

$$\left|u(2\pi)-u(0)-\int_0^{2\pi}e^{is}du(s)\right|\le D\bigvee_0^{2\pi}(u) \qquad (2.10)$$

for any function of bounded variation $u:[0,2\pi]\to\mathbb{C}$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_0^{2\pi}e^{is}du(s)=e^{is}u(s)\Big|_0^{2\pi}-i\int_0^{2\pi}e^{is}u(s)\,ds=u(2\pi)-u(0)-i\int_0^{2\pi}e^{is}u(s)\,ds$$

and the inequality (2.10) becomes

$$\left|\int_0^{2\pi}e^{is}u(s)\,ds\right|\le D\bigvee_0^{2\pi}(u) \qquad (2.11)$$

for any function of bounded variation $u:[0,2\pi]\to\mathbb{C}$.

Now, if we take the function

$$u(s):=\begin{cases}-1 & \text{if } s\in[0,\pi]\\ 1 & \text{if } s\in[\pi,2\pi],\end{cases}$$

then u is of bounded variation, $\bigvee_{0}^{2\pi}(u)=2$ and

$$\int_{0}^{2\pi}e^{is}u(s)\,ds=-\int_{0}^{\pi}e^{is}ds+\int_{\pi}^{2\pi}e^{is}ds=-\frac{1}{i}e^{i\pi}+\frac{1}{i}e^{0}+\frac{1}{i}e^{2\pi}-\frac{1}{i}e^{i\pi}=\frac{4}{i}$$

and the inequality (2.11) becomes $4\leq 2D$ showing that $D\geq 2$. □

Remark 1. *If we take $a=0$ and $b=2\pi$, then we get from (2.4) that*

$$\left|f(1)[u(2\pi)-u(0)]-\int_{0}^{2\pi}f\left(e^{is}\right)du(s)\right|\leq 2K\bigvee_{0}^{2\pi}(u). \tag{2.12}$$

Remark 2. *If $0<b-a\leq\pi$ then*

$$\begin{aligned}\max_{s\in[a,b]}B_{r}(a,b;s)&\leq\max_{s\in[a,b]}\sin^{r}\left(\frac{b-s}{2}\right)+\max_{s\in[a,b]}\sin^{r}\left(\frac{s-a}{2}\right)\\&=2\sin^{r}\left(\frac{b-a}{2}\right)\end{aligned}$$

and by (2.2) we have

$$\left|T_{\mathcal{C}}(f,u;a,b)\right|\leq 2^{r}H\sin^{r}\left(\frac{b-a}{2}\right)\bigvee_{a}^{b}(u). \tag{2.13}$$

Theorem 5. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1). If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$, then*

$$\begin{aligned}\left|T_{\mathcal{C}}(f,u;a,b)\right|&\leq 2^{r-1}LH\int_{a}^{b}\left[\sin^{r}\left(\frac{b-s}{2}\right)+\sin^{r}\left(\frac{s-a}{2}\right)\right]ds\\&\leq LH\frac{(b-a)^{r+1}}{(r+1)}.\end{aligned} \tag{2.14}$$

In particular, if $f:\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $K>0$, then we have

$$\left|T_{\mathcal{C}}(f,u;a,b)\right|\leq 8LK\sin^{2}\left(\frac{b-a}{4}\right)\leq\frac{1}{2}LH(b-a)^{2}. \tag{2.15}$$

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a Riemann integrable function and $v:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $M>0$, then the Riemann-Stieltjes integral $\int_{a}^{b}p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_{a}^{b}p(t)\,dv(t)\right|\leq M\int_{a}^{b}|p(t)|\,dt. \tag{2.16}$$

Taking the modulus in the equality (2.5) and utilizing the property (2.16) we deduce

$$
\begin{aligned}
(2.17)\qquad |T_{\mathcal{C}}(f,u;a,b)| &\le \left|\int_a^b \left[\frac{f(e^{ib})+f(e^{ia})}{2}-f(e^{is})\right]du(s)\right| \\
&\le L\int_a^b \left|\frac{f(e^{ib})+f(e^{ia})}{2}-f(e^{is})\right| ds \\
&\le \frac{1}{2}L\int_a^b \left[\left|f(e^{ib})-f(e^{is})\right|+\left|f(e^{is})-f(e^{ia})\right|\right] ds \\
&\le \frac{1}{2}LH\int_a^b \left[\left|e^{ib}-e^{is}\right|^r+\left|e^{is}-e^{ia}\right|^r\right] ds \\
&= 2^{r-1}LH\int_a^b \left[\sin^r\left(\frac{b-s}{2}\right)+\sin^r\left(\frac{s-a}{2}\right)\right] ds
\end{aligned}
$$

which proves the first inequality in (2.14).

On making use of the elementary inequality $\sin x \le x, x\in[0,\pi]$ we have

$$
\begin{aligned}
&\int_a^b \left[\sin^r\left(\frac{b-s}{2}\right)+\sin^r\left(\frac{s-a}{2}\right)\right] ds \\
&\le \int_a^b \left(\frac{b-s}{2}\right)^r ds+\left(\frac{s-a}{2}\right)^r ds \\
&= \frac{(b-a)^{r+1}+(b-a)^{r+1}}{(r+1)2^r}=\frac{(b-a)^{r+1}}{(r+1)2^{r-1}}.
\end{aligned}
$$

This proves the second part of the inequality (2.14).

For $r=1$ we have

$$
\begin{aligned}
&\int_a^b \left[\sin\left(\frac{b-s}{2}\right)+\sin\left(\frac{s-a}{2}\right)\right] ds \\
&= 2\sin\left(\frac{b-a}{4}\right)\int_a^b \cos\left(\frac{s-\frac{a+b}{2}}{2}\right) ds = 8\sin^2\left(\frac{b-a}{4}\right)
\end{aligned}
$$

Using (2.14) for $r=1$ we deduce (2.15). □

Remark 3. *For $a=0$ and $b=2\pi$ we have by (2.15) that*

$$
(2.18)\qquad \left|f(1)\left[u(2\pi)-u(0)\right]-\int_0^{2\pi} f(e^{is})\,du(s)\right| \le 8LK.
$$

The case of monotonic nondecreasing integrators that is important for applications for unitary operators is as follows.

Theorem 6. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1). If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$
\begin{aligned}
(2.19)\qquad |T_{\mathcal{C}}(f,u;a,b)| &\le 2^{r-1}H\int_a^b \left[\sin^r\left(\frac{b-s}{2}\right)+\sin^r\left(\frac{s-a}{2}\right)\right] du(s) \\
&\le \frac{1}{2}H\int_a^b \left[(b-s)^r+(s-a)^r\right] du(s).
\end{aligned}
$$

In particular, if $f:\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $K>0$, then we have

$$\begin{aligned}(2.20)\qquad &|T_{\mathcal{C}}(f,u;a,b)| \\ &\le 2^{1/2}K\sin\left(\frac{b-a}{4}\right)\int_a^b\left[1+\cos\left(s-\frac{a+b}{2}\right)\right]^{1/2}du(s) \\ &\le \frac{1}{2}K(b-a)[u(b)-u(a)].\end{aligned}$$

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$(2.21)\qquad \left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t).$$

Utilising the property (2.21), we have from (2.5) that

$$\begin{aligned}(2.22)\qquad |T_{\mathcal{C}}(f,u;a,b)| &\le \left|\int_a^b\left[\frac{f(e^{ib})+f(e^{ia})}{2}-f(e^{is})\right]du(s)\right| \\ &\le \int_a^b\left|\frac{f(e^{ib})+f(e^{ia})}{2}-f(e^{is})\right|du(s) \\ &\le \frac{1}{2}\int_a^b\left[\left|f(e^{ib})-f(e^{is})\right|+\left|f(e^{is})-f(e^{ia})\right|\right]du(s) \\ &\le \frac{1}{2}H\int_a^b\left[\left|e^{ib}-e^{is}\right|^r+\left|e^{is}-e^{ia}\right|^r\right]du(s) \\ &= 2^{r-1}H\int_a^b\left[\sin^r\left(\frac{b-s}{2}\right)+\sin^r\left(\frac{s-a}{2}\right)\right]du(s),\end{aligned}$$

which proves the first part of (2.19). The second part is obvious.

For $r=1$ we have

$$\begin{aligned}&\int_a^b\left[\sin\left(\frac{b-s}{2}\right)+\sin\left(\frac{s-a}{2}\right)\right]du(s) \\ &= 2\sin\left(\frac{b-a}{4}\right)\int_a^b\cos\left(\frac{s-\frac{a+b}{2}}{2}\right)u(s) \\ &= 2^{1/2}\sin\left(\frac{b-a}{4}\right)\int_a^b\left[1+\cos\left(s-\frac{a+b}{2}\right)\right]^{1/2}du(s).\end{aligned}$$

This proves (2.20). □

Corollary 1. *Assume that f is as in Theorem 6. If the function $u:[0,2\pi]\to\mathbb{R}$ is monotonic nondecreasing on $[0,2\pi]$, then*

$$\begin{aligned}(2.23)\qquad &\left|f(1)[u(2\pi)-u(0)]-\int_0^{2\pi}f(e^{is})\,du(s)\right| \\ &\le 2^rH\int_0^{2\pi}\sin^r\left(\frac{s}{2}\right)du(s)=2^{r/2}H\int_0^{2\pi}(1-\cos s)^{r/2}\,du(s).\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\int_0^{2\pi}\left[\sin^r\left(\frac{2\pi-s}{2}\right)+\sin^r\left(\frac{s}{2}\right)\right]du(s)\\
&=\int_0^{2\pi}\left[\sin^r\left(\pi-\frac{s}{2}\right)+\sin^r\left(\frac{s}{2}\right)\right]du(s)\\
&=\int_t^{2\pi}\left[\sin^r\left(\frac{s}{2}\right)+\sin^r\left(\frac{s}{2}\right)\right]du(s)\\
&=2\int_0^{2\pi}\sin^r\left(\frac{s}{2}\right)du(s)
\end{aligned}$$

and by (2.19) we get (2.23).

Since for $s\in[0,2\pi]$ we have

$$\sin\left(\frac{s}{2}\right)=\left(\frac{1-\cos s}{2}\right)^{1/2},$$

then the last part of (2.23) is obtained. □

3. A Quadrature Rule

We consider the following *partition* of the interval $[a,b]$

$$\Delta_n : a=x_0<x_1<...<x_{n-1}<x_n=b$$

where $0\le k\le n-1$. Define $h_k:=x_{k+1}-x_k$, $0\le k\le n-1$ and $\nu(\Delta_n)=\max\{h_k : 0\le k\le n-1\}$ the norm of the partition Δ_n.

For the continuous function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and the function $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation on $[a,b]$, define the *trapezoid quadrature rule*

$$T_n(f,u,\Delta_n):=\sum_{k=0}^{n-1}\frac{f\left(e^{ix_{k+1}}\right)+f\left(e^{ix_k}\right)}{2}\left[u(x_{k+1})-u(x_k)\right] \tag{3.1}$$

and the remainder $R_n(f,u,\Delta_n)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right)du(t)$ by $T_n(f,u,\Delta_n)$. Then we have

$$\int_a^b f\left(e^{it}\right)du(t)=T_n(f,u,\Delta_n)+R_n(f,u,\Delta_n). \tag{3.2}$$

The following result provides *a priory* bounds for $R_n(f,u,\Delta_n)$ in several instances of f and u as above.

Proposition 1. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z)-f(w)|\le H|z-w|^r$$

for any $w,z\in\mathcal{C}(0,1)$, where $H>0$ and $r\in(0,1]$ are given.

If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n : a=x_0<x_1<...<x_{n-1}<x_n=b$ with the norm

$\nu(\Delta_n) \le \pi$ we have the error bound

$$|R_n(f,u,\Delta_n)| \le 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{x_{k+1}-x_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \le 2^r H \sin^r\left(\frac{\nu(\Delta_n)}{2}\right) \bigvee_a^b (u) \le \nu^r(\Delta_n) H \bigvee_a^b (u). \tag{3.3}$$

Proof. Since $\nu(\Delta_n) \le \pi$, then on writing inequality (2.13) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \le k \le n-1$, we have

$$\left| \int_{x_k}^{x_{k+1}} f(e^{it})\, du(t) - \frac{f(e^{ix_{k+1}}) + f(e^{ix_k})}{2} [u(x_{k+1}) - u(x_k)] \right| \le 2^r H \sin^r\left(\frac{x_{k+1}-x_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \le 2^r H \sin^r\left(\frac{\nu(\Delta_n)}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \le \nu^r(\Delta_n) H \bigvee_{x_k}^{x_{k+1}}(u). \tag{3.4}$$

Summing over k from 0 to $n-1$ in (3.4) and utilizing the generalized triangle inequality, we deduce (3.3). □

Remark 4. *If the function $f : \mathcal{C}(0,1) \to \mathbb{C}$ is Lipschitzian with the constant $K > 0$, then by (2.4) we have a better error bound, namely*

$$|R_n(f,u,\Delta_n)| \le 2K \sum_{k=0}^{n-1} \sin\left(\frac{x_{k+1}-x_k}{4}\right) \bigvee_{x_k}^{x_{k+1}}(u) \le 2K \sin\left(\frac{\nu(\Delta_n)}{4}\right) \bigvee_a^b (u) \le \frac{1}{2}\nu(\Delta_n) K \bigvee_a^b (u). \tag{3.5}$$

Remark 5. *The inequality (3.5) has some particular cases of interest as follows.*
1. If we take $\Delta_2 : a = x_0 = 0,\ x_1 = \pi,\ x_2 = b = 2\pi$, then

$$\begin{aligned} T_2(f,u,\Delta_2) &= \frac{f(-1)+f(1)}{2}[u(\pi) - u(0)] \\ &\quad + \frac{f(1)+f(-1)}{2}[u(2\pi) - u(\pi)] \\ &= \frac{f(1)+f(-1)}{2}[u(2\pi) - u(0)] \end{aligned}$$

and writing the inequality (3.5) for this case we get

$$\left| \int_0^{2\pi} f(e^{it})\, du(t) - \frac{f(1)+f(-1)}{2}[u(2\pi) - u(0)] \right| \le \sqrt{2} K \bigvee_0^{2\pi}(u). \tag{3.6}$$

2. If we take $\Delta_4 : a = x_0 = 0,\ x_1 = \frac{\pi}{2},\ x_2 = \pi,\ x_3 = \frac{3\pi}{2},\ x_4 = b = 2\pi,$ *then*

$$\begin{aligned}
&T_4(f,u,\Delta_4)\\
&= \frac{f(i)+f(1)}{2}\left[u\left(\frac{\pi}{2}\right)-u(0)\right]+\frac{f(i)+f(-1)}{2}\left[u(\pi)-u\left(\frac{\pi}{2}\right)\right]\\
&+\frac{f(-1)+f(-i)}{2}\left[u\left(\frac{3\pi}{2}\right)-u(\pi)\right]+\frac{f(-i)+f(1)}{2}\left[u(2\pi)-u\left(\frac{3\pi}{2}\right)\right]\\
&= \frac{f(1)}{2}\left[u(2\pi)-u\left(\frac{3\pi}{2}\right)+u\left(\frac{\pi}{2}\right)-u(0)\right]+\frac{f(i)}{2}\left[u(\pi)-u(0)\right]\\
&+\frac{f(-1)}{2}\left[u\left(\frac{3\pi}{2}\right)-u\left(\frac{\pi}{2}\right)\right]+\frac{f(-i)}{2}\left[u(2\pi)-u(\pi)\right]
\end{aligned}$$

and writing the inequality (3.5) for this case we get

$$\begin{aligned}
&\left|\int_0^{2\pi} f\left(e^{it}\right)du(t)-\frac{f(1)}{2}\left[u(2\pi)-u\left(\frac{3\pi}{2}\right)+u\left(\frac{\pi}{2}\right)-u(0)\right]\right.\\
&-\frac{f(i)}{2}\left[u(\pi)-u(0)\right]-\frac{f(-1)}{2}\left[u\left(\frac{3\pi}{2}\right)-u\left(\frac{\pi}{2}\right)\right]\\
&\left.-\frac{f(-i)}{2}\left[u(2\pi)-u(\pi)\right]\right|\\
&\leq \sqrt{2-\sqrt{2}}K\bigvee_0^{2\pi}(u).
\end{aligned} \tag{3.7}$$

We consider the following partition of the interval $[0,2\pi]$

$$\Gamma_n : 0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \leq k \leq n-1$. Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \leq k \leq n-1$ and $\nu(\Gamma_n) = \max\{h_k : 0 \leq k \leq n-1\}$ the norm of the partition Γ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$T_n(f,u,\Gamma_n;x,y) := \sum_{k=0}^{n-1}\frac{f\left(e^{ix_{k+1}}\right)+f\left(e^{ix_k}\right)}{2}\left\langle\left(E_{\lambda_{k+1}}-E_{\lambda_k}\right)x,y\right\rangle \tag{3.8}$$

for $x,y \in H$.

For a function $f : \mathcal{C}(0,1) \to \mathbb{C}$ that satisfies a Lipschitz type condition with a constant $K > 0$, we can approximate the function f of unitary operator U as follows

$$\langle f(U)x,y\rangle = T_n(f,\Gamma_n;x,y) + R_n(f,\Gamma_n;x,y) \tag{3.9}$$

for $x,y \in H$, where the reminder satisfies the bounds

$$\begin{aligned}
|R_n(f,\Gamma_n;x,y)| &\leq 2K\sum_{k=0}^{n-1}\sin\left(\frac{\lambda_{k+1}-\lambda_k}{4}\right)\bigvee_{\lambda_k}^{\lambda_{k+1}}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\\
&\leq 2K\sin\left(\frac{\nu(\Gamma_n)}{4}\right)\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq \frac{1}{2}\nu(\Gamma_n)K\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)
\end{aligned} \tag{3.10}$$

for any $x, y \in H$.

Since the following *Total Variation Schwarz type inequality* holds (for a short proof see for instance [15]):

$$\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \|x\|\,\|y\| \tag{3.11}$$

for any $x, y \in H$, then in the bounds above we can replace $\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ with $\|x\|\,\|y\|$.

From (3.6) we have the following trapezoid type inequality for K-Lipshitzian functions $f : \mathcal{C}(0,1) \to \mathbb{C}$ of unitary operators U

$$\begin{aligned}\left|\langle f(U)x, y\rangle - \frac{f(1)+f(-1)}{2}\langle x, y\rangle\right| &\le \sqrt{2}K\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\le \sqrt{2}K\,\|x\|\,\|y\|\end{aligned} \tag{3.12}$$

for any $x, y \in H$.

4. Applications

For $a \neq \pm 1, 0$ consider the function $f : \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. Observe that

$$|f_a(z) - f_a(w)| = \frac{|a|\,|z-w|}{|1-az|\,|1-aw|} \tag{4.1}$$

for any $z, w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ with $t \in [0, 2\pi]$, then we have

$$\begin{aligned}|1-az|^2 &= 1 - 2a\operatorname{Re}(\bar{z}) + a^2|z|^2 = 1 - 2a\cos t + a^2 \\ &\ge 1 - 2|a| + a^2 = (1-|a|)^2\end{aligned}$$

therefore

$$\frac{1}{|1-az|} \le \frac{1}{|1-|a||} \text{ and } \frac{1}{|1-aw|} \le \frac{1}{|1-|a||} \tag{4.2}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilising (4.1) and (4.2) we deduce

$$|f_a(z) - f_a(w)| \le \frac{|a|}{(1-|a|)^2}|z-w| \tag{4.3}$$

for any $z, w \in \mathcal{C}(0,1)$, showing that the function f_a is Lipschitzian with the constant $L_a = \frac{|a|}{(1-|a|)^2}$ on the circle $\mathcal{C}(0,1)$.

If we write the inequality (3.12) for the function f_a, we get

$$\begin{aligned}\left|\left\langle (1-aU)^{-1}x, y\right\rangle - \frac{1}{1-a^2}\langle x, y\rangle\right| &\le \frac{\sqrt{2}\,|a|}{(1-|a|)^2}\bigvee_{0}^{2\pi}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \\ &\le \frac{\sqrt{2}\,|a|}{(1-|a|)^2}K\,\|x\|\,\|y\|\end{aligned} \tag{4.4}$$

for any $x, y \in H$.

Now, for $z,w\in\mathbb{C}$ define the function $f_{z,w}:[0,1]\to\mathbb{C}$, $f_{z,w}(t)=\exp[(1-t)z+tw]$. We observe that $f_{z,w}$ is differentiable on $(0,1)$ and

$$\frac{df_{z,w}(t)}{dt}=(w-z)\exp[(1-t)z+tw]$$

for any $t\in(0,1)$.

We then have

$$\begin{aligned}|\exp(w)-\exp(z)| &= |f_{z,w}(1)-f_{z,w}(0)|=\left|\int_0^1\frac{df_{z,w}(t)}{dt}dt\right| \\ &= \left|(w-z)\int_0^1\exp[(1-t)z+tw]\,dt\right| \\ &\le |w-z|\int_0^1|\exp[(1-t)z+tw]|\,dt \\ &\le |w-z|\int_0^1\exp|(1-t)z+tw|\,dt \\ &\le |w-z|\int_0^1\exp[(1-t)|z|+t|w|]\,dt \end{aligned}\tag{4.5}$$

for any $z,w\in\mathbb{C}$. To obtain this we used the well known inequality $|\exp(u)|\le\exp(|u|)$ for any $u\in\mathbb{C}$.

We observe that if $u\in\mathbb{C}$, then

$$\begin{aligned}|\exp(u)| &= |\exp(\operatorname{Re}u+i\operatorname{Im}u)|=|\exp(\operatorname{Re}u)||\exp(i\operatorname{Im}u)| \\ &= \exp(\operatorname{Re}u)|\cos(\operatorname{Im}u)+i\sin(\operatorname{Im}u)|=\exp(\operatorname{Re}u).\end{aligned}$$

Therefore

$$|\exp[(1-t)z+tw]|=\exp(\operatorname{Re}[(1-t)z+tw])=\exp[(1-t)\operatorname{Re}z+t\operatorname{Re}w]$$

for any $t\in[0,1]$.

From this inequality, we deduce the following result of interest

$$|\exp(w)-\exp(z)|\le|w-z|\int_0^1\exp[(1-t)\operatorname{Re}z+t\operatorname{Re}w]\,dt\tag{4.6}$$

that holds for any $z,w\in\mathbb{C}$.

In the case when $\operatorname{Re}z\neq\operatorname{Re}w$ we have

$$\int_0^1\exp[(1-t)\operatorname{Re}z+t\operatorname{Re}w]\,dt=\frac{\exp(\operatorname{Re}z)-\exp(\operatorname{Re}w)}{\operatorname{Re}z-\operatorname{Re}w},$$

which implies the following inequality of interest:

$$\left|\frac{\exp(w)-\exp(z)}{w-z}\right|\le\frac{\exp(\operatorname{Re}z)-\exp(\operatorname{Re}w)}{\operatorname{Re}z-\operatorname{Re}w}$$

that holds for any $z,w\in\mathbb{C}$ with $\operatorname{Re}z\neq\operatorname{Re}w$.

Now, if $,w\in\mathbb{C}$ with $|w|=|z|=1$, then from (4.5) we have

$$|\exp(w)-\exp(z)|\le e|w-z|$$

which shows that the function $f(z)=\exp(z)$ is Lipschitzian with the constant $L=e$ on the circle $\mathcal{C}(0,1)$.

Utilising the inequality (3.12) we have for any unitary operators U

$$\left|\langle \exp(U)x, y\rangle - \frac{e^2+1}{2e}\langle x, y\rangle\right| \leq \sqrt{2}e\bigvee_0^{2\pi}\left(\langle E_{(\cdot)}x, y\rangle\right) \leq \sqrt{2}e\,\|x\|\,\|y\| \tag{4.7}$$

for any $x, y \in H$.

The interested reader may apply the above results for other Lipschitzian functions. However, the details are not presented here.

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