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# Multivariate Fractional Representation Formula and Ostrowski type inequality

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**Abstract**

Here we derive a multivariate fractional representation formula involving ordinary partial derivatives of first order. Then we prove a related multivariate fractional Ostrowski type inequality with respect to uniform norm.

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**Keywords and Phrases:** Multivariate Fractional integral, Representation formula, Multivariate Ostrowski inequality.

## 1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, then the following Montgomery identity holds [3]:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt + \int_a^b P_1(x,t)\, f'(t)\, dt, \tag{1}$$

where $P_1(x,t)$ is the Peano kernel

$$P_1(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a}, & x < t \le b, \end{cases} \tag{2}$$

The Riemann-Liouville integral operator of order $\alpha > 0$ with anchor point $a \in \mathbb{R}$ is defined by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$$J_a^0 f(x) := f(x), \quad x \in [a,b]. \tag{4}$$

Properties of the above operator can be found in [4].

When $\alpha = 1$, $J_a^1$ reduces to the classical integral.

In [1] we proved the following fractional representation formula of Montgomery identity type.

**Theorem 1** *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$, and $f' : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, $\alpha \geq 1$, $x \in [a,b)$. Then*

$$f(x) = (b-x)^{1-\alpha} \Gamma(\alpha) \left\{ \frac{J_a^{\alpha} f(b)}{b-a} - J_a^{\alpha-1} \left( P_1(x,b) f(b) \right) + J_a^{\alpha} \left( P_1(x,b) f'(b) \right) \right\}. \tag{5}$$

*When $\alpha = 1$ the last (5) reduces to classic Montgomery identity (1).*

We may rewrite (5) as follows

$$f(x) = (b-x)^{1-\alpha} \left[ \frac{1}{b-a} \int_a^b (b-t)^{\alpha-1} f(t)\, dt - \right.$$

$$\left. (\alpha-1) \int_a^b (b-t)^{\alpha-2} P_1(x,t) f(t)\, dt + \int_a^b (b-t)^{\alpha-1} P_1(x,t) f'(t)\, dt \right]. \tag{6}$$

In this article based on (5), we establish a multivariate fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i, b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate fractional Ostrowski type inequality.

# 2 Main Results

We make

**Assumption 2** *Let $f \in C^1 \left( \prod_{i=1}^m [a_i, b_i] \right)$.*

**Assumption 3** *Let $f : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i, b_i] \to \mathbb{R}$, and it is $x_j-$integrable for all $j = 1, ..., m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1, ..., t_i, x_{i+1}, ..., x_m)$ it is integrable on $\prod_{j=1}^i [a_j, b_j]$, for all $i = 1, ..., m$, for any $(x_{i+1}, ..., x_m) \in \prod_{j=i+1}^m [a_j, b_j]$.*

**Convention 4** *We set*

$$\prod_{j=1}^{0} \cdot = 1. \tag{7}$$

**Notation 5** *Here $x = \overrightarrow{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. Likewise $t = \overrightarrow{t} = (t_1, ..., t_m)$, and $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. We denote the kernel*

$$P_1(x_i, t_i) = \begin{cases} \frac{t_i - a_i}{b_i - a_i}, & a_i \leq t_i \leq x_i, \\ \frac{t_i - b_i}{b_i - a_i}, & x_i < t_i \leq b_i, \end{cases} \tag{8}$$

We need

**Definition 6** *(see [2]) Let* $\prod_{i=1}^{m}[a_i,b_i] \subset \mathbb{R}^m$*,* $m \in \mathbb{N}-\{1\}$*,* $a_i < b_i$*,* $a_i, b_i \in \mathbb{R}$*. Let* $\alpha > 0$*,* $f \in L_1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$*. We define the left mixed Riemann-Liouville fractional multiple integral of order* $\alpha$*:*

$$\left(I_{a+}^{\alpha}f\right)(x) := \frac{1}{(\Gamma(\alpha))^m}\int_{a_1}^{x_1}\dots\int_{a_m}^{x_m}\left(\prod_{i=1}^{m}(x_i - t_i)\right)^{\alpha-1} f(t_1,\dots,t_m)\,dt_1\dots dt_m, \tag{9}$$

*where* $x_i \in [a_i,b_i]$*,* $i = 1,\dots,m$*, and* $x = (x_1,\dots,x_m)$*,* $a = (a_1,\dots,a_m)$*,* $b = (b_1,\dots,b_m)$*.*

We present the following multivariate fractional representation formula

**Theorem 7** *Let* $f$ *as in Assumption 2 or Assumption 3,* $\alpha \geq 1$*,* $x_i \in [a_i,b_i)$*,* $i = 1,\dots,m$*. Then*

$$f(x_1,\dots,x_m) = \frac{\left(\prod_{i=1}^{m}(b_i - x_i)\right)^{1-\alpha}(\Gamma(\alpha))^m}{\prod_{i=1}^{m}(b_i - a_i)}\left(I_{a+}^{\alpha}f\right)(b) +$$

$$\sum_{i=1}^{m}A_i(x_1,\dots,x_m) + \sum_{i=1}^{m}B_i(x_1,\dots,x_m), \tag{10}$$

*where for* $i = 1,\dots,m$*:*

$$A_i(x_1,\dots,x_m) := \frac{-(\alpha-1)\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j - a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i-1}(b_j - t_j)\right)^{\alpha-1}\cdot \tag{11}$$

$$(b_i - t_i)^{\alpha-2}P_1(x_i,t_i)\,f(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1\dots dt_i,$$

*and*

$$B_i(x_1,\dots,x_m) := \frac{\left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}(b_j - a_j)}\int_{\prod_{j=1}^{i}[a_j,b_j]}\left(\prod_{j=1}^{i}(b_j - t_j)\right)^{\alpha-1}\cdot \tag{12}$$

$$P_1(x_i,t_i)\frac{\partial f}{\partial x_i}(t_1,\dots,t_i,x_{i+1},\dots,x_m)\,dt_1 dt_2\dots dt_i.$$

**Proof.** By (6) we have

$$f(x_1,\dots,x_m) = (b_1 - x_1)^{1-\alpha}\left[\frac{1}{b_1 - a_1}\int_{a_1}^{b_1}(b_1 - t_1)^{\alpha-1}f(t_1,x_2,\dots,x_m)\,dt_1\right.$$

$$-(\alpha-1)\int_{a_1}^{b_1}(b_1 - t_1)^{\alpha-2}P_1(x_1,t_1)\,f(t_1,x_2,\dots,x_m)\,dt_1$$

$$+\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1\Bigg], \tag{13}$$

and

$$f(t_1,x_2,...,x_m)=(b_2-x_2)^{1-\alpha}\Bigg[\frac{1}{b_2-a_2}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$-(\alpha-1)\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-2}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$+\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2\Bigg]. \tag{14}$$

We plug in (14) into (13). Hence

$$f(x_1,...,x_m)=(b_1-x_1)^{1-\alpha}\Bigg[\frac{1}{b_1-a_1}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}(b_2-x_2)^{1-\alpha}\cdot$$

$$\Bigg[\frac{1}{b_2-a_2}\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$-(\alpha-1)\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-2}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_2 \tag{15}$$

$$+\int_{a_2}^{b_2}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2\Bigg]dt_1$$

$$-(\alpha-1)\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1$$

$$+\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1\Bigg].$$

That is we have so far

$$f(x_1,...,x_m)=\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2- \tag{16}$$

$$\frac{(\alpha-1)(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2$$

$$+\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2$$

$$-(\alpha-1)(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1$$

$$+(b_1-x_1)^{1-\alpha}\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1.$$

Call

$$A_1(x_1,...,x_m):=-(\alpha-1)(b_1-x_1)^{1-\alpha}\cdot$$

$$\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-2}P_1(x_1,t_1)f(t_1,x_2,...,x_m)\,dt_1, \tag{17}$$

$$B_1(x_1,...,x_m):=(b_1-x_1)^{1-\alpha}\cdot$$

$$\int_{a_1}^{b_1}(b_1-t_1)^{\alpha-1}P_1(x_1,t_1)\frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1, \tag{18}$$

$$A_2(x_1,x_2,...,x_m):=-\frac{(\alpha-1)(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot \tag{19}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2,$$

and

$$B_2(x_1,x_2,...,x_m):=\frac{(b_1-x_1)^{1-\alpha}(b_2-x_2)^{1-\alpha}}{(b_1-a_1)}\cdot \tag{20}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-1}P_1(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2.$$

We rewrite (16) as follows

$$f(x_1,...,x_m)=\frac{((b_1-x_1)(b_2-x_2))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}((b_1-t_1)(b_2-t_2))^{\alpha-1}f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2+ \tag{21}$$

$$A_2(x_1,...,x_m)+B_2(x_1,...,x_m)+A_1(x_1,...,x_m)+B_1(x_1,...,x_m).$$

We continue with

$$f(t_1,t_2,x_3,...,x_m)\overset{(6)}{=}\frac{(b_3-x_3)^{1-\alpha}}{b_3-a_3}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3$$

$$-(\alpha-1)(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)\,f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3$$
$$+(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}P_1(x_3,t_3)\,\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_3. \tag{22}$$

Next plug (22) into (21). Hence it holds

$$f(x_1,...,x_m)=\frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)(b_3-a_3)}\cdot$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1}f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3$$
$$-\frac{(\alpha-1)((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2))^{\alpha-1}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)\cdot$$
$$f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3$$
$$+\frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot \tag{23}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1}P_1(x_3,t_3)\cdot$$
$$\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3$$
$$+A_1(x_1,...,x_m)+A_2(x_1,...,x_m)+B_1(x_1,...,x_m)+B_2(x_1,...,x_m).$$

Call

$$A_3(x_1,...,x_m):=-\frac{(\alpha-1)((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot \tag{24}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2))^{\alpha-1}(b_3-t_3)^{\alpha-2}P_1(x_3,t_3)\cdot$$
$$f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3,$$

and

$$B_3(x_1,...,x_m):=\frac{((b_1-x_1)(b_2-x_2)(b_3-x_3))^{1-\alpha}}{(b_1-a_1)(b_2-a_2)}\cdot \tag{25}$$
$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}((b_1-t_1)(b_2-t_2)(b_3-t_3))^{\alpha-1}P_1(x_3,t_3)\cdot$$
$$\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3.$$

Thus we have proved

$$f\left(x_{1},...,x_{m}\right)=\frac{\left(\prod_{i=1}^{3}\left(b_{i}-x_{i}\right)\right)^{1-\alpha}}{\prod_{i=1}^{3}\left(b_{i}-a_{i}\right)}\cdot$$

$$\int_{\prod_{i=1}^{3}[a_{i},b_{i}]}\left(\prod_{i=1}^{3}\left(b_{i}-t_{i}\right)\right)^{\alpha-1}f\left(t_{1},t_{2},t_{3},x_{4},...,x_{m}\right)dt_{1}dt_{2}dt_{3}+$$

$$\sum_{i=1}^{3}A_{i}\left(x_{1},...,x_{m}\right)+\sum_{i=1}^{3}B_{i}\left(x_{1},...,x_{m}\right). \tag{26}$$

Working similarly we finally obtain the fractional representation formula

$$f\left(x_{1},...,x_{m}\right)=\frac{\left(\prod_{i=1}^{m}\left(b_{i}-x_{i}\right)\right)^{1-\alpha}}{\prod_{i=1}^{m}\left(b_{i}-a_{i}\right)}\int_{\prod_{i=1}^{m}[a_{i},b_{i}]}\left(\prod_{i=1}^{m}\left(b_{i}-t_{i}\right)\right)^{\alpha-1}f\left(\overrightarrow{t}\right)d\overrightarrow{t}$$

$$+\sum_{i=1}^{m}A_{i}\left(x_{1},...,x_{m}\right)+\sum_{i=1}^{m}B_{i}\left(x_{1},...,x_{m}\right). \tag{27}$$

The proof of the theorem is now completed. ■

We make

**Remark 8** *Let* $f\in C^{1}\left(\prod_{i=1}^{m}\left[a_{i},b_{i}\right]\right)$, $\alpha\geq1$, $x_{i}\in\left[a_{i},b_{i}\right)$, $i=1,...,m$. *Denote by*

$$\|f\|_{\infty}^{\sup}:=\sup_{x\in\prod_{i=1}^{m}[a_{i},b_{i}]}\left|f\left(x\right)\right|. \tag{28}$$

*We observe that*

$$\left|B_{i}\left(x_{1},...,x_{m}\right)\right|\overset{(12)}{\leq}\frac{\left(\prod_{j=1}^{i}\left(b_{j}-x_{j}\right)\right)^{1-\alpha}}{\prod_{j=1}^{i-1}\left(b_{j}-a_{j}\right)}\left(\int_{\prod_{j=1}^{i}[a_{j},b_{j}]}\left(\prod_{j=1}^{i}\left(b_{j}-t_{j}\right)\right)^{\alpha-1}\cdot\right. \tag{29}$$

$$\left.\left|P_{1}\left(x_{i},t_{i}\right)\right|dt_{1}...dt_{i}\right)\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}^{\sup}=$$

$$\frac{\left(\prod_{j=1}^{i}\left(b_{j}-x_{j}\right)\right)^{1-\alpha}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}^{\sup}}{\prod_{j=1}^{i-1}\left(b_{j}-a_{j}\right)}\left(\prod_{j=1}^{i-1}\int_{a_{j}}^{b_{j}}\left(b_{j}-t_{j}\right)^{\alpha-1}dt_{j}\right)\cdot$$

$$\left(\int_{a_{i}}^{b_{i}}\left(b_{i}-t_{i}\right)^{\alpha-1}\left|P_{1}\left(x_{i},t_{i}\right)\right|dt_{i}\right)= \tag{30}$$

$$\frac{\left(\prod_{j=1}^{i}\left(b_{j}-x_{j}\right)\right)^{1-\alpha}\left\|\frac{\partial f}{\partial x_{i}}\right\|_{\infty}^{\sup}}{\left(b_{i}-a_{i}\right)\prod_{j=1}^{i-1}\left(b_{j}-a_{j}\right)}\frac{\left(\prod_{j=1}^{i-1}\left(b_{j}-a_{j}\right)\right)^{\alpha}}{\alpha^{i-1}}.$$

$$\left[\int_{a_i}^{x_i}(b_i-t_i)^{\alpha-1}(t_i-a_i)\,dt_i+\int_{x_i}^{b_i}(b_i-t_i)^{\alpha-1}(b_i-t_i)\,dt_i\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}\cdot \tag{31}$$

$$\left[\int_{a_i}^{x_i}(b_i-t_i)^{\alpha-1}\left[(b_i-a_i)-(b_i-t_i)\right]dt_i+\frac{(b_i-x_i)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}\cdot \tag{32}$$

$$\left[(b_i-a_i)\left[\frac{(b_i-a_i)^{\alpha}}{\alpha}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]-\frac{(b_i-a_i)^{\alpha+1}}{\alpha+1}+\frac{2(b_i-x_i)^{\alpha+1}}{\alpha+1}\right]=$$

$$\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}(b_i-a_i)}\cdot$$

$$\left[\frac{(b_i-a_i)^{\alpha+1}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{\alpha+1}-(b_i-a_i)\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]. \tag{33}$$

*We have proved for $i=1,...,m$, that*

$$|B_i(x_1,...,x_m)|\leq\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}}\cdot$$

$$\left[\frac{(b_i-a_i)^{\alpha}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{(\alpha+1)(b_i-a_i)}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]. \tag{34}$$

We have established the following multivariate fractional Ostrowski type inequality.

**Theorem 9** *Let $f\in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha\geq 1$, $x_i\in[a_i,b_i)$, $i=1,...,m$. Then*

$$\left|f(x_1,...,x_m)-\frac{\left(\prod_{i=1}^{m}(b_i-x_i)\right)^{1-\alpha}(\Gamma(\alpha))^m\left(I_{a+}^{\alpha}f\right)(b)}{\prod_{i=1}^{m}(b_i-a_i)}-\sum_{i=1}^{m}A_i(x_1,...,x_m)\right|$$

$$\leq\sum_{i=1}^{m}\left\{\frac{\left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}\left(\prod_{j=1}^{i}(b_j-x_j)\right)^{1-\alpha}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\alpha-1}}{\alpha^{i-1}}\cdot\right.$$

$$\left.\left[\frac{(b_i-a_i)^{\alpha}}{\alpha(\alpha+1)}+\frac{2(b_i-x_i)^{\alpha+1}}{(\alpha+1)(b_i-a_i)}-\frac{(b_i-x_i)^{\alpha}}{\alpha}\right]\right\}. \tag{35}$$

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