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SIMPSON TYPE INTEGRAL INEQUALITIES AND APPLICATIONS

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Abstract. In this paper, we obtain Simpson type inequalities for functions whose derivatives in absolute value are $m-$ and $(\alpha, m)-$ logarithmically convex functions. We prove some new bounds based on the celebrated Hermite–Hadamard integral inequality for $m-$ and $(\alpha, m)-$ logarithmically convex functions. Some new error estimations for our results are also given via Simpson's formula.

1. Introduction

Suppose $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_{\infty} = \sup \left| f^{(4)}(x) \right| < \infty$. The following inequality

$$\begin{aligned} &\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq\ & \frac{1}{2880} \left\| f^{(4)} \right\|_{\infty} (b-a)^4 \end{aligned}$$

is well known in the literature as Simpson's inequality.

For some recent results related to Simpson's inequality see [1]-[5] and [7].

The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Convex functions play an important role in many branches of mathematics and the other sciences as engineering, economics and optimization theory. Several extensions, generalizations and refinements have been presented by researchers.

Definition 1. *([6]) A function $f : [0, b] \to (0, \infty)$ is said to be $m-$logarithmically convex if the inequality*

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)} \tag{1.1}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0, 1]$.

Obviously, if putting $m = 1$ in Definition 3, then f is just the ordinary logarithmically convex function on $[0, b]$.

Definition 2. *([6]) A function $f : [0, b] \to (0, \infty)$ is said to be $(\alpha, m)-$logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t^{\alpha}} [f(y)]^{m(1-t^{\alpha})} \tag{1.2}$$

holds for all $x, y \in [0, b]$, $(\alpha, m) \in (0, 1] \times (0, 1]$, and $t \in [0, 1]$.

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Clearly, when taking $\alpha = 1$ in Definition 4, then f becomes the standard $m-$logarithmically convex function on $[0, b]$.

The main purpose of this paper is to prove some new inequalities of Simpson's type for functions whose derivatives are $m-$ and $(\alpha, m)-$logarithmically convex functions by using Lemma 1. Some applications in numerical integration are given.

2. Inequalities of Simpson type and Hadamard type

We have used the following Lemma to obtain our main results.

Lemma 1. *(See [1]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} & \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \\ = \quad & (b-a)\int_0^1 m(t)f'(tb + (1-t)a)dt, \end{aligned}$$

where

$$m(t) = \begin{cases} t - \frac{1}{6}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{5}{6}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Theorem 1. *Let $f : [0, \infty) \to (0, \infty)$ be differentiable mapping with $a, b \in [0, \infty)$ such that $a < b$. If $|f'(x)|$ is $(\alpha, m)-$logarithmically convex function with $f' \in L[a, b]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m K_1(\alpha, m)$$

where

$$\mu = \frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}, \qquad K_1(\alpha, m) = \begin{cases} \frac{5}{36} & , \mu = 1 \\ F_1(\mu, \alpha) & , \mu < 1 \end{cases}$$

and

$$F_1(\mu, \alpha) = \frac{1}{12\alpha^2 \ln \mu}\left[12\mu^{\frac{\alpha}{6}} + 4\alpha\mu^{\frac{\alpha}{2}} + 6(\mu^\alpha - 1) - \alpha \ln \mu (\mu^\alpha + 1)\right].$$

Proof. From Lemma 1 and using the (α,m) −logarithmically convexity of $|f'(x)|$ we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt\right\} \\ \leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt\right. \\ &+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt \\ &\left.+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt\right\}. \end{aligned}$$

If $\mu=1$, we have

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & \frac{5(b-a)}{36}\left|f'\left(\frac{a}{m}\right)\right|^m. \end{aligned}$$

If $\mu<1$, then $\mu^{t^\alpha}\leq\mu^{\alpha t}$, so we can write

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt\right. \\ &+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt \\ &\left.+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt\right\}. \end{aligned}$$

By making use of the neccessary process, the proof is completed. □

Corollary 1. *In Theorem 1, if we choose* $f(a)=f(b)=f\left(\frac{a+b}{2}\right)$, *we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m K_1(\alpha,m)$$

where $K_1(\alpha,m)$ *is as defined in Theorem 1.*

Corollary 2. *Under the assumptions of Theorem 1, if we choose $\alpha = m = 1$, we have the inequality;*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left|f'(a)\right| K_1(1,1)$$

where

$$\mu=\frac{|f'(b)|}{|f'(a)|},\qquad K_1(1,1)=\begin{cases}\frac{5}{36} & ,\mu=1\\ F_1(\mu,1) & ,\mu<1\end{cases}$$

and

$$F_1(\mu,1)=\frac{1}{12\ln\mu}\left[12\mu^{\frac{1}{6}}+4\mu^{\frac{1}{2}}+6(\mu-1)-\ln\mu(\mu+1)\right].$$

Theorem 2. *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is $(\alpha,m)-$logarithmically convex function with $f'\in L[a,b]$ and $(\alpha,m)\in(0,1]\times(0,1]$ for some fixed $q>1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\mu^{\frac{\alpha q}{2}}-1}{\alpha q\ln\mu}\right)^{\frac{1}{q}}+\left(\frac{\mu^{\alpha q}-\mu^{\frac{\alpha q}{2}}}{\alpha q\ln\mu}\right)^{\frac{1}{q}}\right] & ,\mu<1\\ (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} & ,\mu=1\end{cases}$$

where $p=\frac{q}{q-1}$ and $\mu=\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}$.

Proof. From Lemma 1 and using the Hölder inequality, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq (b-a)\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $|f'|^q$ is $(\alpha, m)-$logarithmically convex function, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m \left\{\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt\right)^{\frac{1}{p}}\right. \\
& \times\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}} \\
& +\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)^p dt\right)^{\frac{1}{p}} \\
& \left.\times\left(\int_{\frac{1}{2}}^{1}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}}\right\} \\
=\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
& \times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}=1$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}.
\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}<1$, then $\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}\leq\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt}$, thereby

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
& \times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

Corollary 3. *In Theorem 2, if we choose* $f(a)=f(b)=f\left(\frac{a+b}{2}\right)$, *we obtain the inequality;*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\mu^{\frac{\alpha q}{2}}-1}{\alpha q\ln\mu}\right)^{\frac{1}{q}}+\left(\frac{\mu^{\alpha q}-\mu^{\frac{\alpha q}{2}}}{\alpha q\ln\mu}\right)^{\frac{1}{q}}\right] & ,\mu<1\\ (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} & ,\mu=1\end{cases}$$

where $p=\frac{q}{q-1}$ *and* $\mu=\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}$.

Corollary 4. *Under the assumptios of Theorem 2, if we choose* $\alpha=m=1$, *we obtain the inequality;*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}(b-a)\left|f'(a)\right|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\mu^{\frac{q}{2}}-1}{q\ln\mu}\right)^{\frac{1}{q}}+\left(\frac{\mu^{q}-\mu^{\frac{q}{2}}}{q\ln\mu}\right)^{\frac{1}{q}}\right] & ,\mu<1\\ (b-a)\left|f'(a)\right|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} & ,\mu=1\end{cases}$$

where $p=\frac{q}{q-1}$ *and* $\mu=\frac{|f'(b)|}{|f'(a)|}$.

Theorem 3. *Let* $f:[0,\infty)\to(0,\infty)$ *be differentiable mapping with* $a,b\in[0,\infty)$ *such that* $a<b$. *If* $|f'(x)|^q$ *is* $(\alpha,m)-$*logarithmically convex function with* $f'\in L[a,b]$ *and* $(\alpha,m)\in(0,1]\times(0,1]$ *for some fixed* $q\geq 1$, *then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\begin{cases}(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left[\left(F_1(\mu,\alpha q)\right)^{\frac{1}{q}}+\left(F_2(\mu,\alpha q)\right)^{\frac{1}{q}}\right] & ,\mu<1\\ \frac{5(b-a)}{36}\left|f'\left(\frac{a}{m}\right)\right|^m & ,\mu=1\end{cases}$$

where

$$F_1(\mu,\alpha q)=\frac{1}{12(\alpha q)^2\ln\mu}\left[12\mu^{\frac{\alpha q}{6}}-6\left(1+\mu^{\frac{\alpha q}{2}}\right)+\alpha q\left(2\mu^{\frac{\alpha q}{2}}-\ln\mu\right)\right]$$
$$F_2(\mu,\alpha q)=\frac{1}{12(\alpha q)^2\ln\mu}\left[\mu^{\alpha q}\left(6-\alpha q\ln\mu\right)+\mu^{\frac{\alpha q}{2}}\left(6+2\alpha q\ln\mu\right)\right].$$

Proof. From Lemma 1 and using the power-mean inequality, we have

$$\begin{aligned}
& \left| \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] -\frac{1}{b-a}\int_{a}^{b}f(x)dx\right| \\
\leq & \ (b-a) \\
& \times \left\{ \left(\int_{0}^{\frac{1}{2}}\left| t-\frac{1}{6}\right| dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}\left| t-\frac{1}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
& \left. +\left(\int_{\frac{1}{2}}^{1}\left| t-\frac{5}{6}\right| dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left| t-\frac{5}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\right\} .
\end{aligned}$$

Since $|f'|^q$ is (α, m) $-$logarithmically convex function, we have

$$\begin{aligned}
& \int_{0}^{\frac{1}{2}}\left| t-\frac{1}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt \\
\leq & \ \left| f'\left(\frac{a}{m}\right)\right|^{qm}\left\{ \int_{0}^{\frac{1}{6}}\left(\frac{1}{6}-t\right) \left(\frac{|f'(b)|}{\left| f'\left(\frac{a}{m}\right)\right|^{m}}\right)^{qt^{\alpha}}dt\right. \\
& \left. +\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right) \left(\frac{|f'(b)|}{\left| f'\left(\frac{a}{m}\right)\right|^{m}}\right)^{qt^{\alpha}}dt\right\}
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\frac{1}{2}}^{1}\left| t-\frac{5}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt \\
\leq & \ \left| f'\left(\frac{a}{m}\right)\right|^{qm}\left\{ \int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right) \left(\frac{|f'(b)|}{\left| f'\left(\frac{a}{m}\right)\right|^{m}}\right)^{qt^{\alpha}}dt\right. \\
& \left. +\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right) \left(\frac{|f'(b)|}{\left| f'\left(\frac{a}{m}\right)\right|^{m}}\right)^{qt^{\alpha}}dt\right\} .
\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^{m}}=1$, we obtain

$$\int_{0}^{\frac{1}{2}}\left| t-\frac{1}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt\leq \frac{5}{72}\left| f'\left(\frac{a}{m}\right)\right|^{qm}$$

and

$$\int_{\frac{1}{2}}^{1}\left| t-\frac{5}{6}\right| \left| f'(tb+(1-t)a)\right|^{q}dt\leq \frac{5}{72}\left| f'\left(\frac{a}{m}\right)\right|^{qm}.$$

If we take $\frac{|f'(b)|}{|f'(\frac{a}{m})|^m} < 1$, then $\left(\frac{|f'(b)|}{|f'(\frac{a}{m})|^m}\right)^{qt^\alpha} \leq \left(\frac{|f'(b)|}{|f'(\frac{a}{m})|^m}\right)^{\alpha qt}$, thereby

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left|t - \frac{1}{6}\right| |f'(tb + (1-t)a)|^q \, dt \\ \leq & \left|f'\left(\frac{a}{m}\right)\right|^{qm} \left\{ \int_0^{\frac{1}{6}} \left(\frac{1}{6} - t\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right. \\ & \left. + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t - \frac{1}{6}\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right\} \end{aligned}$$

and

$$\begin{aligned} & \int_{\frac{1}{2}}^{1} \left|t - \frac{5}{6}\right| |f'(tb + (1-t)a)|^q \, dt \\ \leq & \left|f'\left(\frac{a}{m}\right)\right|^{qm} \left\{ \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6} - t\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right. \\ & \left. + \int_{\frac{5}{6}}^{1} \left(t - \frac{5}{6}\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right\}. \end{aligned}$$

Combining all the above inequalities gives us the desired result. □

Corollary 5. *In Theorem 3, if we choose $f(a) = f(b) = f\left(\frac{a+b}{2}\right)$, we obtain the inequality;*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq & \begin{cases} (b-a) \left|f'\left(\frac{a}{m}\right)\right|^m \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left[(F_1(\mu, \alpha q))^{\frac{1}{q}} + (F_2(\mu, \alpha q))^{\frac{1}{q}} \right] & , \mu < 1 \\ \frac{5(b-a)}{36} \left|f'\left(\frac{a}{m}\right)\right|^m & , \mu = 1 \end{cases} \end{aligned}$$

where $F_1(\mu, \alpha q)$ and $F_2(\mu, \alpha q)$ are as defined in Theorem 3.

Corollary 6. *Under the assumptions of Theorem 3, if we choose $\alpha = m = 1$, we have the inequality;*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq & \begin{cases} (b-a) |f'(a)| \left(\frac{5}{72}\right)^{1-\frac{1}{q}} \left[(F_1(\mu, q))^{\frac{1}{q}} + (F_2(\mu, q))^{\frac{1}{q}} \right] & , \mu < 1 \\ \frac{5(b-a)}{36} |f'(a)| & , \mu = 1 \end{cases} \end{aligned}$$

where

$$\begin{aligned} F_1(\mu, q) &= \frac{1}{12q^2 \ln \mu} \left[12\mu^{\frac{q}{6}} - 6\left(1 + \mu^{\frac{q}{2}}\right) + q\left(2\mu^{\frac{q}{2}} - \ln \mu\right) \right] \\ F_2(\mu, q) &= \frac{1}{12q^2 \ln \mu} \left[\mu^q (6 - q \ln \mu) + \mu^{\frac{q}{2}} (6 + 2q \ln \mu) \right]. \end{aligned}$$

3. Applications in numerical integration

Consider a division of the interval $[a,b]$, i.e., $d: a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, $h_i = \frac{x_{i+1}-x_i}{2}$ and consider the Simpson's formula

$$S(f,d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6} (x_{i+1} - x_i).$$

It is known that if the function $f : [a,b] \to \mathbb{R}$, is differentiable function such that $f^{(4)}(x)$ exists on (a,b) and

$$M = \sup_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty,$$

then

$$I = \int_a^b f(x)\, dx = S(f,d) + E_S(f,d), \tag{3.1}$$

where the approximation error $E_S(f,d)$ of the integral I by the Simpson's formula $S(f,d)$ satisfies

$$E_S(f,d) \leq \frac{M}{2880} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5.$$

In the case of the mapping f is not fourth differentiable or the fourth derivative is not bounded on (a,b), then the formula (3.1) can't be applied.

Now, we are in a position to obtain some new estimations for the remainder term $E_S(f,d)$ in terms of logarithmically convex functions.

Proposition 1. *Let $f : [0,\infty) \to (0,\infty)$ be differentiable mapping with $a,b \in [0,\infty)$ such that $a < b$. If $|f'(x)|$ is logarithmically convex function with $f' \in L[a,b]$, then for every division d of $[a,b]$, the following inequality holds:*

$$|E_S(f,d)| \leq \sum_{i=0}^{n-1} (x_{i+1} - x_i) |f'(x_i)| \kappa(1,1)$$

where

$$\mu_1 = \frac{|f'(x_{i+1})|}{|f'(x_i)|}, \qquad \kappa(1,1) = \begin{cases} \frac{5}{36} & ,\mu_1 = 1 \\ F_1(\mu_1, 1) & ,\mu_1 < 1 \end{cases}$$

and

$$F_1(\mu_1,1) = \frac{1}{12 \ln \mu_1} \left[12\mu_1^{\frac{1}{6}} + 4\mu_1^{\frac{1}{2}} + 6(\mu_1 - 1) - \ln \mu_1 (\mu_1 + 1) \right].$$

Proof. By applying Corollary 2 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n-1)$ of the division d, we have

$$\left| \frac{1}{6} \left[f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1}) \right] - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right| \leq (x_{i+1} - x_i) |f'(x_i)| \kappa(1,1).$$

By summing over i from 0 to $n-1$, it is easy to see that

$$\left| S(f,d) - \int_a^b f(x)dx \right| \leq \sum_{i=0}^{n-1} (x_{i+1} - x_i) |f'(x_i)| \kappa(1,1)$$

which completes the proof. □

Proposition 2. *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is ogarithmically convex function with $f'\in L[a,b]$ for some fixed $q>1$, then for every division d of $[a,b]$ the following inequality holds:*

$$\begin{aligned}&|E_S(f,d)|\\ &\leq\begin{cases}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f'(x_i)\right|\left[\left(\frac{\mu_2^{\frac{q}{2}}-1}{q\ln\mu_2}\right)^{\frac{1}{q}}+\left(\frac{\mu_2^q-\mu_2^{\frac{q}{2}}}{q\ln\mu_2}\right)^{\frac{1}{q}}\right] & ,\mu_2<1\\ \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f'(x_i)\right| & ,\mu_2=1\end{cases}\end{aligned}$$

where $p=\frac{q}{q-1}$ and $\mu_2=\frac{\left|f'(x_{i+1})\right|}{|f'(x_i)|}$.

Proof. By a similar argument to the proof of Proposition 1, by applying Corollary 4 on the subintervals $[x_i,x_{i+1}]$, $(i=0,1,...,n-1)$ of the division d, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(x_i)+4f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|\\ &\leq\begin{cases}(x_{i+1}-x_i)\left|f'(x_i)\right|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\mu_2^{\frac{q}{2}}-1}{q\ln\mu_2}\right)^{\frac{1}{q}}+\left(\frac{\mu_2^q-\mu_2^{\frac{q}{2}}}{q\ln\mu_2}\right)^{\frac{1}{q}}\right] & ,\mu_2<1\\ (x_{i+1}-x_i)\left|f'(x_i)\right|\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} & ,\mu_2=1\end{cases}.\end{aligned}$$

By summing over i from 0 to $n-1$, we have

$$\begin{aligned}&\left|S(f,d)-\int_a^b f(x)dx\right|\\ &\leq\begin{cases}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f'(x_i)\right|\left[\left(\frac{\mu_2^{\frac{q}{2}}-1}{q\ln\mu_2}\right)^{\frac{1}{q}}+\left(\frac{\mu_2^q-\mu_2^{\frac{q}{2}}}{q\ln\mu_2}\right)^{\frac{1}{q}}\right] & ,\mu_2<1\\ \left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f'(x_i)\right| & ,\mu_2=1\end{cases}\end{aligned}$$

which is the desired result. □

Proposition 3. *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is logarithmically convex function with $f'\in L[a,b]$ for some fixed $q\geq1$, then for every division d of $[a,b]$ the following inequality holds:*

$$\begin{aligned}&|E_S(f,d)|\\ &\leq\begin{cases}\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|f'(x_i)\right|\left[(F_1(\mu_2,q))^{\frac{1}{q}}+(F_2(\mu_2,q))^{\frac{1}{q}}\right] & ,\mu_2<1\\ \frac{5}{36}\sum_{i=0}^{n-1}\left|f'(x_i)\right|(x_{i+1}-x_i) & ,\mu_2=1\end{cases}\end{aligned}$$

where

$$\begin{aligned}F_1(\mu_2,q)&=\frac{1}{12q^2\ln\mu_2}\left[12\mu_2^{\frac{q}{6}}-6\left(1+\mu_2^{\frac{q}{2}}\right)+q\left(2\mu_2^{\frac{q}{2}}-\ln\mu_2\right)\right]\\ F_2(\mu_2,q)&=\frac{1}{12q^2\ln\mu_2}\left[\mu_2^q(6-q\ln\mu_2)+\mu_2^{\frac{q}{2}}(6+2q\ln\mu_2)\right]\end{aligned}$$

and μ_2 *is as defined in Proposition 2.*

Proof. The proof is immediately follows from Corollary 6 and a similar argument to the proof of the Proposition 2. We omit the details. □

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