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Fractional Representation Formulae under initial conditions and fractional Ostrowski type inequalities

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Abstract

Here we present very general fractional representation formulae for a function in terms of the fractional Riemann-Liouville integrals of different orders of the function and its ordinary derivatives under initial conditions. Based on these we derive general fractional Ostrowski type inequalities with respect to all basic norms.

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1 Introduction

Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$, and $f':[a,b]\to\mathbb{R}$ be integrable on $[a,b]$, then the following Montgomery identity holds [2]:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\int_a^b P_1(x,t)\,f'(t)\,dt, \tag{1}$$

where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t)=\begin{cases}\frac{t-a}{b-a}, & a\le t\le x,\\ \frac{t-b}{b-a}, & x<t\le b,\end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha>0$ with anchor point $a\in\mathbb{R}$ is defined by

$$J_a^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a,b]. \tag{4}$$

Properties of the above operator can be found in [3].

When $\alpha = 1$, J_a^1 reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^\alpha f(b)}{b-a} - J_a^{\alpha-1} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f'(b)) \right\}. \tag{5}$$

When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).

In this article we find higher order fractional representation for $f(x)$, similar to basic (5), and from there we derive interesting fractional Ostrowski type inequalities.

2 Main Results

Next we give higher order fractional representation of f subject to initial conditions.

Theorem 2 *Let $\alpha > 2$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ twice differentiable, with $f'' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f'(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{2-\alpha}}{\alpha - 1} \left[-(b-a)^{\alpha-2} f(a) + \Gamma(\alpha) \left\{ \frac{2}{b-a} J_a^{\alpha-1} f(b) - \right.\right. \tag{6}$$

$$\left.\left. J_a^{\alpha-2} (P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f''(b)) \right\} \right].$$

Proof. Let here $\alpha > 2$ and there exists $f'' : [a,b] \to \mathbb{R}$ that is integrable on $[a,b]$.

We have

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f''(b)) = \int_a^b (b-t)^{\alpha-1} P_1(x,t) f''(t)\, dt = \tag{7}$$

$$\int_a^x \left(\frac{t-a}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt + \int_x^b \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt =$$

$$\int_a^x \left(\frac{t-a}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt + \int_a^b \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt \tag{8}$$

$$- \int_a^x \left(\frac{t-b}{b-a}\right) (b-t)^{\alpha-1} f''(t)\, dt =$$

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt - \frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f''(t)\,dt.$$

That is

$$\Gamma(\alpha)\, J_a^{\alpha} \left(P_1(x,b)\, f''(b)\right) =$$

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt - \frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f''(t)\,dt =: (\xi_1)\,. \tag{9}$$

Next we use integration by parts, plus the assumption $f'(x) = 0$. We have

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt = \int_a^x (b-t)^{\alpha-1} df'(t) =$$

$$-(b-a)^{\alpha-1} f'(a) - \int_a^x f'(t)\, d(b-t)^{\alpha-1} = -(b-a)^{\alpha-1} f'(a) \tag{10}$$

$$+(\alpha-1)\int_a^x (b-t)^{\alpha-2}\, df(t) = -(b-a)^{\alpha-1} f'(a)$$

$$+(\alpha-1)\left[(b-x)^{\alpha-2} f(x) - (b-a)^{\alpha-2} f(a) - \int_a^x f(t)\, d(b-t)^{\alpha-2}\right] =$$

$$-(b-a)^{\alpha-1} f'(a) + (\alpha-1)(b-x)^{\alpha-2} f(x) - (\alpha-1)(b-a)^{\alpha-2} f(a) + \tag{11}$$

$$(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt.$$

That is

$$\int_a^x (b-t)^{\alpha-1} f''(t)\,dt = -(b-a)^{\alpha-1} f'(a) + (\alpha-1)(b-x)^{\alpha-2} f(x) -$$

$$(\alpha-1)(b-a)^{\alpha-2} f(a) + (\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt =: (\lambda_1)\,. \tag{12}$$

Next we observe

$$-\frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f''(t)\,dt = -\frac{1}{(b-a)} \left[\int_a^b (b-t)^{\alpha} df'(t)\right] =$$

$$-\frac{1}{(b-a)} \left[-(b-a)^{\alpha} f'(a) + \alpha \int_a^b (b-t)^{\alpha-1} df(t)\right] = -\frac{1}{(b-a)}\cdot$$

$$\left[-(b-a)^{\alpha} f'(a) - \alpha (b-a)^{\alpha-1} f(a) + \alpha(\alpha-1)\int_a^b (b-t)^{\alpha-2} f(t)\,dt\right] = \tag{13}$$

$$(b-a)^{\alpha-1} f'(a) + \alpha (b-a)^{\alpha-2} f(a) - \frac{\alpha(\alpha-1)}{(b-a)} \int_a^b (b-t)^{\alpha-2} f(t)\,dt.$$

That is

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f''(t)\,dt = (b-a)^{\alpha-1} f'(a) +$$

$$\alpha (b-a)^{\alpha-2} f(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2} f(t)\,dt =: (\lambda_2). \tag{14}$$

We have that

$$(\xi_1) = (\lambda_1) + (\lambda_2).$$

Thus

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f''(b)\right) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) + \tag{15}$$

$$(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt - \frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt.$$

Notice that

$$-\alpha(\alpha-1) = -(\alpha-1)(\alpha-2) - 2(\alpha-1). \tag{16}$$

We split

$$-\frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt = -\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt$$

$$-\frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt. \tag{17}$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt = \tag{18}$$

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)^{\alpha-2} f(t)\,dt + \int_x^b (b-t)^{\alpha-2} f(t)\,dt\right] =$$

$$-\frac{(\alpha-1)(\alpha-2)}{b-a}\left[\int_a^x (b-t)(b-t)^{\alpha-3} f(t)\,dt + \int_x^b (b-t)(b-t)^{\alpha-3} f(t)\,dt\right] \tag{19}$$

$$= -(\alpha-1)(\alpha-2)\cdot$$

$$\left[\int_a^x \left(1-\left(\frac{t-a}{b-a}\right)\right)(b-t)^{\alpha-3} f(t)\,dt - \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt\right] =$$

$$-(\alpha-1)(\alpha-2)\left[\int_a^x (b-t)^{\alpha-3} f(t)\,dt - \left[\int_a^x \left(\frac{t-a}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt \right.\right. \tag{20}$$

$$\left.\left. + \int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-3} f(t)\,dt\right]\right] =$$

$$-(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt+$$

$$(\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)\,dt. \tag{21}$$

Therefore

$$-\frac{\alpha(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt = -\frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt+$$

$$(\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)\,dt \tag{22}$$

$$-(\alpha-1)(\alpha-2)\int_a^x (b-t)^{\alpha-3} f(t)\,dt.$$

Hence it holds

$$\Gamma(\alpha) J_a^\alpha (P_1(x,b) f''(b)) = (\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) -$$

$$\frac{2(\alpha-1)}{b-a}\int_a^b (b-t)^{\alpha-2} f(t)\,dt + (\alpha-1)(\alpha-2)\int_a^b P_1(x,t)(b-t)^{\alpha-3} f(t)\,dt = \tag{23}$$

$$(\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) - \frac{2(\alpha-1)\Gamma(\alpha-1)}{b-a} J_a^{\alpha-1} f(b) + \tag{24}$$

$$(\alpha-1)(\alpha-2)\Gamma(\alpha-2) J_a^{\alpha-2}(P_1(x,b) f(b)) =$$

$$(\alpha-1)(b-x)^{\alpha-2} f(x) + (b-a)^{\alpha-2} f(a) -$$

$$\frac{2\Gamma(\alpha)}{b-a} J_a^{\alpha-1} f(b) + \Gamma(\alpha) J_a^{\alpha-2}(P_1(x,b) f(b)). \tag{25}$$

We have proved that

$$(\alpha-1)(b-x)^{\alpha-2} f(x) = -(b-a)^{\alpha-2} f(a) + \frac{2\Gamma(\alpha)}{b-a} J_a^{\alpha-1} f(b) -$$

$$\Gamma(\alpha) J_a^{\alpha-2}(P_1(x,b) f(b)) + \Gamma(\alpha) J_a^\alpha (P_1(x,b) f''(b)) = \tag{26}$$

$$-(b-a)^{\alpha-2} f(a) +$$

$$\Gamma(\alpha)\left\{\frac{2}{b-a} J_a^{\alpha-1} f(b) - J_a^{\alpha-2}(P_1(x,b) f(b)) + J_a^\alpha (P_1(x,b) f''(b))\right\}. \tag{27}$$

We have produced (6). ■

We continue with

Theorem 3 *Let $\alpha > 3$, $x \in [a, b)$ fixed, $f : [a, b] \to \mathbb{R}$ three times differentiable, with $f''' : [a, b] \to \mathbb{R}$ integrable on $[a, b]$. Assume $f'(x) = f''(x) = 0$. Then*

$$f(x) = \frac{(b-x)^{3-\alpha}}{(\alpha-1)(\alpha-2)} \left\{ -2(\alpha-1)(b-a)^{\alpha-3} f(a) - (b-a)^{\alpha-2} f'(a) + \right.$$

$$\left. \Gamma(\alpha) \left\{ \frac{3}{(b-a)} J_a^{\alpha-2} f(b) - J_a^{\alpha-3} (P_1(x,b) f(b)) + J_a^{\alpha} (P_1(x,b) f'''(b)) \right\} \right\}. \tag{28}$$

Proof. Let here $\alpha > 3$ and there exists $f''' : [a, b] \to \mathbb{R}$ that is integrable on $[a, b]$. We have as before that

$$\Gamma(\alpha) J_a^{\alpha} (P_1(x,b) f'''(b)) =$$

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\, dt - \frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f'''(t)\, dt =: (\xi_2). \tag{29}$$

By assumption we have $f'(x) = f''(x) = 0$. We use repeatedly integration by parts next

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\, dt = \int_a^x (b-t)^{\alpha-1} df''(t) =$$

$$-(b-a)^{\alpha-1} f''(a) + (\alpha-1) \int_a^x (b-t)^{\alpha-2} f''(t)\, dt =$$

$$-(b-a)^{\alpha-1} f''(a) + (\alpha-1) \int_a^x (b-t)^{\alpha-2} df'(t) =$$

$$-(b-a)^{\alpha-1} f''(a) +$$

$$(\alpha-1) \left[-(b-a)^{\alpha-2} f'(a) + (\alpha-2) \int_a^x (b-t)^{\alpha-3} f'(t)\, dt \right] = \tag{30}$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) +$$

$$(\alpha-1)(\alpha-2) \int_a^x (b-t)^{\alpha-3} df(t) =$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2) \left[(b-x)^{\alpha-3} f(x) \right.$$

$$\left. -(b-a)^{\alpha-3} f(a) + (\alpha-3) \int_a^x (b-t)^{\alpha-4} f(t)\, dt \right] =$$

$$-(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) -$$

$$(\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f(a) + (\alpha-1)(\alpha-2)(\alpha-3) \int_a^x (b-t)^{\alpha-4} f(t)\, dt. \tag{31}$$

That is

$$\int_a^x (b-t)^{\alpha-1} f'''(t)\,dt = -(b-a)^{\alpha-1} f''(a) - (\alpha-1)(b-a)^{\alpha-2} f'(a) +$$

$$(\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) - (\alpha-1)(\alpha-2)(b-a)^{\alpha-3} f(a) + \tag{32}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt =: (\omega_1).$$

Similarly we find

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f'''(t)\,dt = -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}\,df''(t) = \tag{33}$$

$$-\frac{1}{(b-a)}\left[-(b-a)^{\alpha} f''(a) + \alpha\int_a^b (b-t)^{\alpha-1} f''(t)\,dt\right] =$$

$$(b-a)^{\alpha-1} f''(a) - \frac{\alpha}{(b-a)}\int_a^b (b-t)^{\alpha-1}\,df'(t) =$$

$$(b-a)^{\alpha-1} f''(a) - \frac{\alpha}{(b-a)}\left[-(b-a)^{\alpha-1} f'(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2} f'(t)\,dt\right] \tag{34}$$

$$= (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2}\,df(t) =$$

$$(b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) - \tag{35}$$

$$\frac{\alpha(\alpha-1)}{(b-a)}\left[-(b-a)^{\alpha-2} f(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3} f(t)\,dt\right] =$$

$$(b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt. \tag{36}$$

That is we found

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f'''(t)\,dt = (b-a)^{\alpha-1} f''(a) + \alpha(b-a)^{\alpha-2} f'(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =: (\omega_2). \tag{37}$$

Notice that

$$(\xi_2) = (\omega_1) + (\omega_2).$$

We have

$$\Gamma(\alpha) J_a^{\alpha}(P_1(x,b) f'''(b)) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) +$$

$$2(\alpha-1)(b-a)^{\alpha-3} f(a) + (a-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt -$$

$$\frac{\alpha(a-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt. \tag{38}$$

We notice that

$$-\alpha(\alpha-1)(\alpha-2) = -3(\alpha-1)(\alpha-2) - (\alpha-1)(\alpha-2)(\alpha-3). \tag{39}$$

Hence

$$-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =$$

$$-\frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt - \tag{40}$$

$$\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt.$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\left[\int_a^x (b-t)^{\alpha-3} f(t)\,dt + \int_x^b (b-t)^{\alpha-3} f(t)\,dt\right] = \tag{41}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$

$$\left[\int_a^x (b-t)(b-t)^{\alpha-4} f(t)\,dt + \int_x^b (b-t)(b-t)^{\alpha-4} f(t)\,dt\right] = \tag{42}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$

$$\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-4} f(t)\,dt - \int_x^b (t-b)(b-t)^{\alpha-4} f(t)\,dt\right] =$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x \left(1-\left(\frac{t-a}{b-a}\right)\right)(b-t)^{\alpha-4} f(t)\,dt -\right.$$

$$\left.\int_x^b \left(\frac{t-b}{b-a}\right)(b-t)^{\alpha-4} f(t)\,dt\right] = \tag{43}$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)\left[\int_a^x (b-t)^{\alpha-4} f(t)\,dt - \int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt\right].$$

We derived that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt = \tag{44}$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt.$$

Therefore we obtain

$$-\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt =$$

$$-\frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt - \tag{45}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^x (b-t)^{\alpha-4} f(t)\,dt +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt.$$

Combining (38) and (45) we find

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f'''(b)\right) = (b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) +$$

$$2(\alpha-1)(b-a)^{\alpha-3} f(a) - \frac{3(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3} f(t)\,dt + \tag{46}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\int_a^b P_1(x,t)(b-t)^{\alpha-4} f(t)\,dt =$$

$$(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a) -$$

$$\frac{3(\alpha-1)(\alpha-2)\Gamma(\alpha-2)}{b-a} J_a^{\alpha-2} f(b) + \tag{47}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3) J_a^{\alpha-3}\left(P_1(x,b) f(b)\right) =$$

$$(b-a)^{\alpha-2} f'(a) + (\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) + 2(\alpha-1)(b-a)^{\alpha-3} f(a) -$$

$$\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) + \Gamma(\alpha) J_a^{\alpha-3}\left(P_1(x,b) f(b)\right). \tag{48}$$

Consequently we get

$$(\alpha-1)(\alpha-2)(b-x)^{\alpha-3} f(x) = -(b-a)^{\alpha-2} f'(a) - 2(\alpha-1)(b-a)^{\alpha-3} f(a)$$

$$+\frac{3\Gamma(\alpha)}{(b-a)} J_a^{\alpha-2} f(b) - \Gamma(\alpha) J_a^{\alpha-3}\left(P_1(x,b) f(b)\right) + \Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f'''(b)\right) = \tag{49}$$

$$-\left(b-a\right)^{\alpha-2}f'\left(a\right)-2\left(\alpha-1\right)\left(b-a\right)^{\alpha-3}f\left(a\right)+$$

$$\Gamma\left(\alpha\right)\left\{\frac{3}{\left(b-a\right)}J_{a}^{\alpha-2}f\left(b\right)-J_{a}^{\alpha-3}\left(P_{1}\left(x,b\right)f\left(b\right)\right)+J_{a}^{\alpha}\left(P_{1}\left(x,b\right)f'''\left(b\right)\right)\right\}, \quad (50)$$

proving the claim. ■

We continue with

Theorem 4 *Let $\alpha>4$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ four times differentiable, with $f^{(4)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f'(x)=f''(x)=f'''(x)=0$. Then*

$$f\left(x\right)=\frac{\left(b-x\right)^{4-\alpha}}{\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)}\Big\{-3\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-4}f\left(\alpha\right)-$$

$$2\left(\alpha-1\right)\left(b-a\right)^{\alpha-3}f'\left(a\right)-\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)+$$

$$\Gamma\left(\alpha\right)\left\{\frac{4J_{a}^{\alpha-3}\left(f\left(b\right)\right)}{\left(b-a\right)}-J_{a}^{\alpha-4}\left(P_{1}\left(x,b\right)f\left(b\right)\right)+J_{a}^{\alpha}\left(P_{1}\left(x,b\right)f^{(4)}\left(b\right)\right)\right\}\Big\}. \quad (51)$$

Proof. Let here $\alpha>4$ and there exists $f^{(4)}:[a,b]\to\mathbb{R}$ that is integrable on $[a,b]$. We have as before that

$$\Gamma\left(\alpha\right)J_{a}^{\alpha}\left(P_{1}\left(x,b\right)f^{(4)}\left(b\right)\right)=\int_{a}^{x}\left(b-t\right)^{\alpha-1}f^{(4)}\left(t\right)dt-$$

$$\frac{1}{\left(b-a\right)}\int_{a}^{b}\left(b-t\right)^{\alpha}f^{(4)}\left(t\right)dt=:\left(\xi_{3}\right).$$

By assumption we have $f'(x)=f''(x)=f'''(x)=0$. We use repeatedly integration by parts next

$$\int_{a}^{x}\left(b-t\right)^{\alpha-1}f^{(4)}\left(t\right)dt=\int_{a}^{x}\left(b-t\right)^{\alpha-1}df^{(3)}\left(t\right)=$$

$$-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)+\left(\alpha-1\right)\int_{a}^{x}\left(b-t\right)^{\alpha-2}df^{(2)}\left(t\right)=-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)+$$

$$\left(\alpha-1\right)\left[-\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)+\left(\alpha-2\right)\int_{a}^{x}\left(b-t\right)^{\alpha-3}df'\left(t\right)\right]= \quad (52)$$

$$-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\int_{a}^{x}\left(b-t\right)^{\alpha-3}df'\left(t\right)=$$

$$-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{a-2}f^{(2)}\left(a\right)+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left[-\left(b-a\right)^{\alpha-3}f'\left(a\right)+\left(\alpha-3\right)\int_{a}^{x}\left(b-t\right)^{\alpha-4}df\left(t\right)\right]= \quad (53)$$

$$-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f'\left(a\right)$$
$$+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\int_a^x\left(b-t\right)^{\alpha-4}df\left(t\right)=$$
$$-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f'\left(a\right)$$
$$+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-x\right)^{\alpha-4}f\left(x\right)-$$
$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4}f\left(a\right)+$$
$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\int_a^x\left(b-t\right)^{\alpha-5}f\left(t\right)dt. \tag{54}$$

We find that

$$\int_a^x\left(b-t\right)^{\alpha-1}f^{(4)}\left(t\right)dt=-\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)-$$
$$\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3}f'\left(a\right)+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-x\right)^{\alpha-4}f\left(x\right)-$$
$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4}f\left(a\right)+$$
$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\int_a^x\left(b-t\right)^{\alpha-5}f\left(t\right)dt=:\left(\theta_1\right). \tag{55}$$

Next we observe that

$$-\frac{1}{(b-a)}\int_a^b\left(b-t\right)^{\alpha}f^{(4)}\left(t\right)dt=-\frac{1}{(b-a)}\int_a^b\left(b-t\right)^{\alpha}df^{(3)}\left(t\right)=$$
$$-\frac{1}{(b-a)}\left[-\left(b-a\right)^{\alpha}f^{(3)}\left(a\right)+\alpha\int_a^b\left(b-t\right)^{\alpha-1}df^{(2)}\left(t\right)\right]= \tag{56}$$
$$\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-\frac{\alpha}{b-a}\int_a^b\left(b-t\right)^{\alpha-1}df^{(2)}\left(t\right)=\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)-$$
$$\frac{\alpha}{b-a}\left[-\left(b-a\right)^{\alpha-1}f^{(2)}\left(a\right)+\left(\alpha-1\right)\int_a^b\left(b-t\right)^{\alpha-2}df'\left(t\right)\right]=$$
$$\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)+\alpha\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)-\frac{\alpha\left(\alpha-1\right)}{(b-a)}\int_a^b\left(b-t\right)^{\alpha-2}df'\left(t\right)=$$
$$\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)+\alpha\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)-$$
$$\frac{\alpha\left(\alpha-1\right)}{(b-a)}\left[-\left(b-a\right)^{\alpha-2}f'\left(a\right)+\left(\alpha-2\right)\int_a^b\left(b-t\right)^{\alpha-3}df\left(t\right)\right]= \tag{57}$$
$$\left(b-a\right)^{\alpha-1}f^{(3)}\left(a\right)+\alpha\left(b-a\right)^{\alpha-2}f^{(2)}\left(a\right)+\alpha\left(\alpha-1\right)\left(b-a\right)^{\alpha-3}f'\left(a\right)-$$
$$\frac{\alpha\left(\alpha-1\right)\left(\alpha-2\right)}{(b-a)}\int_a^b\left(b-t\right)^{\alpha-3}df\left(t\right)=$$

$$(b-a)^{\alpha-1} f^{(3)}(a) + \alpha (b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) - \tag{58}$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)} \left[-(b-a)^{\alpha-3} f(a) + (\alpha-3) \int_a^b (b-t)^{\alpha-4} f(t)\, dt \right] =$$

$$(b-a)^{\alpha-1} f^{(3)}(a) + \alpha (b-a)^{\alpha-2} f^{(2)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$

$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt.$$

That is

$$-\frac{1}{(b-a)} \int_a^b (b-t)^{\alpha} f^{(4)}(t)\, dt = (b-a)^{\alpha-1} f^{(3)}(a) + \alpha (b-a)^{\alpha-2} f^{(2)}(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f'(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) -$$

$$\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt =: (\theta_2). \tag{59}$$

Notice that

$$(\xi_3) = (\theta_1) + (\theta_2). \tag{60}$$

We find that

$$\Gamma(\alpha) J_a^{\alpha} \left(P_1(x,b) f^{(4)}(b) \right) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$

$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) + \tag{61}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4) \int_a^x (b-t)^{\alpha-5} f(t)\, dt$$

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt.$$

We have

$$-\alpha(\alpha-1)(\alpha-2)(\alpha-3) =$$

$$-4(\alpha-1)(\alpha-2)(\alpha-3) - (\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4). \tag{62}$$

and

$$\frac{-\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt = \tag{63}$$

$$-\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)} \int_a^b (b-t)^{\alpha-4} f(t)\, dt.$$

But we see that

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt =$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-5} f(t)\,dt\right.$$

$$\left.-\int_x^b (t-b)(b-t)^{\alpha-5} f(t)\,dt\right] =$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\left[\int_a^x (b-t)^{\alpha-5} f(t)\,dt\right. \tag{64}$$

$$\left.-\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt\right].$$

Therefore it holds

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt =$$

$$-\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt$$

$$-(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^x (b-t)^{\alpha-5} f(t)\,dt$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt. \tag{65}$$

Consequently we get

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right) = (b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) +$$

$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a) + (\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) -$$

$$\frac{4(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4} f(t)\,dt + \tag{66}$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\int_a^b P_1(x,t)(b-t)^{\alpha-5} f(t)\,dt =$$

$$(b-a)^{\alpha-2} f^{(2)}(a) + 2(\alpha-1)(b-a)^{\alpha-3} f'(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x) -$$

$$\frac{4(\alpha-1)(\alpha-2)(\alpha-3)\Gamma(\alpha-3)}{(b-a)} J_a^{\alpha-3}(f(b)) +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4) J_a^{\alpha-4}(P_1(x,b) f(b)) = \tag{67}$$

$$(b-a)^{\alpha-2} f^{(2)}(a)+2(\alpha-1)(b-a)^{\alpha-3} f'(a)+3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)$$
$$+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x)-$$
$$\frac{4\Gamma(\alpha)}{(b-a)} J_a^{\alpha-3}(f(b))+\Gamma(\alpha) J_a^{\alpha-4}(P_1(x,b) f(b)). \tag{68}$$

That is

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(4)}(b)\right)=(b-a)^{\alpha-2} f^{(2)}(a)+2(\alpha-1)(b-a)^{\alpha-3} f'(a)+$$
$$3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f(a)+(\alpha-1)(\alpha-2)(\alpha-3)(b-x)^{\alpha-4} f(x)+$$
$$\Gamma(\alpha)\left\{-\frac{4 J_a^{\alpha-3}(f(b))}{(b-a)}+J_a^{\alpha-4}(P_1(x,b) f(b))\right\}, \tag{69}$$

proving the claim. ■

We continue with

Theorem 5 *Let $\alpha>5$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ five times differentiable, with $f^{(5)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x)=0$, $j=1,2,3,4$. Then*

$$f(x)=\frac{(b-x)^{5-\alpha}}{\prod_{j=1}^{4}(\alpha-j)}\left\{-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5} f(a)-\right.$$
$$3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4} f'(a)-2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)-(b-a)^{\alpha-2} f^{(3)}(a)+$$
$$\left.\Gamma(\alpha)\left\{\frac{5}{(b-a)}\left(J_a^{\alpha-4}(f(b))\right)-J_a^{\alpha-5}(P_1(x,b) f(b))+J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right)\right\}\right\}. \tag{70}$$

Proof. Let here $\alpha>5$ and there exists $f^{(5)}:[a,b]\to\mathbb{R}$ that is integrable on $[a,b]$. We have as before that

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right)=$$
$$\int_a^x (b-t)^{\alpha-1} f^{(5)}(t)\,dt-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt=:(\xi_4). \tag{71}$$

By assumption we have $f^{(j)}(x)=0$, $j=1,2,3,4$. We use repeatedly integration by parts next

$$\int_a^x (b-t)^{\alpha-1} f^{(5)}(t)\,dt=\int_a^x (b-t)^{\alpha-1} df^{(4)}(t)=$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)+\left(\alpha-1\right)\int_a^x \left(b-t\right)^{\alpha-2} df^{(3)}\left(t\right) = -\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)+$$

$$\left(\alpha-1\right)\left\{-\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)+\left(\alpha-2\right)\int_a^x \left(b-t\right)^{\alpha-3} df^{(2)}\left(t\right)\right\} = \tag{72}$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\int_a^x \left(b-t\right)^{\alpha-3} df^{(2)}\left(t\right) =$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left\{-\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)+\left(\alpha-3\right)\int_a^x \left(b-t\right)^{\alpha-4} df'\left(t\right)\right\} = \tag{73}$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\int_a^x \left(b-t\right)^{\alpha-4} df'\left(t\right)$$

$$= -\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)$$

$$-\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)+$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left\{-\left(b-a\right)^{\alpha-4} f'\left(a\right)+\left(\alpha-4\right)\int_a^x \left(b-t\right)^{\alpha-5} df\left(t\right)\right\} =$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4} f'\left(a\right) \tag{74}$$

$$+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\int_a^x \left(b-t\right)^{\alpha-5} df\left(t\right) =$$

$$-\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4} f'\left(a\right)+\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\cdot$$

$$\left\{\left(b-x\right)^{\alpha-5} f\left(x\right)-\left(b-a\right)^{\alpha-5} f\left(a\right)+\left(\alpha-5\right)\int_a^x \left(b-t\right)^{\alpha-6} f\left(t\right) dt\right\}.$$

That is

$$\int_a^x \left(b-t\right)^{\alpha-1} f^{(5)}\left(t\right) dt = -\left(b-a\right)^{\alpha-1} f^{(4)}\left(a\right)-\left(\alpha-1\right)\left(b-a\right)^{\alpha-2} f^{(3)}\left(a\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(b-a\right)^{\alpha-3} f^{(2)}\left(a\right)-\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(b-a\right)^{\alpha-4} f'\left(a\right)+ \tag{75}$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\left(b-x\right)^{\alpha-5} f\left(x\right)-$$

$$\left(\alpha-1\right)\left(\alpha-2\right)\left(\alpha-3\right)\left(\alpha-4\right)\left(b-a\right)^{\alpha-5} f\left(a\right)+$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x (b-t)^{\alpha-6} f(t)\,dt =: (\eta_1).$$

Next we observe that

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt = -\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha}\,df^{(4)}(t) =$$

$$-\frac{1}{(b-a)}\left\{-(b-a)^{\alpha} f^{(4)}(a) + \alpha\int_a^b (b-t)^{\alpha-1}\,df^{(3)}(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) - \frac{\alpha}{(b-a)}\int_a^b (b-t)^{\alpha-1}\,df^{(3)}(t) = \tag{76}$$

$$(b-a)^{\alpha-1} f^{(4)}(a) -$$

$$\frac{\alpha}{(b-a)}\left\{-(b-a)^{\alpha-1} f^{(3)}(a) + (\alpha-1)\int_a^b (b-t)^{\alpha-2}\,df^{(2)}(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) - \frac{\alpha(\alpha-1)}{(b-a)}\int_a^b (b-t)^{\alpha-2}\,df^{(2)}(t) =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) -$$

$$\frac{\alpha(\alpha-1)}{(b-a)}\left\{-(b-a)^{\alpha-2} f^{(2)}(a) + (\alpha-2)\int_a^b (b-t)^{\alpha-3}\,df'(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) -$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\int_a^b (b-t)^{\alpha-3}\,df'(t) =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) -$$

$$\frac{\alpha(\alpha-1)(\alpha-2)}{(b-a)}\left\{-(b-a)^{\alpha-3} f'(a) + (\alpha-3)\int_a^b (b-t)^{\alpha-4}\,df(t)\right\} =$$

$$(b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + \tag{77}$$

$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\int_a^b (b-t)^{\alpha-4}\,df(t)$$

$$= (b-a)^{\alpha-1} f^{(4)}(a) + \alpha(b-a)^{\alpha-2} f^{(3)}(a) + \alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) +$$

$$\alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) - \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{(b-a)}\cdot$$

$$\left\{-(b-a)^{\alpha-4} f(a) + (\alpha-4)\int_a^b (b-t)^{\alpha-5} f(t)\,dt\right\}. \tag{78}$$

We proved that

$$-\frac{1}{(b-a)}\int_a^b (b-t)^{\alpha} f^{(5)}(t)\,dt = (b-a)^{\alpha-1} f^{(4)}(a) + \alpha (b-a)^{\alpha-2} f^{(3)}(a) +$$

$$\alpha(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + \alpha(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a) +$$

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) - \tag{79}$$

$$\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt =: (\eta_2).$$

We have

$$(\xi_4) = (\eta_1) + (\eta_2).$$

Therefore it holds

$$\Gamma(\alpha) J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right) = (b-a)^{\alpha-2} f^{(3)}(a) + \tag{80}$$

$$2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a) + 3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$

$$+(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5} f(x) +$$

$$4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5} f(a) +$$

$$(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)\int_a^x (b-t)^{\alpha-6} f(t)\,dt$$

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt.$$

We see that

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt =$$

$$-\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt \tag{81}$$

$$-\frac{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt.$$

We have

$$-\frac{\prod_{j=1}^{5}(\alpha-j)}{(b-a)}\int_a^b (b-t)^{\alpha-5} f(t)\,dt = -\frac{\prod_{j=1}^{5}(\alpha-j)}{(b-a)}\cdot$$

$$\left[\int_a^x ((b-a)-(t-a))(b-t)^{\alpha-6} f(t)\,dt - \int_x^b (t-b)(b-t)^{\alpha-6} f(t)\,dt\right] =$$

$$-\prod_{j=1}^{5}(\alpha-j)\left[\int_a^x(b-t)^{\alpha-6}f(t)\,dt-\int_a^b P_1(x,t)(b-t)^{\alpha-6}f(t)\,dt\right]. \quad (82)$$

Therefore it holds

$$-\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b(b-t)^{\alpha-5}f(t)\,dt=$$

$$-\frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b(b-t)^{\alpha-5}f(t)\,dt-$$

$$\prod_{j=1}^{5}(\alpha-j)\int_a^x(b-t)^{\alpha-6}f(t)\,dt+\prod_{j=1}^{5}(\alpha-j)\int_a^b P_1(x,t)(b-t)^{\alpha-6}f(t)\,dt. \quad (83)$$

Consequently we get

$$\begin{gathered}\Gamma(\alpha)J_a^{\alpha}\left(P_1(x,b)f^{(5)}(b)\right)=(b-a)^{\alpha-2}f^{(3)}(a)+\\ 2(\alpha-1)(b-a)^{\alpha-3}f^{(2)}(a)+3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4}f'(a)\\ +(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5}f(x)+\\ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5}f(a)-\\ \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}{(b-a)}\int_a^b(b-t)^{\alpha-5}f(t)\,dt\\ +\prod_{j=1}^{5}(\alpha-j)\int_a^b P_1(x,t)(b-t)^{\alpha-6}f(t)\,dt.\end{gathered} \quad (84)$$

So that

$$\begin{gathered}\Gamma(\alpha)J_a^{\alpha}\left(P_1(x,b)f^{(5)}(b)\right)=(b-a)^{\alpha-2}f^{(3)}(a)+\\ 2(\alpha-1)(b-a)^{\alpha-3}f^{(2)}(a)+3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4}f'(a)\\ +(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(b-x)^{\alpha-5}f(x)+\\ 4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5}f(a)-\\ \frac{5(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)\Gamma(\alpha-4)}{(b-a)}\left(J_a^{\alpha-4}(f(b))\right)\\ +\prod_{j=1}^{5}(\alpha-j)\Gamma(\alpha-5)J_a^{\alpha-5}\left(P_1(x,b)f(b)\right).\end{gathered} \quad (85)$$

And finally we derive

$$\prod_{j=1}^{4}(\alpha-j)(b-x)^{\alpha-5}f(x)=-4(\alpha-1)(\alpha-2)(\alpha-3)(b-a)^{\alpha-5}f(a) \quad (86)$$

$$-3(\alpha-1)(\alpha-2)(b-a)^{\alpha-4} f'(a)$$

$$-2(\alpha-1)(b-a)^{\alpha-3} f^{(2)}(a)-(b-a)^{\alpha-2} f^{(3)}(a)+$$

$$\Gamma(\alpha)\left\{\frac{5}{(b-a)}\left(J_a^{\alpha-4}(f(b))\right)-J_a^{\alpha-5}\left(P_1(x,b) f(b)\right)+J_a^{\alpha}\left(P_1(x,b) f^{(5)}(b)\right)\right\},$$

proving the claim. ∎

In general holds the following fractional representation formula

Theorem 6 *Let $\alpha > n$, $n \in \mathbb{N}$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ n-times differentiable, with $f^{(n)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1,...,n-1$. Then*

$$f(x)=\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\left\{-(n-1)\prod_{j=1}^{n-2}(\alpha-j)(b-a)^{\alpha-n} f(a)-\right. \tag{87}$$

$$(n-2)\prod_{j=1}^{n-3}(\alpha-j)(b-a)^{\alpha-n+1} f'(a)-(n-3)\prod_{j=1}^{n-4}(\alpha-j)(b-a)^{\alpha-n+2} f^{(2)}(a)$$

$$-(n-4)\prod_{j=1}^{n-5}(\alpha-j)(b-a)^{\alpha-n+3} f^{(3)}(a)-...$$

$$-(b-a)^{\alpha-2} f^{(n-2)}(a)+\Gamma(\alpha)\left\{\frac{n}{b-a}\left(J_a^{\alpha-n+1}(f(b))\right)-J_a^{\alpha-n}\left(P_1(x,b) f(b)\right)\right.$$

$$\left.\left.+J_a^{\alpha}\left(P_1(x,b) f^{(n)}(b)\right)\right\}\right\}.$$

Above we assume that $\prod_{j=1}^{0}(\alpha-j)=1$, and $\prod_{j=1}^{k}(\alpha-j)=0$ if $k \in \{-1,-2,...\}$. Also set $f^{(-1)}(a) := 0$.

Proof. Based on Theorems 1-5. ∎

Theorems 1-5 are special cases of Theorem 6.

We give applications of Theorem 6 for $n = 6,7$.

Theorem 7 *Let $\alpha > 6$, $x \in [a,b)$ fixed, $f : [a,b] \to \mathbb{R}$ six times differentiable, with $f^{(6)} : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x) = 0$, $j = 1,...,5$. Then*

$$f(x)=\frac{(b-x)^{6-\alpha}}{\prod_{j=1}^{5}(\alpha-j)}\left\{-5\prod_{j=1}^{4}(\alpha-j)(b-a)^{\alpha-6} f(a)-\right.$$

$$4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f'(a)-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(2)}(a)-$$

$$2(\alpha-1)(b-a)^{\alpha-3}f^{(3)}(a)-(b-a)^{\alpha-2}f^{(4)}(a)+ \tag{88}$$

$$\Gamma(\alpha)\left\{\frac{6}{b-a}\left(J_a^{\alpha-5}(f(b))\right)-J_a^{\alpha-6}(P_1(x,b)f(b))+J_a^{\alpha}\left(P_1(x,b)f^{(6)}(b)\right)\right\}\Bigg\}.$$

Theorem 8 *Let $\alpha>7$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ seven times differentiable, with $f^{(7)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x)=0$, $j=1,...,6$. Then*

$$f(x)=\frac{(b-x)^{7-\alpha}}{\prod_{j=1}^{6}(\alpha-j)}\Bigg\{-6\prod_{j=1}^{5}(\alpha-j)(b-a)^{\alpha-7}f(a)- \tag{89}$$

$$5\prod_{j=1}^{4}(\alpha-j)(b-a)^{\alpha-6}f'(a)-4\prod_{j=1}^{3}(\alpha-j)(b-a)^{\alpha-5}f''(a)$$

$$-3\prod_{j=1}^{2}(\alpha-j)(b-a)^{\alpha-4}f^{(3)}(a)-2(\alpha-1)(b-a)^{\alpha-3}f^{(4)}(a)$$

$$-(b-a)^{\alpha-2}f^{(5)}(a)+\Gamma(\alpha)\left\{\frac{7}{b-a}\left(J_a^{\alpha-6}(f(b))\right)-J_a^{\alpha-7}(P_1(x,b)f(b))\right.$$

$$\left.+J_a^{\alpha}\left(P_1(x,b)f^{(7)}(b)\right)\right\}\Bigg\}.$$

We make

Remark 9 *We rewrite (87) as follows:*

$$E_n(f,\alpha,x):=f(x)+\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\Bigg\{(n-1)\prod_{j=1}^{n-2}(\alpha-j)(b-a)^{\alpha-n}f(a)+ \tag{90}$$

$$(n-2)\prod_{j=1}^{n-3}(\alpha-j)(b-a)^{\alpha-n+1}f'(a)+(n-3)\prod_{j=1}^{n-4}(\alpha-j)(b-a)^{\alpha-n+2}f^{(2)}(a)$$

$$+(n-4)\prod_{j=1}^{n-5}(\alpha-j)(b-a)^{\alpha-n+3}f^{(3)}(a)+...+(b-a)^{\alpha-2}f^{(n-2)}(a)+$$

$$+\Gamma(\alpha)\left\{-\frac{n}{b-a}\left(J_a^{\alpha-n+1}(f(b))\right)+J_a^{\alpha-n}(P_1(x,b)f(b))\right\}\Bigg\}=$$

$$\frac{(b-x)^{n-\alpha}\Gamma(\alpha)}{\prod_{j=1}^{n-1}(\alpha-j)}J_a^{\alpha}\left(P_1(x,b)f^{(n)}(b)\right)=$$

$$\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b(b-t)^{\alpha-1}P_1(x,t)f^{(n)}(t)\,dt. \tag{91}$$

We upper bound $E_n(f,\alpha,x)$, that is we upper bound the right hand side of (91).

Consequently we produce fractional Ostrowski type inequalities motivated by [1] done there for $n=1$.

Theorem 10 *Let $\alpha>n$, $n\in\mathbb{N}$, $x\in[a,b)$ fixed, $f:[a,b]\to\mathbb{R}$ n-times differentiable, with $f^{(n)}:[a,b]\to\mathbb{R}$ integrable on $[a,b]$. Assume $f^{(j)}(x)=0$, $j=1,...,n-1$, and $\left\|f^{(n)}\right\|_{\infty}<\infty$. Then*

$$|E_n(f,\alpha,x)|\leq\frac{\left\|f^{(n)}\right\|_{\infty}}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^{\alpha}}{\alpha(\alpha+1)}-\frac{(b-x)^{n}}{\alpha}+\frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right], \tag{92}$$

where $E_n(f,\alpha,x)$ as in (90).

Proof. We have that

$$|E_n(f,\alpha,x)|\leq\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b(b-t)^{\alpha-1}|P_1(x,t)|\left|f^{(n)}(t)\right|dt\leq$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_{\infty}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[\int_a^x(b-t)^{\alpha-1}(t-a)\,dt+\int_x^b(b-t)^{\alpha}\,dt\right]= \tag{93}$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_{\infty}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[\int_a^x(b-t)^{\alpha-1}((b-a)-(b-t))\,dt+\frac{(b-x)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|f^{(n)}\right\|_{\infty}(b-x)^{n-\alpha}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)}\left[(b-a)\int_a^x(b-t)^{\alpha-1}dt-\int_a^x(b-t)^{\alpha}dt+\frac{(b-x)^{\alpha+1}}{\alpha+1}\right]$$

$$= \frac{\left\|f^{(n)}\right\|_\infty (b-x)^{n-\alpha}}{(b-a)\left(\prod_{j=1}^{n-1}(\alpha-j)\right)} \cdot$$

$$\left[(b-a)\left(\frac{(b-a)^\alpha}{\alpha} - \frac{(b-x)^\alpha}{\alpha}\right) - \frac{(b-a)^{\alpha+1}}{\alpha+1} + \frac{2(b-x)^{\alpha+1}}{\alpha+1}\right] = \tag{94}$$

$$\frac{(b-x)^{n-\alpha}\left\|f^{(n)}\right\|_\infty}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-a)^\alpha}{\alpha(\alpha+1)} - \frac{(b-x)^\alpha}{\alpha} + \frac{2(b-x)^{\alpha+1}}{(b-a)(\alpha+1)}\right] = \tag{95}$$

$$\frac{\left\|f^{(n)}\right\|_\infty}{\prod_{j=1}^{n-1}(\alpha-j)}\left[\frac{(b-x)^{n-\alpha}(b-a)^\alpha}{\alpha(\alpha+1)} - \frac{(b-x)^n}{\alpha} + \frac{2(b-x)^{n+1}}{(b-a)(\alpha+1)}\right].$$

■

Theorem 11 *Let all as in Theorem 6. Then*

$$|E_n(f,\alpha,x)| \leq \left(\frac{(b-x)^{n-\alpha}(b-a)^{\alpha-2}}{2\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a+|a+b-2x|)\left\|f^{(n)}\right\|_{L_1([a,b])}. \tag{96}$$

Proof. We have that

$$|E_n(f,\alpha,x)| \leq \frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\int_a^b (b-t)^{\alpha-1}|P_1(x,t)|\left|f^{(n)}(t)\right|dt \leq \tag{97}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\max\{x-a,b-x\}\left\|f^{(n)}\right\|_{L_1([a,b])} = \tag{98}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)}\right)(b-a)^{\alpha-2}\left(\frac{b-a+|a+b-2x|}{2}\right)\left\|f^{(n)}\right\|_{L_1([a,b])}.$$

■

Theorem 12 *Let $p, q, r > 1$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Let all as in Theorem 6, but now $f^{(n)} \in L_r([a,b])$. Then*

$$|E_n(f,\alpha,x)| \leq$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \right) \frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \frac{(b-x)^{q+1}+(x-a)^{q+1}}{(q+1)} \right\}^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])}. \tag{99}$$

Proof. We have

$$|E_n(f,\alpha,x)| \overset{(97)}{\leq} \left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \right) \left(\int_a^b (b-t)^{(\alpha-1)p} dt \right)^{\frac{1}{p}} \cdot$$

$$\left(\int_a^b |P_1(x,t)|^q dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} = \tag{100}$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \right) \frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left(\int_a^x (t-a)^q dt + \int_x^b (b-t)^q dt \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} =$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \right) \frac{(b-a)^{(\alpha-2)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\frac{(x-a)^{q+1}}{(q+1)} + \frac{(b-t)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])} =$$

$$\left(\frac{(b-x)^{n-\alpha}}{\prod_{j=1}^{n-1}(\alpha-j)} \right) \frac{(b-a)^{\alpha-2+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\{ \frac{(b-x)^{q+1}+(x-a)^{q+1}}{(q+1)} \right\}^{\frac{1}{q}} \left\| f^{(n)} \right\|_{L_r([a,b])}, \tag{101}$$

proving the claim. ■

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