Received 18/11/13

Multivariate weighted Fractional Representation Formulae and Ostrowski type inequalities

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we derive multivariate weighted fractional representation formulae involving ordinary partial derivatives of first order. Then we present related multivariate weighted fractional Ostrowski type inequalities with respect to uniform norm.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Multivariate Fractional integral, weighted Representation formula, Multivariate weighted Ostrowski inequality.

1 Introduction

Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$, and $f' : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Suppose now that $w : [a, b] \to [0, \infty)$ is some probability density function, i.e. it is a nonnegative integrable function satisfying $\int_a^b w(t)\,dt = 1$, and $W(t) = \int_a^t w(x)\,dx$ for $t \in [a, b]$, $W(t) = 0$ for $t \leq a$ and $W(t) = 1$ for $t \geq b$. Then, the following identity (Pecarić, [5]) is the weighted generalization of the Montgomery identity ([4])

$$f(x) = \int_a^b w(t) f(t)\,dt + \int_a^b P_w(x, t) f'(t)\,dt, \tag{1}$$

where the weighted Peano Kernel is

$$P_w(x, t) := \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases} \tag{2}$$

In [1] we proved

Theorem 1 *Let $w : [a,b] \to [0,\infty)$ be a probability density function, i.e. $\int_a^b w(t)\,dt = 1$, and set $W(t) = \int_a^t w(x)\,dx$ for $a \le t \le b$, $W(t) = 0$ for $t \le a$ and $W(t) = 1$ for $t \ge b$, $\alpha \ge 1$, and f is as in (1). Then the generalization of the weighted Montgomery identity for fractional integrals is the following*

$$f(x) = (b-x)^{1-\alpha}\,\Gamma(\alpha)\,J_a^{\alpha}(w(b)\,f(b)) -$$

$$J_a^{\alpha-1}(Q_w(x,b)\,f(b)) + J_a^{\alpha}(Q_w(x,b)\,f'(b)), \tag{3}$$

where the weighted fractional Peano Kernel is

$$Q_w(x,t) := \begin{cases} (b-x)^{1-\alpha}\,\Gamma(\alpha)\,W(t), & a \le t \le x, \\ (b-x)^{1-\alpha}\,\Gamma(\alpha)\,(W(t)-1), & x < t \le b, \end{cases} \tag{4}$$

i.e. $Q_w(x,t) = (b-x)^{1-\alpha}\,\Gamma(\alpha)\,P_w(x,t)$.

When $\alpha = 1$ then the weighted generalization of the Montgomery identity for fractional integrals in (3) reduces to the weighted generalization of the Montgomery identity for integrals in (1).

So for $\alpha \ge 1$ and $x \in [a,b)$ we can rewrite (3) as follows

$$f(x) = (b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1}\,w(t)\,f(t)\,dt -$$

$$(b-x)^{1-\alpha}(\alpha-1)\int_a^b (b-t)^{\alpha-2}\,P_w(x,t)\,f(t)\,dt + \tag{5}$$

$$(b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1}\,P_w(x,t)\,f'(t)\,dt.$$

In this article based on (5), we establish a multivariate weighted general fractional representation formula for $f(x)$, $x \in \prod_{i=1}^m [a_i,b_i] \subset \mathbb{R}^m$, and from there we derive an interesting multivariate weighted fractional Ostrowski type inequality. We finish with an application.

2 Main Results

We make

Assumption 2 *Let $f \in C^1\left(\prod_{i=1}^m [a_i,b_i]\right)$.*

Assumption 3 *Let $f : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$ be measurable and bounded, such that there exist $\frac{\partial f}{\partial x_j} : \prod_{i=1}^m [a_i,b_i] \to \mathbb{R}$, and it is x_j-integrable for all $j = 1,...,m$. Furthermore $\frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)$ it is integrable on $\prod_{j=1}^i [a_j,b_j]$, for all $i = 1,...,m$, for any $(x_{i+1},...,x_m) \in \prod_{j=i+1}^m [a_j,b_j]$.*

Convention 4 *We set*

$$\prod_{j=1}^{0} \cdot = 1. \tag{6}$$

Notation 5 *Here* $x = \overrightarrow{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$. *Likewise* $t = \overrightarrow{t} = (t_1, ..., t_m)$, *and* $d\overrightarrow{t} = dt_1 dt_2 ... dt_m$. *Here* w_i, W_i *correspond to* $[a_i, b_i]$, $i = 1, ..., m$, *and are as* w, W *of Theorem 1.*

We need

Definition 6 *(see [2] and [3]) Let* $\prod_{i=1}^{m} [a_i, b_i] \subset \mathbb{R}^m$, $m \in \mathbb{N} - \{1\}$, $a_i < b_i$, $a_i, b_i \in \mathbb{R}$. *Let* $\alpha > 0$, $f \in L_1 \left(\prod_{i=1}^{m} [a_i, b_i] \right)$. *We define the left mixed Riemann-Liouville fractional multiple integral of order* α:

$$\left(I_{a+}^{\alpha} f \right)(x) := \frac{1}{(\Gamma(\alpha))^m} \int_{a_1}^{x_1} ... \int_{a_m}^{x_m} \left(\prod_{i=1}^{m} (x_i - t_i) \right)^{\alpha - 1} f(t_1, ..., t_m) \, dt_1 ... dt_m, \tag{7}$$

where $x_i \in [a_i, b_i]$, $i = 1, ..., m$, *and* $x = (x_1, ..., x_m)$, $a = (a_1, ..., a_m)$, $b = (b_1, ..., b_m)$.

We present the following multivariate weighted fractional representation formula

Theorem 7 *Let* f *as in Assumption 2 or Assumption 3,* $\alpha \geq 1$, $x_i \in [a_i, b_i)$, $i = 1, ..., m$. *Here* P_{w_i} *corresponds to* $[a_i, b_i]$, $i = 1, ..., m$, *and it is as in (2). The probability density function* w_j *is assumed to be bounded for all* $j = 1, ..., m$. *Then*

$$f(x_1, ..., x_m) = \left(\prod_{j=1}^{m} (b_j - x_j) \right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^{\alpha} \left(\prod_{j=1}^{m} w_j \right) f \right)(b) +$$

$$\sum_{i=1}^{m} A_i(x_1, ..., x_m) + \sum_{i=1}^{m} B_i(x_1, ..., x_m), \tag{8}$$

where for $i = 1, ..., m$:

$$A_i(x_1, ..., x_m) := -(\alpha - 1) \left(\prod_{j=1}^{i} (b_j - x_j) \right)^{1-\alpha} \int_{\prod_{j=1}^{i} [a_j, b_j]} \left(\prod_{j=1}^{i-1} (b_j - t_j) \right)^{\alpha - 1} \cdot \tag{9}$$

$$(b_i - t_i)^{\alpha - 2} \left(\prod_{j=1}^{i-1} w_j(t_j) \right) P_{w_i}(x_i, t_i) f(t_1, ..., t_i, x_{i+1}, ..., x_m) \, dt_1 ... dt_i,$$

and

$$B_i(x_1,...,x_m) := \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \int_{\prod_{j=1}^{i}[a_j,b_j]} \left(\prod_{j=1}^{i}(b_j - t_j)\right)^{\alpha-1} \cdot \tag{10}$$

$$\left(\prod_{j=1}^{i-1} w_j(t_j)\right) P_{w_i}(x_i,t_i) \frac{\partial f}{\partial x_i}(t_1,...,t_i,x_{i+1},...,x_m)\,dt_1...dt_i.$$

Proof. We have that

$$f(x_1,x_2,...,x_m) \overset{(5)}{=} (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} w_1(t_1) f(t_1,x_2,...,x_m)\,dt_1 +$$

$$A_1(x_1,...,x_m) + B_1(x_1,...,x_m), \tag{11}$$

where

$$A_1(x_1,...,x_m) := -(\alpha - 1)(b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-2} \cdot \tag{12}$$

$$P_{w_1}(x_1,t_1) f(t_1,x_2,...,x_m)\,dt_1,$$

and

$$B_1(x_1,...,x_m) := (b_1 - x_1)^{1-\alpha} \int_{a_1}^{b_1} (b_1 - t_1)^{\alpha-1} \cdot \tag{13}$$

$$P_{w_1}(x_1,t_1) \frac{\partial f}{\partial x_1}(t_1,x_2,...,x_m)\,dt_1.$$

Similarly it holds

$$f(t_1,x_2,...,x_m) \overset{(5)}{=} (b_2 - x_2)^{1-\alpha} \int_{a2}^{b_2} (b_2 - t_2)^{\alpha-1} w_2(t_2) f(t_1,t_2,x_3,...,x_m)\,dt_2$$

$$-(\alpha - 1)(b_2 - x_2)^{1-\alpha} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-2} P_{w_2}(x_2,t_2) f(t_1,t_2,x_3,...,x_m)\,dt_2 +$$

$$(b_2 - x_2)^{1-\alpha} \int_{a_2}^{b_2} (b_2 - t_2)^{\alpha-1} P_{w_2}(x_2,t_2) \frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_2. \tag{14}$$

Next we plug (14) into (11).

We get

$$f(x_1,...,x_m) = ((b_1 - x_1)(b_2 - x_2))^{1-\alpha} \cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} ((b_1 - t_1)(b_2 - t_2))^{\alpha-1} w_1(t_1) w_2(t_2) f(t_1,t_2,x_3,...,x_m)\,dt_1 dt_2 +$$

$$A_2(x_1,...,x_m) + B_2(x_1,...,x_m) + A_1(x_1,...,x_m) + B_1(x_1,...,x_m), \tag{15}$$

where

$$A_2(x_1,...,x_m) := -(\alpha-1)\left((b_1-x_1)(b_2-x_2)\right)^{1-\alpha}\cdot \tag{16}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}(b_1-t_1)^{\alpha-1}(b_2-t_2)^{\alpha-2}w_1(t_1)P_{w_2}(x_2,t_2)f(t_1,t_2,x_3,...,x_m)\,dt_1dt_2,$$

and

$$B_2(x_1,...,x_m) := \left((b_1-x_1)(b_2-x_2)\right)^{1-\alpha}\cdot \tag{17}$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left((b_1-t_1)(b_2-t_2)\right)^{\alpha-1}w_1(t_1)P_{w_2}(x_2,t_2)\frac{\partial f}{\partial x_2}(t_1,t_2,x_3,...,x_m)\,dt_1dt_2.$$

We continue as above.

We also have

$$\begin{aligned}
f(t_1,t_2,x_3,...,x_m) &\overset{(5)}{=} (b_3-x_3)^{1-\alpha}\cdot \\
&\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}w_3(t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3 \\
&-(\alpha-1)(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-2}P_{w_3}(x_3,t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_3 \\
&+(b_3-x_3)^{1-\alpha}\int_{a_3}^{b_3}(b_3-t_3)^{\alpha-1}P_{w_3}(x_3,t_3)\frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\,dt_3.
\end{aligned} \tag{18}$$

We plug (18) into (15). Therefore it holds

$$\begin{aligned}
f(x_1,...,x_m) &= \left(\prod_{j=1}^{3}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{3}(b_j-t_j)\right)^{\alpha-1}\cdot \\
&\left(\prod_{j=1}^{3}w_j(t_j)\right)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3+ \\
&\sum_{j=1}^{3}A_j(x_1,...,x_m)+\sum_{j=1}^{3}B_j(x_1,...,x_m),
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
A_3(x_1,...,x_m) &:= -(\alpha-1)\left(\prod_{j=1}^{3}(b_j-x_j)\right)^{1-\alpha}\int_{\prod_{j=1}^{3}[a_j,b_j]}\left(\prod_{j=1}^{2}(b_j-t_j)\right)^{\alpha-1}\cdot \\
&(b_3-t_3)^{\alpha-2}\left(\prod_{j=1}^{2}w_j(t_j)\right)P_{w_3}(x_3,t_3)f(t_1,t_2,t_3,x_4,...,x_m)\,dt_1dt_2dt_3,
\end{aligned} \tag{20}$$

and

$$B_3(x_1,...,x_m) := \left(\prod_{j=1}^{3}(b_j - x_j)\right)^{1-\alpha} \int_{\prod_{j=1}^{3}[a_j,b_j]} \left(\prod_{j=1}^{3}(b_j - t_j)\right)^{\alpha-1} \cdot \tag{21}$$

$$\left(\prod_{j=1}^{2} w_j(t_j)\right) P_{w_3}(x_3,t_3) \frac{\partial f}{\partial x_3}(t_1,t_2,t_3,x_4,...,x_m)\, dt_1 dt_2 dt_3.$$

Continuing similarly we finally obtain

$$f(x_1,...,x_m) = \left(\prod_{j=1}^{m}(b_j - x_j)\right)^{1-\alpha} \cdot$$

$$\int_{\prod_{j=1}^{m}[a_j,b_j]} \left(\prod_{j=1}^{m}(b_j - t_j)\right)^{\alpha-1} \left(\prod_{j=1}^{m} w_j(t_j)\right) f\left(\overrightarrow{t}\right) d\overrightarrow{t}$$

$$+\sum_{i=1}^{m} A_i(x_1,...,x_m) + \sum_{i=1}^{m} B_i(x_1,...,x_m), \tag{22}$$

that is proving the claim. ■

We make

Remark 8 *Let* $f \in C^1\left(\prod_{i=1}^{m}[a_i,b_i]\right)$, $\alpha \geq 1$, $x_i \in [a_i,b_i)$, $i = 1,...,m$. *Denote by*

$$\|f\|_{\infty}^{\sup} := \sup_{x \in \prod_{i=1}^{m}[a_i,b_i]} |f(x)|. \tag{23}$$

From (2) we get that

$$|P_w(x,t)| \leq \begin{Bmatrix} W(x), & a \leq t \leq x, \\ 1 - W(x), & x < t \leq b \end{Bmatrix}$$

$$\leq \max\{W(x), 1 - W(x)\} = \frac{1 + |2W(x) - 1|}{2}. \tag{24}$$

That is

$$|P_w(x,t)| \leq \frac{1 + |2W(x) - 1|}{2}, \tag{25}$$

for all $t \in [a,b]$, *where* $x \in [a,b]$ *is fixed.*

Consequently it holds

$$|P_{w_i}(x_i,t_i)| \leq \frac{1 + |2W_i(x_i) - 1|}{2}, \quad i = 1,...,m. \tag{26}$$

Assume here that

$$w_j(t_j) \leq K_j, \tag{27}$$

for all $t_j \in [a_j, b_j]$*, where* $K_j > 0$*,* $j = 1, ..., m$*.*

Therefore we derive

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\prod_{j=1}^{i-1} K_j\right) \cdot$$

$$\left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup} \prod_{j=1}^{i} \left(\int_{a_j}^{b_j} (b_j - t_j)^{\alpha-1} dt_j\right). \tag{28}$$

That is

$$|B_i(x_1, ..., x_m)| \leq \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^{i}(b_j - a_j)^{\alpha}}{\alpha^i}\right) \left(\prod_{j=1}^{i-1} K_j\right) \cdot$$

$$\left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup}, \tag{29}$$

for all $i = 1, ..., m$*.*

Based on the above and Theorem 7 we have established the following multivariate weighted fractional Ostrowski type inequality.

Theorem 9 *Let* $f \in C^1\left(\prod_{i=1}^{m}[a_i, b_i]\right)$*,* $\alpha \geq 1$*,* $x_i \in [a_i, b_i)$*,* $i = 1, ..., m$*. Here* P_{w_i} *corresponds to* $[a_i, b_i]$*,* $i = 1, ..., m$*, and it is as in (2). Assume that* $w_j(t_j) \leq K_j$*, for all* $t_j \in [a_j, b_j]$*, where* $K_j > 0$*,* $j = 1, ..., m$*. And* $A_i(x_1, ..., x_m)$ *is as in (9),* $i = 1, ..., m$*. Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^{m}(b_j - x_j)\right)^{1-\alpha} (\Gamma(\alpha))^m \left(I_{a+}^{\alpha}\left(\prod_{j=1}^{m} w_j\right) f\right)(b) \right.$$

$$\left. - \sum_{i=1}^{m} A_i(x_1, ..., x_m) \right| \leq \sum_{i=1}^{m} \left\{ \left(\prod_{j=1}^{i}(b_j - x_j)\right)^{1-\alpha} \left(\frac{\prod_{j=1}^{i}(b_j - a_j)^{\alpha}}{\alpha^i}\right) \cdot \right.$$

$$\left. \left(\prod_{j=1}^{i-1} K_j\right) \left(\frac{1 + |2W_i(x_i) - 1|}{2}\right) \left\|\frac{\partial f}{\partial x_i}\right\|_{\infty}^{\sup} \right\}. \tag{30}$$

3 Application

Here we operate on $[0, 1]^m$, $m \in \mathbb{N} - \{1\}$. We notice that

$$\int_0^1 \left(\frac{e^{-x}}{1 - e^{-1}}\right) dx = 1, \tag{31}$$

and

$$\frac{e^{-x}}{1-e^{-1}} \leq \frac{1}{1-e^{-1}}, \text{ for all } x \in [0,1]. \tag{32}$$

So here we choose as w_j the probability density function

$$w_j^*(t_j) := \frac{e^{-t_j}}{1-e^{-1}}, \tag{33}$$

$j = 1, ..., m$, $t_j \in [0,1]$.

So we have the corrsponding W_j as

$$W_j^*(t_j) = \frac{1-e^{-t_j}}{1-e^{-1}}, \quad t_j \in [0,1], \tag{34}$$

and the corresponding P_{w_j} as

$$P_{w_j}^*(x_j, t_j) = \begin{cases} \frac{1-e^{-t_j}}{1-e^{-1}}, & 0 \leq t_j \leq x_j, \\ \frac{e^{-1}-e^{-t_j}}{1-e^{-1}}, & x_j < t_j \leq 1, \end{cases} \tag{35}$$

$j = 1, ..., m$.

Set $\overrightarrow{0} = (0, ..., 0)$ and $\overrightarrow{1} = (1, ..., 1)$.

First we apply Theorem 7.

We have

Theorem 10 *Let* $f \in C^1([0,1]^m)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. *Then*

$$f(x_1, ..., x_m) = \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha} \left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m \left(I_{\overrightarrow{0}+}^{\alpha}\left(e^{-\sum_{j=1}^{m} t_j} f(\cdot)\right)\right)\left(\overrightarrow{1}\right)$$

$$+\sum_{i=1}^{m} A_i^*(x_1, ..., x_m) + \sum_{i=1}^{m} B_i^*(x_1, ..., x_m), \tag{36}$$

where for $i = 1, ..., m$:

$$A_i^*(x_1, ..., x_m) := \frac{-(\alpha-1)}{(1-e^{-1})^{i-1}} \left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha} \int_{[0,1]^i} \left(\prod_{j=1}^{i-1}(1-t_j)\right)^{\alpha-1} \cdot$$

$$(1-t_i)^{\alpha-2} e^{-\sum_{j=1}^{i-1} t_j} P_{w_i}^*(x_i, t_i) f(t_1, ..., t_i, x_{i+1}, ..., x_m) dt_1 ... dt_i, \tag{37}$$

and

$$B_i^*(x_1, ..., x_m) := \frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{(1-e^{-1})^{i-1}} \int_{[0,1]^i} \left(\prod_{j=1}^{i}(1-t_j)\right)^{\alpha-1} \cdot \tag{38}$$

$$e^{-\sum_{j=1}^{i-1} t_j} P_{w_i}^*(x_i, t_i) \frac{\partial f}{\partial x_i}(t_1, ..., t_i, x_{i+1}, ..., x_m) dt_1 ... dt_i.$$

Above we set $\sum_{i=1}^{0} \cdot = 0$.

Finally we apply Theorem 9.

Theorem 11 *Let $f \in C^1\left([0,1]^m\right)$, $\alpha \geq 1$, $x_i \in [0,1)$, $i = 1, ..., m$. Here $P^*_{w_i}$ is as in (35) and $A^*_i(x_1, ..., x_m)$ as in (37), $i = 1, ..., m$. Then*

$$\left| f(x_1, ..., x_m) - \left(\prod_{j=1}^{m}(1-x_j)\right)^{1-\alpha}\left(\frac{\Gamma(\alpha)}{1-e^{-1}}\right)^m \left(I^{\alpha}_{\overrightarrow{0}+}\left(e^{-\sum_{j=1}^{m} t_j} f(\cdot)\right)\right)\left(\overrightarrow{1}\right) \right. \tag{39}$$

$$\left. - \sum_{i=1}^{m} A^*_i(x_1, ..., x_m) \right| \leq \sum_{i=1}^{m}\left\{ \frac{\left(\prod_{j=1}^{i}(1-x_j)\right)^{1-\alpha}}{\alpha^i\left(1-e^{-1}\right)^{i-1}} \cdot \right.$$

$$\left. \left(\frac{1+\left|2W^*_i(x_i)-1\right|}{2}\right)\left\|\frac{\partial f}{\partial x_i}\right\|^{\sup}_{\infty} \right\}.$$

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