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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR h-CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Some inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)<\int_a^b f(x)dx<(b-a)\frac{f(a)+f(b)}{2},\quad a,b\in\mathbb{R}. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [41]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [41]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[24], [31]-[34] and [44].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [45, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

Motivated by the above results, in this paper we extend the concept of h-convexity introduced for functions of a real variable in [51] to functions defined on convex subsets of real or complex linear spaces and provide some Hermite-Hadamard type inequalities generalizing and improving (1.2). Natural applications that refine the norm inequality (1.3) are also given.

2. h-Convex Functions on Linear Spaces

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([36])**.** *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{2.1}$$

Some further properties of this class of functions can be found in [27], [28], [30], [42], [45] and [46]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (2.1) is satisfied for any vectors $x, y \in C$ and $t \in (0, 1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([30])**.** *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{2.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{2.3}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P-functions see [30] and [43] while for quasi convex functions, the reader can consult [29].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2.2) (or (2.3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [37], [39] and [48].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([51])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{2.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [51], [6], [40], [49], [47] and [50].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{2.5}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We can prove now the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces.

Theorem 1. *Assume that the function $f : C \subseteq X \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq [f(x) + f(y)] \int_0^1 h(t)\, dt. \tag{2.6}$$

Proof. By the h-convexity of f we have

$$f(tx+(1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{2.7}$$

for any $t \in [0,1]$.

Integrating (2.7) on $[0,1]$ over t, we get

$$\int_0^1 f(tx+(1-t)y)\,dt \le f(x)\int_0^1 h(t)\,dt + f(y)\int_0^1 h(1-t)\,dt$$

and since $\int_0^1 h(t)\,dt = \int_0^1 h(1-t)\,dt$, we get the second part of (2.6).

From the h-convexity of f we have

$$f\left(\frac{z+w}{2}\right) \le h\left(\frac{1}{2}\right)[f(z)+f(w)] \tag{2.8}$$

for any $z, w \in C$.

If we take in (2.8) $z = tx+(1-t)y$ and $w = (1-t)x+ty$, then we get

$$f\left(\frac{x+y}{2}\right) \le h\left(\frac{1}{2}\right)[f(tx+(1-t)y)+f((1-t)x+ty)] \tag{2.9}$$

for any $t \in [0,1]$.

Integrating (2.9) on $[0,1]$ over t and taking into account that

$$\int_0^1 f(tx+(1-t)y)\,dt = \int_0^1 f((1-t)x+ty)\,dt$$

we get the first inequality in (2.6). □

Remark 1. *If $f : I \to [0,\infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a, b \in I, a < b$, then from (2.6) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in* [47]

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \int_a^b f(u)\,du \le [f(a)+f(b)]\int_0^1 h(t)\,dt.$$

If we write (2.6) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions 1.2.

If we write (2.6) for the case of P-type functions $f : C \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]\,dt \le f(x)+f(y), \tag{2.10}$$

that has been obtained for functions of real variable in [30].

If f is Breckner s-convex on C, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (2.6) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]\,dt \le \frac{f(x)+f(y)}{s+1}, \tag{2.11}$$

that was obtained for functions of a real variable in [25].

Since the function $g(x) = \|x\|^s$ is Breckner s-convex on on the normed linear space X, $s \in (0,1)$, then for any $x, y \in X$ we have

$$\frac{1}{2}\|x+y\|^s \le \int_0^1 \|(1-t)x+ty\|^s\,dt \le \frac{\|x\|^s+\|x\|^s}{s+1}. \tag{2.12}$$

If $f : C \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1)$, then

$$\frac{1}{2^{s+1}} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt \leq \frac{f(x) + f(y)}{1-s}. \tag{2.13}$$

We notice that for $s = 1$ the first inequality in (2.13) still holds, i.e.

$$\frac{1}{4} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]\, dt. \tag{2.14}$$

The case for functions of real variables was obtained for the first time in [30].

3. Refinements

The following representation result holds.

Lemma 1. *Let $f : C \subseteq X \to \mathbb{C}$ where C is a convex subset of the real or complex linear space X. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then for any $\lambda \in [0, 1]$ we have the representation*

$$\begin{aligned} \int_0^1 f[(1-t)x + ty]\, dt &= (1-\lambda) \int_0^1 f[(1-t)((1-\lambda)x + \lambda y) + ty]\, dt \\ &\quad + \lambda \int_0^1 f[(1-t)x + t((1-\lambda)x + \lambda y)]\, dt. \end{aligned} \tag{3.1}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (3.1) is obvious.

Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 f[(1-t)(\lambda y + (1-\lambda)x) + ty]\, dt = \int_0^1 f[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\, dt$$

and

$$\int_0^1 f[t(\lambda y + (1-\lambda)x) + (1-t)x]\, dt = \int_0^1 f[t\lambda y + (1-\lambda t)x]\, dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)du$. Then

$$\int_0^1 f[((1-t)\lambda + t)y + (1-t)(1-\lambda)x]\, dt = \frac{1}{1-\lambda} \int_\lambda^1 f[uy + (1-u)x]\, du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f[t\lambda y + (1-\lambda t)x]\, dt = \frac{1}{\lambda} \int_0^\lambda f[uy + (1-u)x]\, du.$$

Therefore

$$\begin{aligned} &(1-\lambda) \int_0^1 f[(1-t)(\lambda y + (1-\lambda)x) + ty]\, dt + \lambda \int_0^1 f[t(\lambda y + (1-\lambda)x) + (1-t)x]\, dt \\ &= \int_\lambda^1 f[uy + (1-u)x]\, du + \int_0^\lambda f[uy + (1-u)x]\, du = \int_0^1 f[uy + (1-u)x]\, du \end{aligned}$$

and the identity (3.1) is proved. $\square$

Theorem 2. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then for any $\lambda \in [0,1]$ we have the inequalities*

$$
\begin{aligned}
(3.2)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda) f\left[\frac{(1-\lambda)x + (\lambda+1)y}{2}\right] + \lambda f\left[\frac{(2-\lambda)x + \lambda y}{2}\right]\right\} \\
& \leq \int_0^1 f[(1-t)x + ty]\,dt \\
& \leq [f((1-\lambda)x + \lambda y) + (1-\lambda) f(y) + \lambda f(x)] \int_0^1 h(t)\,dt \\
& \leq \{[h(1-\lambda) + \lambda] f(x) + [h(\lambda) + 1 - \lambda] f(y)\} \int_0^1 h(t)\,dt.
\end{aligned}
$$

Proof. Since $f : C \subseteq X \to [0, \infty)$ is an h-convex function, then by Theorem 1 we have

$$
\begin{aligned}
(3.3)\qquad \frac{1}{2h\left(\frac{1}{2}\right)} f\left[\frac{(1-\lambda)x + (\lambda+1)y}{2}\right] & \leq \int_0^1 f[(1-t)((1-\lambda)x + \lambda y) + ty]\,dt \\
& \leq [f((1-\lambda)x + \lambda y) + f(y)] \int_0^1 h(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
(3.4)\qquad \frac{1}{2h\left(\frac{1}{2}\right)} f\left[\frac{(2-\lambda)x + \lambda y}{2}\right] & \leq \int_0^1 f[(1-t)x + t((1-\lambda)x + \lambda y)]\,dt \\
& \leq [f(x) + f((1-\lambda)x + \lambda y)] \int_0^1 h(t)\,dt.
\end{aligned}
$$

Now, if we multiply the inequality (3.3) by $1 - \lambda \geq 0$ and (3.4) by $\lambda \geq 0$ and add the obtained inequalities, then we get

$$
\begin{aligned}
(3.5)\qquad & \frac{1-\lambda}{2h\left(\frac{1}{2}\right)} f\left[\frac{(1-\lambda)x + (\lambda+1)y}{2}\right] + \frac{\lambda}{2h\left(\frac{1}{2}\right)} f\left[\frac{(2-\lambda)x + \lambda y}{2}\right] \\
& \leq (1-\lambda) \int_0^1 f[(1-t)((1-\lambda)x + \lambda y) + ty]\,dt \\
& + \lambda \int_0^1 f[(1-t)x + t((1-\lambda)x + \lambda y)]\,dt \\
& \leq (1-\lambda)[f((1-\lambda)x + \lambda y) + f(y)] \int_0^1 h(t)\,dt \\
& + \lambda [f(x) + f((1-\lambda)x + \lambda y)] \int_0^1 h(t)\,dt
\end{aligned}
$$

and by (3.1) we obtain the first two inequalities in (3.2).

The last part is obvious. □

Remark 2. *With the assumptions from Theorem 2, we observe that if we take either $\lambda = 0$ or $\lambda = 1$ in the first two inequalities in (3.2), then we get (2.6).*

If we take $\lambda = \frac{1}{2}$ and use the h-convexity of f, then we get from (3.2) that

$$
\begin{aligned}
\frac{1}{4h^2\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) &\le \frac{1}{4h\left(\frac{1}{2}\right)}\left\{f\left(\frac{x+3y}{4}\right)+f\left(\frac{3x+y}{4}\right)\right\} \\
&\le \int_0^1 f\left[(1-t)x+ty\right]dt \\
&\le \left[f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{2}\right]\int_0^1 h(t)\,dt \\
&\le \left[h\left(\frac{1}{2}\right)+\frac{1}{2}\right]\left[f(x)+f(y)\right]\int_0^1 h(t)\,dt.
\end{aligned}
\tag{3.6}
$$

In general, if $h(\lambda) > 0$ for $\lambda \in (0,1)$, then

$$
\begin{aligned}
&(1-\lambda) f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+\lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right] \\
&= \frac{1-\lambda}{h(1-\lambda)} h(1-\lambda) f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right] \\
&+ \frac{\lambda}{h(\lambda)} h(\lambda) f\left[\frac{(2-\lambda)x+\lambda y}{2}\right] \\
&\ge \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} \\
&\times \left\{h(1-\lambda) f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+h(\lambda) f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right\} \\
&\ge \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} \\
&\times f\left[(1-\lambda)\frac{(1-\lambda)x+(\lambda+1)y}{2}+\lambda\frac{(2-\lambda)x+\lambda y}{2}\right] \\
&= \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and from (3.2) we get the sequence of inequalities

$$
\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} \min\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} f\left(\frac{x+y}{2}\right) \\
&\le \frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda) f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+\lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right\} \\
&\le \int_0^1 f\left[(1-t)x+ty\right]dt \\
&\le \left[f\left((1-\lambda)x+\lambda y\right)+(1-\lambda) f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt \\
&\le \left\{\left[h(1-\lambda)+\lambda\right] f(x)+\left[h(\lambda)+1-\lambda\right] f(y)\right\}\int_0^1 h(t)\,dt.
\end{aligned}
\tag{3.7}
$$

Corollary 1. *Let $f : C \subseteq X \to [0,\infty)$ be a convex function on the convex set C in a linear space X. Then for any $y, x \in C$ and for any $\lambda \in [0,1]$ we have the* *inequalities*

$$f\left(\frac{x+y}{2}\right)\leq(1-\lambda)f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+\lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right] \tag{3.8}$$
$$\leq\int_0^1 f[(1-t)x+ty]\,dt$$
$$\leq f((1-\lambda)x+\lambda y)+(1-\lambda)f(y)+\lambda f(x)$$
$$\leq\frac{f(y)+f(x)}{2}.$$

Remark 3. *The inequality (3.8) has been obtained for functions of a real variable by A. El Farissi in* [35].

The inequality (3.8) provides the following norm inequality:

$$\left\|\frac{x+y}{2}\right\|^p\leq(1-\lambda)\left\|\frac{(1-\lambda)x+(\lambda+1)y}{2}\right\|^p+\lambda\left\|\frac{(2-\lambda)x+\lambda y}{2}\right\|^p \tag{3.9}$$
$$\leq\int_0^1\|(1-t)x+ty\|^p\,dt$$
$$\leq\|(1-\lambda)x+\lambda y\|^p+(1-\lambda)\|y\|^p+\lambda\|x\|^p\leq\frac{\|y\|^p+\|x\|^p}{2},$$

that holds for any $x,y\in X$, a normed space, $p\geq1$ and $\lambda\in[0,1]$.

Corollary 2. *Assume that the function $f:C\subseteq X\to[0,\infty)$ is a Breckner s-convex function with $s\in(0,1)$. Let $y,x\in C$ with $y\neq x$ and assume that the mapping $[0,1]\ni t\mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then for any $\lambda\in(0,1)$ we have the inequalities*

$$2^{s-1}\left(\frac{1}{2}-\left|\frac{1}{2}-\lambda\right|\right)^{1-s}f\left(\frac{x+y}{2}\right) \tag{3.10}$$
$$\leq 2^{s-1}\left\{(1-\lambda)f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+\lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right\}$$
$$\leq\int_0^1 f[(1-t)x+ty]\,dt$$
$$\leq\frac{1}{s+1}[f((1-\lambda)x+\lambda y)+(1-\lambda)f(y)+\lambda f(x)]$$
$$\leq\frac{1}{s+1}\{[(1-\lambda)^s+\lambda]f(x)+(\lambda^s+1-\lambda)f(y)\}.$$

The inequality (3.10) provides the following norm inequality:

$$\begin{aligned}
&2^{s-1}\left(\frac{1}{2}-\left|\frac{1}{2}-\lambda\right|\right)^{1-s}\left\|\frac{x+y}{2}\right\|^{s} \\
&\leq 2^{s-1}\left\{(1-\lambda)\left\|\frac{(1-\lambda)x+(\lambda+1)y}{2}\right\|^{s}+\lambda\left\|\frac{(2-\lambda)x+\lambda y}{2}\right\|^{s}\right\} \\
&\leq \int_{0}^{1}\|(1-t)x+ty\|^{s}dt \\
&\leq \frac{1}{s+1}\left[\|(1-\lambda)x+\lambda y\|^{s}+(1-\lambda)\|y\|^{s}+\lambda\|x\|^{s}\right] \\
&\leq \frac{1}{s+1}\left\{\left[(1-\lambda)^{s}+\lambda\right]\|x\|^{s}+\left(\lambda^{s}+1-\lambda\right)\|y\|^{s}\right\}
\end{aligned} \tag{3.11}$$

that holds for any $x,y\in X$, a normed space, $s\in(0,1)$ and $\lambda\in(0,1)$.

In particular, we have

$$\begin{aligned}
&4^{s-1}\left\|\frac{x+y}{2}\right\|^{s}\leq 2^{s-2}\left\{\left\|\frac{x+3y}{4}\right\|^{s}+\left\|\frac{3x+y}{4}\right\|^{s}\right\} \\
&\leq \int_{0}^{1}\|(1-t)x+ty\|^{s}dt \\
&\leq \frac{1}{s+1}\left[\left\|\frac{x+y}{2}\right\|^{s}+\frac{\|y\|^{s}+\|x\|^{s}}{2}\right]\leq\frac{1+2^{1-s}}{(s+1)2^{s}}\left(\|x\|^{s}+\|y\|^{s}\right),
\end{aligned} \tag{3.12}$$

for any $x,y\in X$ and $s\in(0,1)$.

Remark 4. *Similar inequalities can be stated for functions of s-Godunova-Levin type, with $s\in[0,1)$, however the details are omitted.*

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