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DOUBLE INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR h-CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Some double integral inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [41]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [41]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[24], [31]-[34] and [44].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,y \in X,\ x \neq y$. Define the segment

$$[x,y] := \{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t) := f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]dt \le \frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Since $f(x)=\|x\|^p$ ($x\in X$ and $1\le p<\infty$) is a convex function, then for any $x,y\in X$ we have the following norm inequality from (1.2) (see [45, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p\le\int_0^1\|(1-t)x+ty\|^p dt\le\frac{\|x\|^p+\|y\|^p}{2}. \tag{1.3}$$

Motivated by the above results, in this paper we obtain double integral inequalities of Hermite-Hadamard type in which upper and lower bounds for the quantity

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha$$

are provided for some classes of h-convex functions defined on linear spaces. Applications for norm inequalities and for Godunova-Levin type of functions are also given.

2. A Double Integral Inequality for Convex Functions

For $a,b,c,d\ge 0$ with $b>a$ and $d>c$ we define the positive quantity

$$I(a,b;c,d):=\int_a^b\left(\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\right)d\alpha. \tag{2.1}$$

We have the following representation:

Lemma 1. *Let $a,b,c,d\ge 0$ with $b>a$ and $d>c$. We have the equality*

$$I(a,b;c,d)=I_d(a,b)-I_c(a,b), \tag{2.2}$$

where $I_z(x,y)$ is defined for $x,y,z\ge 0$ with $y>x$ by

$$\begin{aligned}&I_z(x,y)\\&:=\frac{1}{2}\left[\left(y^2-z^2\right)\ln(y+z)+\left(z^2-x^2\right)\ln(x+z)+(y-x)\left(z-\frac{x+y}{2}\right)\right].\end{aligned} \tag{2.3}$$

In particular,

$$I(a,b;a,b)=I_b(a,b)-I_a(a,b)=\frac{1}{2}(b-a)^2. \tag{2.4}$$

Proof. We have

$$\begin{aligned}I(a,b;c,d)&=\int_a^b\left(\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\right)d\alpha\\&=\int_a^b\alpha\left(\int_c^d\frac{d\beta}{\alpha+\beta}\right)d\alpha=\int_a^b\alpha\left[\ln(\alpha+d)-\ln(\alpha+d)\right]d\alpha\\&=\int_a^b\alpha\ln(\alpha+d)\,d\alpha-\int_a^b\alpha\ln(\alpha+d)\,d\alpha\\&=\int_{a+d}^{b+d}(u-d)\ln u du-\int_{a+c}^{b+c}(u-c)\ln u du.\end{aligned} \tag{2.5}$$

Utilising the integration by parts formula, we have

$$\int_{a+d}^{b+d}(u-d)\ln u du=\left.\frac{(u-d)^2}{2}\ln u\right|_{a+d}^{b+d}-\frac{1}{2}\int_{a+d}^{b+d}\frac{(u-d)^2}{u}du \tag{2.6}$$
$$=\frac{b^2}{2}\ln(b+d)-\frac{a^2}{2}\ln(a+d)-\frac{1}{2}\int_{a+d}^{b+d}\frac{(u-d)^2}{u}du.$$

Observe that

$$\begin{aligned}&\int_{a+d}^{b+d}\frac{(u-d)^2}{u}du \\ &=\int_{a+d}^{b+d}\frac{u^2-2du+d^2}{u}du=\int_{a+d}^{b+d}\left(u-2d+\frac{d^2}{u}\right)du \\ &=\left.\frac{u^2}{2}\right|_{a+d}^{b+d}-2d(b-a)+d^2\ln(b+d)-d^2\ln(a+d) \\ &=\frac{(b+d)^2-(a+d)^2}{2}-2d(b-a)+d^2\ln(b+d)-d^2\ln(a+d) \\ &=\frac{(b-a)(b+a+2d)}{2}-2d(b-a)+d^2\ln(b+d)-d^2\ln(a+d) \\ &=(b-a)\left(\frac{a+b}{2}-d\right)+d^2\ln(b+d)-d^2\ln(a+d).\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we have

$$\begin{aligned}&\int_{a+d}^{b+d}(u-d)\ln u du \\ &=\frac{b^2}{2}\ln(b+d)-\frac{a^2}{2}\ln(a+d) \\ &-\frac{1}{2}\left[(b-a)\left(\frac{a+b}{2}-d\right)+d^2\ln(b+d)-d^2\ln(a+d)\right] \\ &=I_d(a,b).\end{aligned}$$

Similarly,

$$\int_{a+c}^{b+c}(u-c)\ln u du=I_c(a,b)$$

and by (2.5) we get the desired identity (2.2).

We have

$$\begin{aligned}&I_b(a,b) \\ &=\frac{1}{2}\left[\left(b^2-b^2\right)\ln(b+b)+\left(b^2-a^2\right)\ln(a+b)+(b-a)\left(b-\frac{a+b}{2}\right)\right] \\ &=\frac{1}{2}\left(b^2-a^2\right)\ln(a+b)+\frac{1}{4}(b-a)^2\end{aligned}$$

and

$$\begin{aligned}
&I_a(a,b)\\
&=\frac{1}{2}\left[\left(b^2-a^2\right)\ln(b+a)+\left(a^2-a^2\right)\ln(a+a)+(b-a)\left(a-\frac{a+b}{2}\right)\right]\\
&=\frac{1}{2}\left(b^2-a^2\right)\ln(a+b)-\frac{1}{4}(b-a)^2,
\end{aligned}$$

which gives the desired equality (2.4). □

The following double integral inequality for convex functions holds.

Theorem 1. *Let $f:C\subseteq X\to[0,\infty)$ be a convex function on the convex set C in a linear space X. Then for any $x,y\in C$ with $x\neq y$ and for any $a,b,c,d\geq 0$ with $b>a$ and $d>c$ we have*

$$\begin{aligned}
&f\left(\frac{I(a,b;c,d)}{(b-a)(d-c)}x+\frac{I(c,d;a,b)}{(b-a)(d-c)}y\right)\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha\\
&\leq\frac{I(a,b;c,d)}{(b-a)(d-c)}f(x)+\frac{I(c,d;a,b)}{(b-a)(d-c)}f(y),
\end{aligned}\tag{2.8}$$

where

$$I(a,b;c,d):=\int_a^b\left(\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta\right)d\alpha$$

and

$$I(c,d;a,b):=\int_a^b\left(\int_c^d\left(\frac{\beta}{\alpha+\beta}\right)d\beta\right)d\alpha.$$

Proof. Consider the function $g_{x,y}:[0,1]\to\mathbb{R}$, $g_{x,y}(s)=f(sx+(1-s)y)$. This function is convex on $[0,1]$ and by Jensen's double integral inequality for real functions of real variable we have

$$g_{x,y}\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}\right)\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g_{x,y}\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha,$$

which is equivalent with

$$\begin{aligned}
&f\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}x+\left(1-\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}\right)y\right)\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha}{\alpha+\beta}x+\left(1-\frac{\alpha}{\alpha+\beta}\right)y\right)d\beta d\alpha.
\end{aligned}$$

By simple calculation we obtain

$$\begin{aligned}
&f\left(\frac{\int_a^b\int_c^d\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}x+\frac{\int_a^b\int_c^d\left(\frac{\beta}{\alpha+\beta}\right)d\beta d\alpha}{(b-a)(d-c)}y\right)\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(\frac{\alpha}{\alpha+\beta}x+\frac{\beta}{\alpha+\beta}y\right)d\beta d\alpha
\end{aligned}$$

and the first part of (2.8) is proved.

By the convexity of f we have

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\alpha}{\alpha+\beta}f\left(x\right)+\frac{\beta}{\alpha+\beta}f\left(y\right)$$

for any $x,y\in C$ and for all $\alpha,\beta\geq 0$ with $\alpha+\beta>0$.

Integrating on the rectangle $[a,b]\times[c,d]$ we have

$$\int_a^b\int_c^d f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha\leq f\left(x\right)\int_a^b\int_c^d\frac{\alpha}{\alpha+\beta}d\beta d\alpha+f\left(y\right)\int_a^b\int_c^d\frac{\beta}{\alpha+\beta}d\beta d\alpha,$$

which proves the second part of (2.8). □

Corollary 1. *Let $f:C\subseteq X\to[0,\infty)$ be a convex function on the convex set C in a linear space X. Then for any $x,y\in C$ with $x\neq y$ and for any $b>a\geq 0$ we have*

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{\left(b-a\right)^2}\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha\leq\frac{f\left(x\right)+f\left(y\right)}{2}. \tag{2.9}$$

The proof is obvious from (2.8) noticing that $I\left(a,b;a,b\right)=\frac{1}{2}\left(b-a\right)^2.$

Remark 1. *Let $(X,\|\cdot\|)$ be a real or complex linear spaces and $p\geq 1$. Then for any $x,y\in X$ we have*

$$\begin{aligned}&\left\|\frac{I\left(a,b;c,d\right)}{\left(b-a\right)\left(d-c\right)}x+\frac{I\left(c,d;a,b\right)}{\left(b-a\right)\left(d-c\right)}y\right\|^p \\ &\leq\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d\left\|\frac{\alpha x+\beta y}{\alpha+\beta}\right\|^p d\beta d\alpha \\ &\leq\frac{I\left(a,b;c,d\right)}{\left(b-a\right)\left(d-c\right)}\left\|x\right\|^p+\frac{I\left(c,d;a,b\right)}{\left(b-a\right)\left(d-c\right)}\left\|y\right\|^p\end{aligned} \tag{2.10}$$

for any $a,b,c,d\geq 0$ with $b>a$ and $d>c$.

In particular, we have

$$\left\|\frac{x+y}{2}\right\|^p\leq\frac{1}{\left(b-a\right)^2}\int_a^b\int_a^b\left\|\frac{\alpha x+\beta y}{\alpha+\beta}\right\|^p d\beta d\alpha\leq\frac{\left\|x\right\|^p+\left\|y\right\|^p}{2} \tag{2.11}$$

for any $b>a\geq 0$.

3. Double Integral Inequalities for h-Convex Functions

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([36])**.** *We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q\left(I\right)$ if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have*

$$f\left(tx+\left(1-t\right)y\right)\leq\frac{1}{t}f\left(x\right)+\frac{1}{1-t}f\left(y\right). \tag{3.1}$$

Some further properties of this class of functions can be found in [27], [28], [30], [42], [45] and [46]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f:C\subseteq X\to[0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (3.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([30])**.** *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{3.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{3.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [30] and [43] while for quasi convex functions, the reader can consult [29].

If $f : C \subseteq X \to [0,\infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (3.2) (or (3.3)) holds true for $x, y \in C$ and $t \in [0,1]$.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [37], [39] and [48].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([51])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{3.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [51], [6], [40], [49], [47] and [50].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{3.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \le s_1 \le s_2 \le 1$.

We can prove now the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces.

Theorem 2. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is an h-convex function with $h \in L[0, 1]$. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0, 1]$. Then*

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \\ &\le \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left[f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right) \right] d\beta d\alpha \\ &\le \frac{f(x) + f(y)}{2(b-a)(d-c)} \int_a^b \int_c^d \left[h\left(\frac{\alpha}{\alpha + \beta}\right) + h\left(\frac{\beta}{\alpha + \beta}\right) \right] d\beta d\alpha \end{aligned} \tag{3.6}$$

for any $a, b, c, d \ge 0$ with $b > a$ and $d > c$.

Proof. By the h-convexity of f we have

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{3.7}$$

and

$$f((1-t)x + ty) \le h(1-t) f(x) + h(t) f(y) \tag{3.8}$$

for any $t \in [0, 1]$.

Summing the inequalities (3.7) and (3.8) and dividing by 2 we get

$$\frac{1}{2} [f(tx + (1-t)y) + f((1-t)x + ty)] \le \frac{1}{2} [h(1-t) + h(t)] [f(x) + f(y)] \tag{3.9}$$

for any $t \in [0, 1]$.

Taking $t=\frac{\alpha}{\alpha+\beta}$ in (3.9) we get

$$
\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right] \\
&\leq \frac{1}{2}\left[h\left(\frac{\alpha}{\alpha+\beta}\right)+h\left(\frac{\beta}{\alpha+\beta}\right)\right][f(x)+f(y)]
\end{aligned} \tag{3.10}
$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$.

Since the mapping $[0,1]\ni t\mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$, then the double integrals

$$
\int_a^b\int_c^d f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha \text{ and } \int_a^b\int_c^d f\left(\frac{\alpha y+\beta x}{\alpha+\beta}\right)d\beta d\alpha
$$

exist and integrating the inequality on the rectangle $[a,b]\times[c,d]$ over (α,β) we get the second inequality in (3.6).

From the h-convexity of f we also have

$$
f\left(\frac{z+w}{2}\right)\leq h\left(\frac{1}{2}\right)[f(z)+f(w)] \tag{3.11}
$$

for any $z,w\in C$.

If we take in (3.11) $z=\frac{\alpha x+\beta y}{\alpha+\beta}$ and $w=\frac{\beta x+\alpha y}{\alpha+\beta}$, then we get

$$
f\left(\frac{x+y}{2}\right)\leq h\left(\frac{1}{2}\right)\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right] \tag{3.12}
$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$.

Integrating the inequality on the rectangle $[a,b]\times[c,d]$ over (α,β) we get the first inequality in (3.6). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{x+y}{2}\right) &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\beta d\alpha \\
&\leq \frac{f(x)+f(y)}{(b-a)^2}\int_a^b\int_a^b h\left(\frac{\alpha}{\alpha+\beta}\right)d\beta d\alpha
\end{aligned} \tag{3.13}
$$

for any $b>a\geq 0$.

The following result holds for convex functions.

Corollary 3. *Let $f:C\subseteq X\to[0,\infty)$ be a convex function on the convex set C in a linear space X. Then for any $x,y\in C$ with $x\neq y$ and for any $a,b,c,d\geq 0$ with $b>a$ and $d>c$ we have*

$$
\begin{aligned}
f\left(\frac{x+y}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\left[\frac{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)}{2}\right]d\beta d\alpha \\
&\leq \frac{I(a,b;c,d)+I(c,d;a,b)}{(b-a)(d-c)}\cdot\frac{f(x)+f(y)}{2},
\end{aligned} \tag{3.14}
$$

where $I(a,b;c,d)$ and $I(c,d;a,b)$ are defined by (2.1).

For two distinct positive numbers p and q we consider the *Logarithmic mean*

$$
L(p,q):=\frac{p-q}{\ln p-\ln q}.
$$

Corollary 4. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is of Godunova-Levin type on C. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then for any $a, b, c, d > 0$ with $b > a$ and $d > c$ we have*

$$\begin{aligned}
\frac{1}{4} f\left(\frac{x+y}{2}\right) \tag{3.15}\\
&\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left[f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) + f\left(\frac{\beta x + \alpha y}{\alpha + \beta}\right)\right] d\beta d\alpha \\
&\leq \frac{f(x) + f(y)}{2} \left[2 + \frac{A(c,d)}{L(a,b)} + \frac{A(a,b)}{L(c,d)} \right],
\end{aligned}$$

where L is the logarithmic mean and A is the arithmetic mean of the numbers involved.

Proof. We take $h(t) = \frac{1}{t}, t \in (0,1)$ in (3.6) and have to integrate the double integral

$$\int_a^b \int_c^d \left(\frac{\alpha+\beta}{\alpha} + \frac{\alpha+\beta}{\beta} \right) d\beta d\alpha.$$

Observe that

$$\begin{aligned}
\int_a^b \int_c^d \frac{\alpha+\beta}{\alpha} d\beta d\alpha &= \int_a^b \int_c^d \left(1 + \frac{\beta}{\alpha}\right) d\beta d\alpha \\
&= (b-a)(d-c) + (\ln b - \ln a) \frac{d^2 - c^2}{2} \\
&= (b-a)(d-c) \left(1 + \frac{\ln b - \ln a}{b-a} \cdot \frac{c+d}{2}\right) \\
&= (b-a)(d-c) \left[1 + \frac{A(c,d)}{L(a,b)}\right]
\end{aligned}$$

and

$$\int_a^b \int_c^d \frac{\alpha+\beta}{\beta} d\beta d\alpha = (b-a)(d-c) \left[1 + \frac{A(a,b)}{L(c,d)}\right],$$

which produce the second part of (3.15). □

Remark 2. *With the assumptions of Corollary 4 we have the inequalities*

$$\begin{aligned}
\frac{1}{4} f\left(\frac{x+y}{2}\right) &\leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) d\beta d\alpha \tag{3.16}\\
&\leq \left[1 + \frac{A(a,b)}{L(a,b)}\right] [f(x) + f(y)]
\end{aligned}$$

for any $b > a > 0$.

Corollary 5. *Assume that the function $f : C \subseteq X \to [0, \infty)$ is of P-type on C. Let $y, x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then for any a, b, c, d with $b > a \geq 0$ and $d > c \geq 0$*

we have

$$\begin{aligned} &\frac{1}{2} f\left(\frac{x+y}{2}\right) \\ &\leq \frac{1}{2(b-a)(d-c)} \int_a^b \int_c^d \left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right] d\beta d\alpha \\ &\leq f(x)+f(y) \end{aligned} \tag{3.17}$$

and, in particular

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) d\beta d\alpha \leq f(x)+f(y). \tag{3.18}$$

The interested reader may obtain similar results for other h-convex functions as provided above. The details are omitted.

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