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SOME INEQUALITIES INVOLVING THE BESSEL FUNCTIONS OF THE FIRST KIND

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ABSTRACT. In this paper, in view of the integral representation of Bessel functions of the first kind and the inequalities for concave and r-concave functions, we establish some inequalities for the Bessel functions of the first kind.

INTRODUCTION

The Bessel function of the first kind of order ν, denoted by $J_\nu(x)$, is defined as a particular solution of the second order differential equation

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

which is also called the Bessel equation with index ν. It is known (see [4]) that

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!\Gamma(\nu+m+1)}, \quad x \in \mathbb{R}.$$

In [1], M. Abramowitz and I. A. Stegun mentioned the integral representation of the function under the form

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \cos(xt)dt. \tag{1}$$

From this expression we can define the following function

$$f_\nu(x) = \begin{cases} \dfrac{J_\nu(x)}{x^\nu}, & \text{if } x \neq 0 \\ \lim\limits_{x\to 0} \dfrac{J_\nu(x)}{x^\nu}, & \text{if } x = 0. \end{cases} \tag{2}$$

It is easy to see that $f_\nu(-x) = f_\nu(x)$ for every $x \in \mathbb{R}$ and $f_\nu(0) = \dfrac{1}{2^\nu\Gamma(\nu+1)}$. Moreover, f_ν is a differentiable and continuous function on $\mathbb{R}$. In addition, we also have

$$f'_\nu(x) = -xf_{\nu+1}(x), \tag{3}$$

for all $x \in \mathbb{R}$.

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In this paper we use the equality (1) to advance some new properties and inequalities for f_ν based on the properties of concave and r-concave function.

1. Preliminaries

Here we recall some definitions and results related our main results.

Definition 1.1 ([9])**.** A function f is called to be *concave* on $[a,b]$ if and only if

$$f(\lambda x+(1-\lambda)y)\geq \lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

for every $\lambda\in[0,1]$ and $x,y\in[a,b]$.

In [9], A. W. Roberts and D. E. Vargerg referred to the condition for a twice differentiable function f is concave on I to be

$$f''(x)\leq 0,\quad \text{for all } x\in I.$$

Definition 1.2 ([10])**.** A positive valued function f is called to be r*-concave* on $[a,b]$, if for each $x,y\in[a,b]$ and $\lambda\in[0,1]$,

$$f(\lambda x+(1-\lambda)y)\geq \begin{cases}[\lambda f^r(x)+(1-\lambda)f^r(y)]^{1/r}, & r\neq 0,\\ [f(x)]^\lambda[f(y)]^{1-\lambda}, & r=0.\end{cases} \tag{1.2}$$

It is obvious 0-concave functions are simply log-concave functions and 1-concave functions are ordinary concave functions. One should note that if f is a r-concave on $[a,b]$, then f^r is concave function with $r>0$.

Definition 1.3 ([11])**.** A function $f:[a,b]\subset(0,+\infty)\to(0,+\infty)$ is said to be *geometrically concave* if and only if

$$f(x^\alpha y^{1-\alpha})\geq f^\alpha(x)f^{1-\alpha}(y) \tag{1.3}$$

for all $\alpha\in[0,1]$ and $x,y\in[a,b]$.

In [11], X. Zhang and N. Zheng referred to the condition for a twice differentiable function f is geometrically concave on interval I to be

$$x[f''(x)f(x)-[f'(x)]^2]+f(x)f'(x)\leq 0,\quad \text{for all } x\in I. \tag{1.4}$$

Remark 1.1. Suppose that a positive function f defined on $[a,b]$ is to be concave. Then by using Lemma 2.5 in [10] we have the following inequalities

$$f(\lambda x+(1-\lambda)y)\geq \lambda f(x)+(1-\lambda)f(y)\geq \left[\lambda f^r(x)+(1-\lambda)f^r(y)\right]^{1/r}\geq [f(x)]^\lambda[f(y)]^{1-\lambda} \tag{1.5}$$

hold for all $r\in(0,1]$. This gives us the related between above functional classes.

Remark 1.2. Let $0 \le r \le s$ and suppose that f is a s-concave function on $[a,b]$. Then by using Lemma 2.5 in [10], it's easy to deduce that f is also r-concave on the interval $[a,b]$.

S. S. Dragomir and C. E. M. Pearce [5] referred to two well-known results for a convex function. Here we present these results for concave function.

Theorem 1.3 ([5]). *Let p,q be given positive numbers and f is a continuous concave function on $[a_1,b_1]$. Then for $a_1 \le a < b \le b_1$ the following inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \ge \frac{1}{2y}\int_{A-y}^{A+y} f(t)dt \ge \frac{1}{2}(f(A-y)+f(A+y)) \ge \frac{pf(a)+qf(b)}{p+q} \tag{1.6}$$

hold for $A = \frac{pa+qb}{p+q}$ and $0 < y \le \frac{b-a}{p+q}\min\{p,q\}$.

Theorem 1.4 ([5]). *Let f be a concave function on $[a,b]$. Then for all $t \in [a,b]$ we have the following inequality*

$$\frac{1}{b-a}\int_a^b f(x)dx \ge \frac{f(t)}{2} + \frac{1}{2}\frac{bf(b)-af(a)-t[f(b)-f(a)]}{b-a}. \tag{1.7}$$

2. Main results

In this section, firstly we advance some properties of the function f_ν. Then we use it to advance some new inequalities.

Theorem 2.1. *For $\nu \ge 0$ we have the following statements:*

(i) *f_ν is concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$;*
(ii) *f_ν is r-concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $r \in [0,1]$;*
(iii) *f_ν is geometrically concave on $(0, \frac{\pi}{2})$.*

Proof. (i) It's easy to check that for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have

$$f''_\nu(x) = \frac{-1}{2^\nu\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^1 t^2(1-t^2)^{\nu-1/2}\cos(xt)dt \le 0. \tag{2.1}$$

So the function f_ν is concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) This case is directly consequence of the statement (i) and the inequality (1.5).

(iii) For every $x \in (0, \frac{\pi}{2})$, we have $f_\nu(x) \ge 0$ and

$$f'_\nu(x) = \frac{-1}{2^\nu\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^1 t(1-t^2)^{\nu-1/2}\cos(xt)dt \le 0. \tag{2.2}$$

Thus, combining (2.1) and (2.2) give us, for all $x \in (0, \frac{\pi}{2})$,

$$x[f''_\nu(x)f_\nu(x) - [f'_\nu(x)]^2] + f_\nu(x)f'_\nu(x) \le 0.$$

Hence f_ν satisfies the condition (1.4) and therefore is geometrically concave on $(0, \frac{\pi}{2})$. $\square$

Remark 2.2. For $\nu \geq 0$ we have, in view of Jensen inequality,

$$\frac{f_\nu(x)+f_\nu(y)}{2} \leq f_\nu\left(\frac{x+y}{2}\right), \quad x,y \in [-\pi/2, \pi/2]. \tag{2.3}$$

Theorem 2.3. *Suppose that* $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ *and* $p, q > 0$. *Let* $A = \frac{pa+qb}{p+q}$ *and* $0 < y \leq \frac{b-a}{p+q}\min\{p,q\}$. *Then, for all* $\nu \geq 0$ *and* $r \in (0,1]$, *we have*

$$f_\nu^r\left(\frac{pa+qb}{p+q}\right) \geq \frac{1}{(2y)^r}\left(\int_{A-y}^{A+y} f_\nu(t)dt\right)^r \geq \frac{1}{2y}\int_{A-y}^{A+y} f_\nu^r(t)dt$$
$$\geq \frac{1}{2}[f_\nu^r(A-y)+f_\nu^r(A+y)] \geq \frac{pf_\nu^r(a)+qf_\nu^r(b)}{p+q}. \tag{2.4}$$

Proof. It's easy to see that the function $f_\nu(t) \geq 0$ for all $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore, by Hölder inequality for $r \in (0,1]$, we have

$$\frac{1}{2y}\int_{A-y}^{A+y} f_\nu^r(t)dt \leq \frac{1}{(2y)^r}\left(\int_{A-y}^{A+y} f_\nu(t)dt\right)^r. \tag{2.5}$$

By (ii) of Theorem 2.1, the function f_ν is r-concave on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with $r \in (0,1]$. Hence, applying the inequalities (1.14) in [5] and Theorem 2.1, we get

$$\frac{1}{2y}\int_{A-y}^{A+y} f_\nu^r(t)dt \leq \frac{1}{(2y)^r}\left(\int_{A-y}^{A+y} f_\nu(t)dt\right)^r \leq f_\nu^r\left(\frac{pa+qb}{p+q}\right), \tag{2.6}$$

and

$$\frac{1}{2y}\int_{A-y}^{A+y} f_\nu^r(t)dt \geq \frac{1}{2}[f_\nu^r(A-y)+f_\nu^r(A+y)] \geq \frac{pf_\nu^r(a)+qf_\nu^r(b)}{p+q}. \tag{2.7}$$

Combining (2.6) and (2.7) give us the proved. □

Theorem 2.4. *Suppose that* $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$ *and* $r \in (0,1]$. *Then, for every* $t \in [a,b]$ *and* $\nu \geq 0$ *we have the following inequalities*

$$\frac{1}{(b-a)^r}\left(\int_a^b f_\nu(t)dt\right)^r \geq \frac{1}{b-a}\int_a^b f_\nu^r(t)dt$$
$$\geq \frac{f_\nu^r(t)}{2} + \frac{1}{2}\frac{bf_\nu^r(b)-af_\nu^r(a)-t[f_\nu^r(b)-f_\nu^r(a)]}{b-a}. \tag{2.8}$$

Proof. The proof is run analogously to Theorem 2.3 but applying Theorem 19 in [5] and Theorem 2.1. □

Theorem 2.5. *Let* $-\frac{\pi}{2} \leq a < b \leq \frac{\pi}{2}$. *Then for all* $t \in [a,b]$ *one has the inequality*

$$f_\nu(t) + tf_{\nu+1}(t)\left(t - \frac{a+b}{2}\right) \geq \frac{1}{b-a}\int_a^b f_\nu(x)dx. \tag{2.9}$$

Proof. Directly applying Theorem 18 in [5] and Theorem 2.1. □

Theorem 2.6. *Let* $-\frac{\pi}{2} \le a < b \le \frac{\pi}{2}$ *and* $q = \dfrac{p}{p-1}$ *where* $p > 1$. *Then one has the inequality*

$$\left|\frac{f_\nu(a)+f_\nu(b)}{2} - \frac{1}{b-a}\int_a^b f_\nu(x)dx\right| \ge \frac{1}{2}\frac{(b-a)^{1/p}}{(p+1)^{1/p}}\left(\int_a^b |x|^q |f_{\nu+1}(x)|^q dx\right)^{1/q}. \quad (2.10)$$

Proof. Directly applying Theorem 26 in [5] and Theorem 2.1. □

Corollary 2.7. *For* $-\frac{\pi}{2} \le a < b \le \frac{\pi}{2}$ *and* $q = \dfrac{p}{p-1}$ *where* $p > 1$ *we have the inequality*

$$\frac{1}{b-a}\int_a^b f_\nu(x)dx - \frac{f_\nu(a)+f_\nu(b)}{2} \ge \frac{1}{2}\frac{(b-a)^{1/p}}{(p+1)^{1/p}}\left(\int_a^b |x|^q |f_{\nu+1}(x)|^q dx\right)^{1/q}. \quad (2.11)$$

Proof. Directly applying the inequality (2.10) and Theorem 2.1. □

Corollary 2.8. *For* $-\frac{\pi}{2} \le a < b \le \frac{\pi}{2}$ *we have the inequality*

$$\frac{1}{b-a}\int_a^b f_\nu(x)dx - \frac{f_\nu(a)+f_\nu(b)}{2} \ge \frac{[af_{\nu+1}(a) - bf_{\nu+1}(b)](b-a)}{4}. \quad (2.12)$$

Proof. Directly applying Corollary 10 in [5] and Theorem 2.1. □

Theorem 2.9. *Let* $-\frac{\pi}{2} \le a < b \le \frac{\pi}{2}$. *Then for all* $y \in [a,b]$ *we have the following inequalities*

$$\frac{1}{b-a}\int_a^b f_\nu(x)dx \ge \frac{1}{b-a}\int_y^b f_\nu(x)dx + \frac{y-a}{b-a}\frac{f_\nu(a)+f_\nu(y)}{2} \ge \frac{f_\nu(a)+f_\nu(b)}{2} \quad (2.13)$$

and

$$f_\nu\Big(\frac{a+b}{2}\Big) \ge f_\nu\Big(\frac{a+b}{2}\Big) - \frac{y-a}{b-a}f_\nu\Big(\frac{a+y}{2}\Big) + \frac{1}{b-a}\int_a^y f_\nu(x)dx \ge \frac{1}{b-a}\int_a^b f_\nu(x)dx. \quad (2.14)$$

Proof. Directly applying Theorem 67 in [5] and Theorem 2.1. □

Theorem 2.10. *Let* $-\frac{\pi}{2} \le a < b \le \frac{\pi}{2}$ *and* $0 \le s \le r \le 1$. *Then for* $\nu \ge 0$ *we have the following inequalities*

$$\frac{1}{b-a}\int_a^b f_\nu(x)dx \ge \frac{r}{r+1}\frac{f_\nu^{r+1}(b)-f_\nu^{r+1}(a)}{f_\nu^r(b)-f_\nu^r(a)} \ge \frac{s}{s+1}\frac{f_\nu^{s+1}(b)-f_\nu^{s+1}(a)}{f_\nu^s(b)-f_\nu^s(a)}$$
$$\ge [f_\nu(b)-f_\nu(a)][\ln f_\nu(b) - \ln f_\nu(a)]. \quad (2.15)$$

Proof. Directly applying Theorem 2.6 in [10] and Theorem 2.1. □

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