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n-POINTS INEQUALITIES OF HERMITE-HADAMARD TYPE FOR h-CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Some n-points inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([37]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [28], [29], [31], [43], [46] and [47]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([31]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [31] and [44] while for quasi convex functions, the reader can consult [30].

If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0,1]$.

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Definition 3 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f\left(tx + (1-t)y\right) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [26], [27], [38], [40] and [49].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0,1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g\left(tx + (1-t)y\right) &= \|tx + (1-t)y\|^s \leq \left(t\|x\| + (1-t)\|y\|\right)^s \\ &\leq \left(t\|x\|\right)^s + \left[(1-t)\|y\|\right]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0,1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([52])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f\left(tx + (1-t)y\right) \leq h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [52], [6], [41], [50], [48] and [51].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f\left(tx + (1-t)y\right) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0,1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a,b \in \mathbb{R}. \tag{1.6}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [42]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [42]. Since (1.6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[25], [32]-[35] and [45].

We can state the following generalization of the Hermite-Hadamard inequality for h-convex functions defined on convex subsets of linear spaces [24].

Theorem 1. *Assume that the function $f : C \subseteq X \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y,x \in C$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq [f(x)+f(y)] \int_0^1 h(t)\,dt. \tag{1.7}$$

Remark 1. *If $f : I \to [0,\infty)$ is an h-convex function on an interval I of real numbers with $h \in L[0,1]$ and $f \in L[a,b]$ with $a,b \in I, a < b$, then from (1.7) we get the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [48]*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(u)\,du \leq [f(a)+f(b)] \int_0^1 h(t)\,dt.$$

If we write (1.7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{2}. \tag{1.8}$$

If we write (1.7) for the case of P-type functions $f : C \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq f(x)+f(y), \tag{1.9}$$

that has been obtained for functions of real variable in [31].

If f is Breckner s-convex on C, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (1.7) we get

$$2^{s-1} f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x+ty]\,dt \leq \frac{f(x)+f(y)}{s+1}, \tag{1.10}$$

that was obtained for functions of a real variable in [26].

Since the function $g(x) = \|x\|^s$ is Breckner s-convex on on the normed linear space X, $s \in (0,1)$, then for any $x,y \in X$ we have

$$\frac{1}{2}\|x+y\|^s \leq \int_0^1 \|(1-t)x+ty\|^s\,dt \leq \frac{\|x\|^s+\|x\|^s}{s+1}. \tag{1.11}$$

If $f:C\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1)$, then

$$\frac{1}{2^{s+1}}f\left(\frac{x+y}{2}\right)\le\int_0^1 f\left[(1-t)x+ty\right]dt\le\frac{f(x)+f(y)}{1-s}. \tag{1.12}$$

We notice that for $s=1$ the first inequality in (1.12) still holds, i.e.

$$\frac{1}{4}f\left(\frac{x+y}{2}\right)\le\int_0^1 f\left[(1-t)x+ty\right]dt. \tag{1.13}$$

The case for functions of real variables was obtained for the first time in [31].

Motivated by the above results, in this paper some n-points inequalities of Hermite-Hadamard type for h-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

2. Some New Results

In [24] we also obtained the following result:

Theorem 2. *Assume that the function $f:C\subseteq X\to[0,\infty)$ is an h-convex function with $h\in L[0,1]$. Let $y,x\in C$ with $y\neq x$ and assume that the mapping $[0,1]\ni t\mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\left\{(1-\lambda)f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+\lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]\right\}\\
&\le\int_0^1 f\left[(1-t)x+ty\right]dt\\
&\le\left[f\left((1-\lambda)x+\lambda y\right)+(1-\lambda)f(y)+\lambda f(x)\right]\int_0^1 h(t)\,dt\\
&\le\left\{\left[h(1-\lambda)+\lambda\right]f(x)+\left[h(\lambda)+1-\lambda\right]f(y)\right\}\int_0^1 h(t)\,dt.
\end{aligned} \tag{2.1}$$

We can state the following new corollary as well:

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)}\\
&\times\int_0^1(1-\lambda)\left\{f\left[\frac{(1-\lambda)x+(\lambda+1)y}{2}\right]+f\left[\frac{(1-\lambda)y+(\lambda+1)x}{2}\right]\right\}d\lambda\\
&\le\int_0^1 f\left[(1-t)x+ty\right]dt\\
&\le\left[\int_0^1 f\left((1-\lambda)x+\lambda y\right)d\lambda+\frac{f(y)+f(x)}{2}\right]\int_0^1 h(t)\,dt\\
&\le\left[f(x)+f(y)\right]\left[\int_0^1 h(\lambda)\,d\lambda+\frac{1}{2}\right]\int_0^1 h(t)\,dt.
\end{aligned} \tag{2.2}$$

Proof. It follows by (2.1) by integration on $[0,1]$ over λ and by taking into account that, by changing the variable $1-\lambda=\mu$, we have

$$\int_0^1 \lambda f\left[\frac{(2-\lambda)x+\lambda y}{2}\right]d\lambda=\int_0^1 (1-\mu) f\left[\frac{(1+\mu)x+(1-\mu)y}{2}\right]d\mu.$$

□

The following result for double integral also holds:

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)(b-a)^2} \\
&\times\int_a^b\int_a^b \frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\}d\alpha d\beta \\
&\leq \int_0^1 f\left[(1-t)x+ty\right]dt \\
&\leq \left[\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta+\frac{f(y)+f(x)}{2}\right]\int_0^1 h(t)\,dt \\
&\leq \left[\frac{1}{(b-a)^2}\int_a^b\int_a^b h\left(\frac{\beta}{\alpha+\beta}\right)d\alpha d\beta+\frac{1}{2}\right]\left[f(x)+f(y)\right]\int_0^1 h(t)\,dt,
\end{aligned}\tag{2.3}$$

for any $b>a\geq 0$.

Proof. If we take $\lambda=\frac{\alpha}{\alpha+\beta}$ we have

$$\begin{aligned}
&\frac{1}{2h\left(\frac{1}{2}\right)} \\
&\times\left\{\frac{\beta}{\alpha+\beta}f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right]+\frac{\alpha}{\alpha+\beta}f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\} \\
&\leq \int_0^1 f\left[(1-t)x+ty\right]dt \\
&\leq \left[f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}f(y)+\frac{\alpha}{\alpha+\beta}f(x)\right]\int_0^1 h(t)\,dt \\
&\leq \left\{\left[h\left(\frac{\beta}{\alpha+\beta}\right)+\frac{\alpha}{\alpha+\beta}\right]f(x)+\left[h\left(\frac{\alpha}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}\right]f(y)\right\} \\
&\times\int_0^1 h(t)\,dt,
\end{aligned}\tag{2.4}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$.

Since the mapping $[0,1]\ni t\mapsto f\left[(1-t)x+ty\right]$ is Lebesgue integrable on $[0,1]$, then the double integral $\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta$ exists for any $b>a\geq 0$. The same for the other integrals in (2.3).

If we integrate the inequality (2.4) on the square $[a,b]^2$ over (α,β) we have

$$\begin{aligned}
(2.5)\quad & \frac{1}{2h\left(\frac{1}{2}\right)(b-a)^2} \\
& \times \int_a^b \int_a^b \left\{ \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] + \frac{\alpha}{\alpha+\beta} f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\} d\alpha d\beta \\
& \leq \int_0^1 f\left[(1-t)x+ty\right]dt \\
& \leq \int_a^b \int_a^b \left[f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right) + \frac{\beta}{\alpha+\beta} f(y) + \frac{\alpha}{\alpha+\beta} f(x)\right] d\alpha d\beta \int_0^1 h(t)\,dt \\
& \leq \frac{1}{(b-a)^2}\int_0^1 h(t)\,dt \\
& \times \int_a^b \int_a^b \left\{\left[h\left(\frac{\beta}{\alpha+\beta}\right)+\frac{\alpha}{\alpha+\beta}\right] f(x) + \left[h\left(\frac{\alpha}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}\right] f(y)\right\} d\alpha d\beta.
\end{aligned}$$

Observe that

$$\begin{aligned}
& \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] d\alpha d\beta \\
& = \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right] d\alpha d\beta
\end{aligned}$$

and then

$$\begin{aligned}
& \int_a^b \int_a^b \left\{ \frac{\beta}{\alpha+\beta} f\left[\frac{\beta x+(2\alpha+\beta)y}{2(\alpha+\beta)}\right] + \frac{\alpha}{\alpha+\beta} f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\} d\alpha d\beta \\
& = \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right] + f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\} d\alpha d\beta.
\end{aligned}$$

Also

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} d\alpha d\beta$$

and since

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta + \int_a^b \int_a^b \frac{\beta}{\alpha+\beta} d\alpha d\beta = \int_a^b \int_a^b \frac{\alpha+\beta}{\alpha+\beta} d\alpha d\beta = (b-a)^2,$$

then we have

$$\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \frac{1}{2}(b-a)^2.$$

Moreover, we have

$$\int_a^b \int_a^b h\left(\frac{\alpha}{\alpha+\beta}\right) d\alpha d\beta = \int_a^b \int_a^b h\left(\frac{\beta}{\alpha+\beta}\right) d\alpha d\beta.$$

Utilising (2.5), we get the desired result (2.3). □

Remark 2. *Let $f : C \subseteq X \to \mathbb{C}$ be a convex function on the convex subset C of a real or complex linear space X. Then for any $x, y \in C$ and $b > a \geq 0$ we have*

$$
\begin{aligned}
(2.6) \qquad & f\left(\frac{x+y}{2}\right) \\
& \leq \frac{1}{(b-a)^2} \\
& \times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ f\left[\frac{\alpha x + (2\beta+\alpha) y}{2(\alpha+\beta)}\right] + f\left[\frac{(2\beta+\alpha) x + \alpha y}{2(\alpha+\beta)}\right] \right\} d\alpha d\beta \\
& \leq \int_0^1 f[(1-t)x + ty]\, dt \\
& \leq \frac{1}{2}\left[\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{\beta x + \alpha y}{\alpha+\beta}\right) d\alpha d\beta + \frac{f(y)+f(x)}{2}\right] \\
& \leq \frac{f(y)+f(x)}{2}.
\end{aligned}
$$

The second and third inequalities are obvious from (2.3) for $h(t) = t$.

By the convexity of f we have

$$
\begin{aligned}
& \frac{1}{2}\left\{ f\left[\frac{\alpha x + (2\beta+\alpha) y}{2(\alpha+\beta)}\right] + f\left[\frac{(2\beta+\alpha) x + \alpha y}{2(\alpha+\beta)}\right] \right\} \\
& \geq f\left[\frac{1}{2}\left\{\left[\frac{\alpha x + (2\beta+\alpha) y}{2(\alpha+\beta)}\right] + \left[\frac{(2\beta+\alpha) x + \alpha y}{2(\alpha+\beta)}\right]\right\}\right] \\
& = f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$.

If we multiply this inequality by $\frac{2\alpha}{\alpha+\beta} \geq 0$ and integrate on the square $[a, b]^2$ we get

$$
\begin{aligned}
& \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ f\left[\frac{\alpha x + (2\beta+\alpha) y}{2(\alpha+\beta)}\right] + f\left[\frac{(2\beta+\alpha) x + \alpha y}{2(\alpha+\beta)}\right] \right\} d\alpha d\beta \\
& \geq 2 f\left(\frac{x+y}{2}\right) \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = (b-a)^2 f\left(\frac{x+y}{2}\right),
\end{aligned}
$$

since we know that

$$
\int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} d\alpha d\beta = \frac{1}{2}(b-a)^2.
$$

This proves the first inequality in (2.6).

By the convexity of f we also have

$$
f\left(\frac{\beta x + \alpha y}{\alpha+\beta}\right) \leq \frac{\beta}{\alpha+\beta} f(x) + \frac{\alpha}{\alpha+\beta} f(y)
$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$. Integrating on the square $[a,b]^2$ we get

$$\begin{aligned}
&\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta\\
&\leq f(x)\int_a^b\int_a^b\frac{\beta}{\alpha+\beta}d\alpha d\beta+f(y)\int_a^b\int_a^b\frac{\alpha}{\alpha+\beta}d\alpha d\beta\\
&=\frac{1}{2}(b-a)^2\left[f(y)+f(x)\right],
\end{aligned}$$

which proves the last inequality in (2.6).

Let $(X,\|\cdot\|)$ be a normed linear space over the real or complex number fields. Then for any $x,y\in X$, $p\geq 1$ and $b>a\geq 0$ we have:

$$\begin{aligned}
&\left\|\frac{x+y}{2}\right\|^p\\
&\leq\frac{1}{(b-a)^2}\\
&\times\int_a^b\int_a^b\frac{\alpha}{\alpha+\beta}\left\{\left\|\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right\|^p+\left\|\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right\|^p\right\}d\alpha d\beta\\
&\leq\int_0^1\|(1-t)x+ty\|^p dt\\
&\leq\frac{1}{2}\left[\frac{1}{(b-a)^2}\int_a^b\int_a^b\left\|\frac{\beta x+\alpha y}{\alpha+\beta}\right\|^p d\alpha d\beta+\frac{\|y\|^p+\|x\|^p}{2}\right]\\
&\leq\frac{\|y\|^p+\|x\|^p}{2}.
\end{aligned}\tag{2.7}$$

The case of Breckner s-convexity is as follows:

Remark 3. *Assume that the function $f:C\subseteq X\to[0,\infty)$ is a Breckner s-convex function with $s\in(0,1)$. Let $y,x\in C$ with $y\neq x$ and assume that the mapping $[0,1]\ni t\mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then for any $b>a\geq 0$ we have*

$$\begin{aligned}
&\frac{2^{s-1}}{(b-a)^2}\\
&\times\int_a^b\int_a^b\frac{\alpha}{\alpha+\beta}\left\{f\left[\frac{\alpha x+(2\beta+\alpha)y}{2(\alpha+\beta)}\right]+f\left[\frac{(2\beta+\alpha)x+\alpha y}{2(\alpha+\beta)}\right]\right\}d\alpha d\beta\\
&\leq\int_0^1 f[(1-t)x+ty]\,dt\\
&\leq\frac{1}{s+1}\left[\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)d\alpha d\beta+\frac{f(y)+f(x)}{2}\right].
\end{aligned}\tag{2.8}$$

We also have the norm inequalities:

$$\begin{aligned}(2.9)\qquad & \frac{2^{s-1}}{(b-a)^2} \\ & \times \int_a^b \int_a^b \frac{\alpha}{\alpha+\beta} \left\{ \left\| \frac{\alpha x + (2\beta+\alpha) y}{2(\alpha+\beta)} \right\|^s + \left\| \frac{(2\beta+\alpha) x + \alpha y}{2(\alpha+\beta)} \right\|^s \right\} d\alpha d\beta \\ & \leq \int_0^1 \|(1-t) x + ty\|^s dt \\ & \leq \frac{1}{2} \left[\frac{1}{(b-a)^2} \int_a^b \int_a^b \left\| \frac{\beta x + \alpha y}{\alpha+\beta} \right\|^s d\alpha d\beta + \frac{\|y\|^s + \|x\|^s}{2} \right],\end{aligned}$$

for any $x, y \in X$, a normed linear space.

3. Inequalities for n-Points

In order to extend the above results for n-points, we need the following representation of the integral that is of interest in itself.

Theorem 3. *Let $f : C \subseteq X \to \mathbb{C}$ be defined on the convex subset C of a real or complex linear space X. Assume that for $x, y \in C$ with $x \neq y$ the mapping $[0,1] \mapsto f((1-t)x + ty) \in \mathbb{C}$ is Lebesgue integrable on $[0,1]$. Then for any division*

$$0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 1 \text{ with } n \geq 1,$$

we have the representation

$$(3.1)\qquad \int_0^1 f((1-t)x + ty)\, dt$$
$$= \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \int_0^1 f\{(1-u)[(1-\lambda_j)x + \lambda_j y] + u[(1-\lambda_{j+1})x + \lambda_{j+1} y]\}\, du.$$

Proof. We have

$$(3.2)\qquad \int_0^1 f((1-t)x + ty)\, dt = \sum_{j=0}^{n-1} \int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x + ty)\, dt.$$

In the integral

$$\int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x + ty)\, dt,\ j \in \{0, ..., n-1\},$$

consider the change of variable

$$u := \frac{1}{\lambda_{j+1} - \lambda_j} (t - \lambda_j), t \in [\lambda_j, \lambda_{j+1}].$$

Then

$$du = \frac{1}{\lambda_{j+1} - \lambda_j} dt,$$

$u=0$ for $t=\lambda_j$, $u=1$ for $t=\lambda_{j+1}$, $t=(1-u)\lambda_j+u\lambda_{j+1}$ and

$$
\begin{aligned}
(3.3)\qquad & \int_{\lambda_j}^{\lambda_{j+1}} f((1-t)x+ty)\,dt \\
&=(\lambda_{j+1}-\lambda_j) \\
&\times\int_0^1 f[(1-(1-u)\lambda_j-u\lambda_{j+1})x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du \\
&=(\lambda_{j+1}-\lambda_j) \\
&\times\int_0^1 f[(1-u+u-(1-u)\lambda_j-u\lambda_{j+1})x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du \\
&=(\lambda_{j+1}-\lambda_j) \\
&\times\int_0^1 f[((1-u)(1-\lambda_j)+u(1-\lambda_{j+1}))x+((1-u)\lambda_j+u\lambda_{j+1})y]\,du \\
&=\int_0^1 f\{(1-u)[(1-\lambda_j)x+\lambda_j y]+u[(1-\lambda_{j+1})x+\lambda_{j+1}y]\}\,du
\end{aligned}
$$

for any $j\in\{0,...,n-1\}$.

Making use of (3.2) and (3.3) we deduce the desired result (3.1). □

The following particular case is of interest and has been obtained in [24].

Corollary 3. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
(3.4)\qquad \int_0^1 f((1-t)x+ty)\,dt &=\lambda\int_0^1 f\{(1-u)x+u[(1-\lambda)x+\lambda y]\}\,du \\
&+(1-\lambda)\int_0^1 f\{(1-u)[(1-\lambda)x+\lambda y]+uy\}\,du
\end{aligned}
$$

for any $\lambda\in[0,1]$.

Proof. Follows from (3.1) by choosing $0=\lambda_0\le\lambda_1=\lambda\le\lambda_2=1$. □

The following result holds for h-convex functions:

Theorem 4. *Let $f:C\subseteq X\to\mathbb{C}$ be defined on the convex subset C of a real or complex linear space X and f is h-convex on C with $h\in L[0,1]$. Assume that for $x,y\in C$ with $x\neq y$ the mapping $[0,1]\mapsto f((1-t)x+ty)\in\mathbb{R}$ is Lebesgue integrable on $[0,1]$. Then for any division*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\ge 1,$$

we have the inequalities

$$\begin{aligned}
(3.5)\qquad & \frac{1}{2h\left(\frac{1}{2}\right)}\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right)f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\} \\
& \leq \int_0^1 f\left(\left(1-t\right)x+ty\right)dt \\
& \leq \sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right)\left[f\left(\left(1-\lambda_j\right)x+\lambda_j y\right)+f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)\right] \\
& \times \int_0^1 h\left(u\right)du.
\end{aligned}$$

Proof. Since f is h-convex, then

$$\begin{aligned}
& f\left\{\left(1-u\right)\left[\left(1-\lambda_j\right)x+\lambda_j y\right]+u\left[\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right]\right\} \\
& \leq h\left(1-u\right)f\left(\left(1-\lambda_j\right)x+\lambda_j y\right)+h\left(u\right)f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)
\end{aligned}$$

for any $u\in[0,1]$ and for any $j\in\{0,...,n-1\}$.

Integrating this inequality over $u\in[0,1]$ we get

$$\begin{aligned}
& \int_0^1 f\left\{\left(1-u\right)\left[\left(1-\lambda_j\right)x+\lambda_j y\right]+u\left[\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right]\right\}du \\
& \leq \int_0^1 \left\{h\left(1-u\right)f\left(\left(1-\lambda_j\right)x+\lambda_j y\right)+h\left(u\right)f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)\right\}du \\
& = f\left(\left(1-\lambda_j\right)x+\lambda_j y\right)\int_0^1 h\left(1-u\right)du+f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)\int_0^1 h\left(u\right)du \\
& = \left[f\left(\left(1-\lambda_j\right)x+\lambda_j y\right)+f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)\right]\int_0^1 h\left(u\right)du,
\end{aligned}$$

for any $j\in\{0,...,n-1\}$.

Multiplying this inequality by $\lambda_{j+1}-\lambda_j\geq 0$ and summing over j from 0 to $n-1$ we get, via the equality (3.1), the second inequality in (3.5).

Since f is h-convex, then for any $v,w\in C$ we also have

$$f\left(v\right)+f\left(w\right)\geq\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{v+w}{2}\right).$$

If we write this inequality for

$$v=\left(1-u\right)\left[\left(1-\lambda_j\right)x+\lambda_j y\right]+u\left[\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right]$$

and

$$w=u\left[\left(1-\lambda_j\right)x+\lambda_j y\right]+\left(1-u\right)\left[\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right]$$

and take into account that

$$\begin{aligned}
\frac{v+w}{2} & = \frac{1}{2}\left\{\left[\left(1-\lambda_j\right)x+\lambda_j y\right]+\left[\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right]\right\} \\
& = \left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y,
\end{aligned}$$

then we get

$$
\begin{aligned}
(3.6)\qquad & f\left\{(1-u)\left[(1-\lambda_j)x+\lambda_j y\right]+u\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\} \\
& +f\left\{u\left[(1-\lambda_j)x+\lambda_j y\right]+(1-u)\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\} \\
& \geq \frac{1}{h\left(\frac{1}{2}\right)} f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\}
\end{aligned}
$$

for any $u\in[0,1]$ and $j\in\{0,...,n-1\}$.

Integrating the inequality (3.6) over $u\in[0,1]$ we get

$$
\begin{aligned}
(3.7)\qquad & \int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x+\lambda_j y\right]+u\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du \\
& +\int_0^1 f\left\{u\left[(1-\lambda_j)x+\lambda_j y\right]+(1-u)\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du \\
& \geq \frac{1}{h\left(\frac{1}{2}\right)} f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\}
\end{aligned}
$$

for any $j\in\{0,...,n-1\}$.

Since

$$
\begin{aligned}
& \int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x+\lambda_j y\right]+u\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du \\
& =\int_0^1 f\left\{u\left[(1-\lambda_j)x+\lambda_j y\right]+(1-u)\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du,
\end{aligned}
$$

then by (3.7) we get

$$
\begin{aligned}
& \int_0^1 f\left\{(1-u)\left[(1-\lambda_j)x+\lambda_j y\right]+u\left[(1-\lambda_{j+1})x+\lambda_{j+1}y\right]\right\}du \\
& \geq \frac{1}{2h\left(\frac{1}{2}\right)} f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\}
\end{aligned}
$$

for any $j\in\{0,...,n-1\}$.

Multiplying this inequality by $\lambda_{j+1}-\lambda_j\geq 0$ and summing over j from 0 to $n-1$ we get, via the equality (3.1), the first inequality in (3.5). □

Remark 4. *If we take in (3.5)* $0=\lambda_0\leq\lambda_1=\lambda\leq\lambda_2=1$, *then we get the first two inequalities in (2.1).*

The case of convex functions is as follows:

Corollary 4. *Let* $f:C\subseteq X\to\mathbb{R}$ *be a convex function on the convex subset* C *of a real or complex linear space* X. *Then for any division*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\geq 1,$$

and for any $x, y \in C$ we have the inequalities

$$
\begin{aligned}
(3.8)\qquad & f\left(\frac{x+y}{2}\right) \\
& \leq \sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right) f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right) x+\frac{\lambda_j+\lambda_{j+1}}{2} y\right\} \\
& \leq \int_0^1 f((1-t) x+t y) \, dt \\
& \leq \frac{1}{2} \sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right)\left[f\left(\left(1-\lambda_j\right) x+\lambda_j y\right)+f\left(\left(1-\lambda_{j+1}\right) x+\lambda_{j+1} y\right)\right] \\
& \leq \frac{f(x)+f(y)}{2}.
\end{aligned}
$$

Proof. The second and third inequalities in (3.8) follows from (3.5) by taking $h(t) = t$.

By the Jensen discrete inequality

$$
\sum_{j=1}^{m} p_j f(z_j) \geq f\left(\sum_{j=1}^{m} p_j z_j\right),
$$

where $p_j \geq 0$, $j \in \{1, ..., m\}$ with $\sum_{j=1}^{m} p_j = 1$ and $z_j \in C$, $j \in \{1, ..., m\}$ we have

$$
\begin{aligned}
& \sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right) f\left\{\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right) x+\frac{\lambda_j+\lambda_{j+1}}{2} y\right\} \\
& \geq f\left\{\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right)\left[\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right) x+\frac{\lambda_j+\lambda_{j+1}}{2} y\right]\right\} \\
& = f\left\{\left(\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_j\right)-\frac{\sum_{j=0}^{n-1}\left(\lambda_{j+1}^2-\lambda_j^2\right)}{2}\right) x+\frac{\sum_{j=0}^{n-1}\left(\lambda_{j+1}^2-\lambda_j^2\right)}{2} y\right\} \\
& = f\left\{\left(1-\frac{1}{2}\right) x+\frac{1}{2} y\right\} = f\left(\frac{x+y}{2}\right)
\end{aligned}
$$

and the first part of (3.8) is proved.

By the convexity of f we also have

$$
\begin{aligned}
&\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[f\left((1-\lambda_j)x+\lambda_j y\right)+f\left((1-\lambda_{j+1})x+\lambda_{j+1}y\right)\right]\\
&\leq \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[(1-\lambda_j)f(x)+\lambda_j f(y)+(1-\lambda_{j+1})f(x)+\lambda_{j+1}f(y)\right]\\
&=\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[(2-(\lambda_j+\lambda_{j+1}))f(x)+(\lambda_j+\lambda_{j+1})f(y)\right]\\
&=\left(2\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)-\sum_{j=0}^{n-1}\left(\lambda_{j+1}^2-\lambda_j^2\right)\right)f(x)+\sum_{j=0}^{n-1}\left(\lambda_{j+1}^2-\lambda_j^2\right)f(y)\\
&=f(x)+f(y),
\end{aligned}
$$

which proves the last part of (3.8). □

Remark 5. *Let $(X,\|\cdot\|)$ be a normed linear space over the real or complex number fields. Then for any division*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\geq 1,$$

and for any $x,y\in X$ we have the inequalities

$$
\begin{aligned}
&\left\|\frac{x+y}{2}\right\|^p\\
&\leq \sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left\|\left(1-\frac{\lambda_j+\lambda_{j+1}}{2}\right)x+\frac{\lambda_j+\lambda_{j+1}}{2}y\right\|^p\\
&\leq \int_0^1\|(1-t)x+ty\|^p\,dt\\
&\leq \frac{1}{2}\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\left[\|(1-\lambda_j)x+\lambda_j y\|^p+\|(1-\lambda_{j+1})x+\lambda_{j+1}y\|^p\right]\\
&\leq \frac{\|x\|^p+\|y\|^p}{2},
\end{aligned}
\tag{3.9}
$$

where $p\geq 1$.

Corollary 5. *Let $f:C\subseteq X\to\mathbb{R}$ be defined on the convex subset C of a real or complex linear space X and f is Breckner s-convex on C with $s\in(0,1)$. Assume that for $x,y\in C$ with $x\neq y$ the mapping $[0,1]\mapsto f((1-t)x+ty)\in\mathbb{R}$ is Lebesgue integrable on $[0,1]$. Then for any division*

$$0=\lambda_0<\lambda_1<...<\lambda_{n-1}<\lambda_n=1 \text{ with } n\geq 1,$$

we have the inequalities

$$2^{s-1}\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_{j}\right)f\left\{\left(1-\frac{\lambda_{j}+\lambda_{j+1}}{2}\right)x+\frac{\lambda_{j}+\lambda_{j+1}}{2}y\right\} \tag{3.10}$$
$$\leq\int_{0}^{1}f\left(\left(1-t\right)x+ty\right)dt$$
$$\leq\frac{1}{s+1}\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_{j}\right)\left[f\left(\left(1-\lambda_{j}\right)x+\lambda_{j}y\right)+f\left(\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right)\right].$$

Since, for $s\in(0,1)$, the function $f(x)=\|x\|^{s}$ is Breckner s-convex on the normed linear space X, then by (3.10) we get for any $x,y\in X$

$$2^{s-1}\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_{j}\right)\left\|\left(1-\frac{\lambda_{j}+\lambda_{j+1}}{2}\right)x+\frac{\lambda_{j}+\lambda_{j+1}}{2}y\right\|^{s} \tag{3.11}$$
$$\leq\int_{0}^{1}\left\|\left(1-t\right)x+ty\right\|^{s}dt$$
$$\leq\frac{1}{s+1}\sum_{j=0}^{n-1}\left(\lambda_{j+1}-\lambda_{j}\right)\left[\left\|\left(1-\lambda_{j}\right)x+\lambda_{j}y\right\|^{s}+\left\|\left(1-\lambda_{j+1}\right)x+\lambda_{j+1}y\right\|^{s}\right].$$

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