

**EXPONENTIAL POMPEIU'S TYPE INEQUALITIES WITH
APPLICATIONS TO OSTROWSKI'S INEQUALITY**

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ABSTRACT. In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

1. INTRODUCTION

In 1946, Pompeiu [6] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Lemma 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$(1.1) \quad \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

The following inequality is useful to derive some Ostrowski type inequalities.

Corollary 1 (Pompeiu's Inequality). *With the assumptions of Lemma 1 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a, b)} |f(t) - t f'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$(1.2) \quad |t f(x) - x f(t)| \leq \|f - \ell f'\|_\infty |x - t|$$

for any $t, x \in [a, b]$.

The inequality (1.2) was obtained by the author in [3].

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 1 (Ostrowski, 1938 [4]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $|f'(t)| \leq M < \infty$ for all $t \in (a, b)$. Then for any $x \in [a, b]$, we have the inequality*

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a).$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

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Theorem 2 (Dragomir, 2005 [3]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$(1.4) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty,$$

where $\ell(t) = t$, $t \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [7], E. C. Popa using a mean value theorem obtained a generalization of (1.4) as follows:

Theorem 3 (Popa, 2007 [7]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Assume that $\alpha \notin [a, b]$. Then for any $x \in [a, b]$, we have the inequality*

$$(1.5) \quad \left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f - \ell_\alpha f'\|_\infty,$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p -norm, $1 \leq p \leq \infty$ which for $p = \infty$ give Dragomir's result.

Theorem 4 (Pečarić & Ungar, 2006 [5]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality*

$$(1.6) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq PU(x, p) \|f - \ell f'\|_p,$$

for $x \in [a, b]$, where

$$PU(x, p) : = (b-a)^{\frac{1}{p}-1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right].$$

In the cases $(p, q) = (1, \infty), (\infty, 1)$ and $(2, 2)$ the quantity $PU(x, p)$ has to be taken as the limit as $p \rightarrow 1, \infty$ and 2, respectively.

For other inequalities in terms of the p -norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$ and $\alpha \notin [a, b]$ see [1] and [2].

In this paper, some exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type inequalities.

2. EXPONENTIAL INEQUALITIES

We can provide some similar results for complex-valued functions with the exponential instead of ℓ .

Lemma 2. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) \neq 0$. Then for any $t, x \in [a, b]$ we have*

$$(2.1) \quad \left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| \leq \begin{cases} |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_\infty \times \left| \frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right| & \text{if } f' - \alpha f \in L_\infty[a, b], \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \times \left| \frac{1}{\exp(tq \operatorname{Re}(\alpha))} - \frac{1}{\exp(xq \operatorname{Re}(\alpha))} \right|^{1/q} & \text{if } f' - \alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 \frac{1}{\min\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\}} \end{cases}$$

or, equivalently

$$(2.2) \quad |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| \leq \begin{cases} |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_\infty \times |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| & \text{if } f' - \alpha f \in L_\infty[a, b], \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \times |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} & \text{if } f' - \alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 \max\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\}. \end{cases}$$

Proof. If f is absolutely continuous, then $f/\exp(\alpha \cdot)$ is absolutely continuous on the interval $[a, b]$ and

$$\int_t^x \left(\frac{f(s)}{\exp(\alpha s)} \right)' ds = \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\begin{aligned} \int_t^x \left(\frac{f(s)}{\exp(\alpha s)} \right)' ds &= \int_t^x \frac{f'(s) \exp(\alpha s) - \alpha f(s) \exp(\alpha s)}{\exp(2\alpha s)} ds \\ &= \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds, \end{aligned}$$

then we get the following identity

$$(2.3) \quad \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} = \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds$$

for any $t, x \in [a, b]$ with $x \neq t$.

Taking the modulus in (2.3) we have

$$(2.4) \quad \left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| = \left| \int_t^x \frac{f'(s) - \alpha f(s)}{\exp(\alpha s)} ds \right| \\ \leq \left| \int_t^x \frac{|f'(s) - \alpha f(s)|}{|\exp(\alpha s)|} ds \right| := I$$

and utilizing Hölder's integral inequality we deduce

$$(2.5) \quad I \leq \begin{cases} \sup_{s \in [t, x] \setminus \{x, t\}} |f'(s) - \alpha f(s)| \left| \int_t^x \frac{1}{|\exp(\alpha s)|} ds \right|, \\ \left| \int_t^x |f'(s) - \alpha f(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds \right|^{1/q}, \\ \left| \int_t^x |f'(s) - \alpha f(s)| ds \right| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\}, \\ \left\| f' - \alpha f \right\|_\infty \left| \int_t^x \frac{1}{|\exp(\alpha s)|} ds \right|, \\ \left\| f' - \alpha f \right\|_p \left| \int_t^x \frac{1}{|\exp(\alpha s)|^q} ds \right|^{1/q}, \\ \left\| f' - \alpha f \right\|_1 \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\}. \end{cases}$$

Now, since $\alpha = \operatorname{Re}(\alpha) + i \operatorname{Im}(\alpha)$ and $s \in [a, b]$, then

$$|\exp(\alpha s)| = \exp(s \operatorname{Re}(\alpha) + i s \operatorname{Im}(\alpha)) = |\exp(s \operatorname{Re}(\alpha)) \exp(i s \operatorname{Im}(\alpha))| \\ = |\exp(s \operatorname{Re}(\alpha))| |\exp(i s \operatorname{Im}(\alpha))| = \exp(s \operatorname{Re}(\alpha)).$$

We have

$$\int_t^x \frac{1}{|\exp(\alpha s)|} ds = \int_t^x \frac{1}{\exp(s \operatorname{Re}(\alpha))} ds = \int_t^x \exp(-s \operatorname{Re}(\alpha)) ds \\ = -\operatorname{Re}(\alpha) \exp(-s \operatorname{Re}(\alpha)) \Big|_t^x \\ = -\operatorname{Re}(\alpha) \exp(-x \operatorname{Re}(\alpha)) + \operatorname{Re}(\alpha) \exp(-t \operatorname{Re}(\alpha)) \\ = \operatorname{Re}(\alpha) [\exp(-t \operatorname{Re}(\alpha)) - \exp(-x \operatorname{Re}(\alpha))] \\ = \operatorname{Re}(\alpha) \left[\frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right]$$

and by (2.4) and (2.5) we get

$$\left| \frac{f(x)}{\exp(\alpha x)} - \frac{f(t)}{\exp(\alpha t)} \right| \leq \|f' - \alpha f\|_\infty |\operatorname{Re}(\alpha)| \left| \frac{1}{\exp(t \operatorname{Re}(\alpha))} - \frac{1}{\exp(x \operatorname{Re}(\alpha))} \right|$$

and the first part of (2.1) is proved.

We have

$$\int_t^x \frac{1}{|\exp(\alpha s)|^q} ds = \int_t^x \frac{1}{\exp(sq \operatorname{Re}(\alpha))} ds \\ = q \operatorname{Re}(\alpha) \left[\frac{1}{\exp(tq \operatorname{Re}(\alpha))} - \frac{1}{\exp(xq \operatorname{Re}(\alpha))} \right]$$

and by (2.4) and (2.5) we get the second part of (2.1).

We have

$$\sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(\alpha s)|} \right\} = \frac{1}{\min \{ \exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha)) \}}$$

and by (2.4) and (2.5) we get the last part of (2.1).

The inequality (2.2) follows by (2.1) on multiplying with $|\exp(\alpha x) \exp(\alpha t)|$ and performing the required calculation. \square

The following particular case is of interest.

Corollary 2. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $t, x \in [a, b]$ we have*

$$(2.6) \quad \left| \frac{f(x)}{\exp(x)} - \frac{f(t)}{\exp(t)} \right| \leq \begin{cases} \|f' - f\|_{\infty} \left| \frac{1}{\exp(t)} - \frac{1}{\exp(x)} \right| & \text{if } f' - f \in L_{\infty}[a, b], \\ q^{1/q} \|f' - f\|_p \left| \frac{1}{\exp(tq)} - \frac{1}{\exp(xq)} \right|^{1/q} & \text{if } f' - f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 \frac{1}{\min\{\exp(t), \exp(x)\}}, & \end{cases}$$

or, equivalently

$$(2.7) \quad |\exp(t) f(x) - f(t) \exp(x)| \leq \begin{cases} \|f' - f\|_{\infty} |\exp(x) - \exp(t)| & \text{if } f' - f \in L_{\infty}[a, b], \\ q^{1/q} \|f' - f\|_p |\exp(xq) - \exp(tq)|^{1/q} & \text{if } f' - f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 \max\{\exp(t), \exp(x)\}. & \end{cases}$$

Remark 1. *If $\operatorname{Re}(\alpha) = 0$ then the inequality (2.5) becomes*

$$I \leq \begin{cases} \sup_{s \in [t, x] \setminus \{x, t\}} |f'(s) - i \operatorname{Im}(\alpha) f(s)| \left| \int_t^x \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|} ds \right|, \\ \left| \int_t^x |f'(s) - i \operatorname{Im}(\alpha) f(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|^q} ds \right|^{1/q}, \\ \left| \int_t^x |f'(s) - i \operatorname{Im}(\alpha) f(s)| ds \right| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|\exp(i \operatorname{Im}(\alpha) s)|} \right\}, \end{cases}$$

$$\leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_{\infty} \left| \int_t^x ds \right|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p \left| \int_t^x ds \right|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1, \end{cases} = \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_{\infty} |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1. \end{cases}$$

Therefore we have

$$(2.8) \quad \left| \frac{f(x)}{\exp(i \operatorname{Im}(\alpha)x)} - \frac{f(t)}{\exp(i \operatorname{Im}(\alpha)t)} \right| \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_{\infty} |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1, \end{cases}$$

or, equivalently

$$(2.9) \quad |\exp(i \operatorname{Im}(\alpha)t) f(x) - f(t) \exp(i \operatorname{Im}(\alpha)x)| \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_{\infty} |x - t|, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p |x - t|^{1/q}, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1 \end{cases}$$

for any $t, x \in [a, b]$.

In particular, we have

$$(2.10) \quad \left| \frac{f(x)}{\exp(ix)} - \frac{f(t)}{\exp(it)} \right| \leq \begin{cases} \|f' - if\|_{\infty} |x - t|, \\ \|f' - if\|_p |x - t|^{1/q}, \\ \|f' - if\|_1, \end{cases}$$

or, equivalently

$$(2.11) \quad |\exp(it) f(x) - f(t) \exp(ix)| \leq \begin{cases} \|f' - if\|_{\infty} |x - t|, \\ \|f' - if\|_p |x - t|^{1/q}, \\ \|f' - if\|_1, \end{cases}$$

for any $t, x \in [a, b]$.

3. INEQUALITIES OF OSTROWSKI TYPE

The following result holds:

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$. Then for any $x \in [a, b]$ we have*

$$(3.1) \quad \left| f(x) \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x) \int_a^b f(t) dt \right| \leq \begin{cases} |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_{\infty} B_1(a, b, x, \alpha) & \text{if } f' - \alpha f \in L_{\infty}[a, b], \\ q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} (b - a)^{1/p} \times \|f' - \alpha f\|_p |B_q(a, b, x, \alpha)|^{1/q} & \text{if } f' - \alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - \alpha f\|_1 B_{\infty}(a, b, x, \alpha) \end{cases}$$

where

$$B_q(a, b, x, \alpha) := 2 \left[\exp(xq \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) + \frac{1}{q \operatorname{Re}(\alpha)} \left(\frac{\exp(bq \operatorname{Re}(\alpha)) + \exp(aq \operatorname{Re}(\alpha))}{2} - \exp(xq \operatorname{Re}(\alpha)) \right) \right]$$

for $q \geq 1$ and

$$B_\infty(a, b, x, \alpha) := \exp(x \operatorname{Re}(\alpha)) (x - a) + \frac{\exp(b \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}.$$

Proof. Utilising the first inequality in (2.2) we have

$$(3.2) \quad \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| \leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \leq |\operatorname{Re}(\alpha)| \|f' - \alpha f\|_\infty \int_a^b |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| dt$$

for any $x \in [a, b]$.

Observe that, since $\operatorname{Re}(\alpha) > 0$, then

$$\begin{aligned} & \int_a^b |\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))| dt \\ &= \int_a^x (\exp(x \operatorname{Re}(\alpha)) - \exp(t \operatorname{Re}(\alpha))) dt \\ &+ \int_x^b (\exp(t \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))) dt \\ &= \exp(x \operatorname{Re}(\alpha)) (x - a) - \frac{\exp(t \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)} \Big|_a^x \\ &+ \frac{\exp(t \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)} \Big|_x^b - (b - x) \exp(x \operatorname{Re}(\alpha)) \\ &= \exp(x \operatorname{Re}(\alpha)) (2x - a - b) - \frac{1}{\operatorname{Re}(\alpha)} (\exp(x \operatorname{Re}(\alpha)) - \exp(a \operatorname{Re}(\alpha))) \\ &+ \frac{1}{\operatorname{Re}(\alpha)} (\exp(b \operatorname{Re}(\alpha)) - \exp(x \operatorname{Re}(\alpha))) \\ &= \exp(x \operatorname{Re}(\alpha)) (2x - a - b) \\ &+ \frac{1}{\operatorname{Re}(\alpha)} (\exp(b \operatorname{Re}(\alpha)) + \exp(a \operatorname{Re}(\alpha)) - 2 \exp(x \operatorname{Re}(\alpha))) \\ &= 2 \left[\exp(x \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) + \frac{1}{\operatorname{Re}(\alpha)} \left(\frac{\exp(b \operatorname{Re}(\alpha)) + \exp(a \operatorname{Re}(\alpha))}{2} - \exp(x \operatorname{Re}(\alpha)) \right) \right] \end{aligned}$$

for any $x \in [a, b]$.

Also

$$\begin{aligned} & f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \\ &= f(x) \frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha} - \exp(\alpha x) \int_a^b f(t) dt \end{aligned}$$

for any $x \in [a, b]$ and by (3.2) we get the first inequality in (3.1).

Using the second inequality in (2.2) we have

$$\begin{aligned} (3.3) \quad & \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| \\ & \leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \\ & \leq q^{1/q} |\operatorname{Re}(\alpha)|^{1/q} \|f' - \alpha f\|_p \int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} dt \end{aligned}$$

for any $x \in [a, b]$.

By Hölder's integral inequality we also have

$$\begin{aligned} & \int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q} dt \\ & \leq \left(\int_a^b dt \right)^{1/p} \left[\int_a^b (|\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))|^{1/q})^q dt \right]^{1/q} \\ & = (b-a)^{1/p} \left[\int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))| dt \right]^{1/q}, \end{aligned}$$

for any $x \in [a, b]$.

Observe that, as above, we have

$$\begin{aligned} & \int_a^b |\exp(xq \operatorname{Re}(\alpha)) - \exp(tq \operatorname{Re}(\alpha))| dt \\ &= 2 \left[\exp(xq \operatorname{Re}(\alpha)) \left(x - \frac{a+b}{2} \right) \right. \\ & \left. + \frac{1}{q \operatorname{Re}(\alpha)} \left(\frac{\exp(bq \operatorname{Re}(\alpha)) + \exp(aq \operatorname{Re}(\alpha))}{2} - \exp(xq \operatorname{Re}(\alpha)) \right) \right] \\ &= B_q(a, b, x, \alpha) \end{aligned}$$

for any $x \in [a, b]$ and by (3.3) we get the second part of (3.1).

Using the third inequality in (2.2) we have

$$\begin{aligned} (3.4) \quad & \left| f(x) \int_a^b \exp(\alpha t) dt - \exp(\alpha x) \int_a^b f(t) dt \right| \\ & \leq \int_a^b |\exp(\alpha t) f(x) - f(t) \exp(\alpha x)| dt \\ & \leq \|f' - \alpha f\|_1 \int_a^b \max\{\exp(t \operatorname{Re}(\alpha)), \exp(x \operatorname{Re}(\alpha))\} dt \end{aligned}$$

for any $x \in [a, b]$.

Observe that,

$$\begin{aligned}
& \int_a^b \max \{ \exp (t \operatorname{Re}(\alpha)), \exp (x \operatorname{Re}(\alpha)) \} dt \\
&= \int_a^x \max \{ \exp (t \operatorname{Re}(\alpha)), \exp (x \operatorname{Re}(\alpha)) \} dt \\
&+ \int_x^b \max \{ \exp (t \operatorname{Re}(\alpha)), \exp (x \operatorname{Re}(\alpha)) \} dt \\
&= \int_a^x \exp (x \operatorname{Re}(\alpha)) dt + \int_x^b \exp (t \operatorname{Re}(\alpha)) dt = \\
&= \exp (x \operatorname{Re}(\alpha)) (x-a) + \frac{\exp (b \operatorname{Re}(\alpha)) - \exp (x \operatorname{Re}(\alpha))}{\operatorname{Re}(\alpha)}
\end{aligned}$$

and by (3.4) we get the third part of (3.1). \square

Remark 2. *If $\operatorname{Re}(\alpha) < 0$, then a similar result may be stated. However the details are left to the interested reader.*

Corollary 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $x \in [a, b]$ we have*

$$(3.5) \quad \left| f(x) [\exp(b) - \exp(a)] - \exp(x) \int_a^b f(t) dt \right|$$

$$\leq \begin{cases} \|f' - f\|_\infty B_1(a, b, x) & \text{if } f' - f \in L_\infty[a, b], \\ q^{1/q} (b-a)^{1/p} \|f' - f\|_p & \text{if } f' - f \in L_p[a, b] \\ \quad \times |B_q(a, b, x)|^{1/q} & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - f\|_1 B_\infty(a, b, x) & \end{cases}$$

where

$$\begin{aligned}
& B_q(a, b, x) \\
&:= 2 \left[\left(x - \frac{a+b}{2} \right) \exp(xq) + \frac{1}{q} \left(\frac{\exp(bq) + \exp(aq)}{2} - \exp(xq) \right) \right]
\end{aligned}$$

for $q \geq 1$ and

$$B_\infty(a, b, x) := (x-a) \exp(x) + \exp(b) - \exp(x).$$

Remark 3. *The midpoint case is as follows:*

$$(3.6) \quad \left| f\left(\frac{a+b}{2}\right) [\exp(b) - \exp(a)] - \exp\left(\frac{a+b}{2}\right) \int_a^b f(t) dt \right|$$

$$\leq \begin{cases} \|f' - f\|_\infty B_1(a, b) & \text{if } f' - f \in L_\infty[a, b], \\ q^{1/q} (b-a)^{1/p} \|f' - f\|_p & \text{if } f' - f \in L_p[a, b] \\ \quad \times |B_q(a, b)|^{1/q} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \|f' - f\|_1 B_\infty(a, b) & \end{cases}$$

where

$$B_q(a, b, x) := \frac{2}{q} \left(\frac{\exp(bq) + \exp(aq)}{2} - \exp\left(\frac{a+b}{2}q\right) \right)$$

for $q \geq 1$ and

$$B_\infty(a, b) := \frac{b-a}{2} \exp\left(\frac{a+b}{2}\right) + \exp(b) - \exp\left(\frac{a+b}{2}\right).$$

The case $\operatorname{Re}(\alpha) = 0$ is different and may be stated as follows.

Theorem 6. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) = 0$ and $\operatorname{Im}(\alpha) \neq 0$. Then for any $x \in [a, b]$ we have*

$$(3.7) \quad \left| f(x) \frac{\exp(i \operatorname{Im}(\alpha) b) - \exp(i \operatorname{Im}(\alpha) a)}{i \operatorname{Im}(\alpha)} - \exp(i \operatorname{Im}(\alpha) x) \int_a^b f(t) dt \right|$$

$$\leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty & \text{if } f' - i \operatorname{Im}(\alpha) f \in L_\infty[a, b], \\ \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \\ \frac{q}{q+1} \|f' - i \operatorname{Im}(\alpha) f\|_p & \text{if } f' - i \operatorname{Im}(\alpha) f \in L_p[a, b] \\ \times \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \|f' - i \operatorname{Im}(\alpha) f\|_1 (b-a). & \end{cases}$$

Proof. Utilizing the inequality (2.9) we have

$$\begin{aligned}
(3.8) \quad & \left| f(x) \frac{\exp(i \operatorname{Im}(\alpha) b) - \exp(i \operatorname{Im}(\alpha) a)}{i \operatorname{Im}(\alpha)} - \exp(i \operatorname{Im}(\alpha) x) \int_a^b f(t) dt \right| \\
& \leq \int_a^b |\exp(i \operatorname{Im}(\alpha) t) f(x) - f(t) \exp(i \operatorname{Im}(\alpha) x)| dt \\
& \leq \begin{cases} \|f' - i \operatorname{Im}(\alpha) f\|_\infty \int_a^b |x-t| dt, \\ \|f' - i \operatorname{Im}(\alpha) f\|_p \int_a^b |x-t|^{1/q} dt, \\ \|f' - i \operatorname{Im}(\alpha) f\|_1 \int_a^b dt. \end{cases}
\end{aligned}$$

Since

$$\int_a^b |x-t| dt = \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2$$

and

$$\int_a^b |x-t|^{1/q} dt = \frac{q}{q+1} \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}},$$

then we get from (3.8) the desired result (3.7). \square

Corollary 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$. Then for any $x \in [a, b]$ we have*

$$\begin{aligned}
(3.9) \quad & \left| f(x) \frac{\exp(ib) - \exp(ia)}{i} - \exp(ix) \int_a^b f(t) dt \right| \\
& \leq \begin{cases} \|f' - if\|_\infty \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)^2 & \text{if } f' - if \in L_\infty[a, b], \\ \frac{q}{q+1} \|f' - if\|_p \left[\left(\frac{b-x}{b-a} \right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a} \right)^{\frac{q+1}{q}} \right] (b-a)^{\frac{q+1}{q}} & \text{if } f' - if \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - if\|_1 (b-a). & \end{cases}
\end{aligned}$$

Remark 4. *The midpoint case is as follows*

$$(3.10) \quad \left| f\left(\frac{a+b}{2}\right) \frac{\exp(ib) - \exp(ia)}{i} - \exp\left(i\frac{a+b}{2}\right) \int_a^b f(t) dt \right|$$

$$\leq \begin{cases} \frac{1}{4} \|f' - if\|_\infty (b-a)^2, & \text{if } f' - if \in L_\infty[a, b], \\ \frac{q}{(q+1)2^{1/q}} \|f' - if\|_p (b-a)^{\frac{q+1}{q}}, & \text{if } f' - if \in L_p[a, b]. \end{cases}$$

Similar inequalities may be stated if one uses (2.1) and integrates over t on $[a, b]$. The details are left to the interested reader.

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