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Basic and s-convexity Ostrowski and Grüss type inequalities involving several functions

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Abstract

Using the harmonic polynomials representation formula due to Dedic, Pečaric and Ujevic [5], we establish Ostrowski and Grüss type inequalities involving several functions. The estimates are with respect all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, and also take into account the s-convexity and s-concavity in the second sense of the involved functions.

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1 Introduction

The following results motivate our work.

Theorem 1 *(1938, Ostrowski [11]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Theorem 2 *(1935, Grüss [9]) Let f, g be integrable functions from $[a,b]$ into $\mathbb{R}$, that satisfy the conditions*

$$m \leq f(x) \leq M, \quad n \leq g(x) \leq N, \quad x \in [a,b],$$

where $m, M, n, N \in \mathbb{R}$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$

$$\leq \frac{1}{4} (M-m)(N-n). \quad (2)$$

Theorem 3 *(1998, Dragomir and Wang [7]) Let* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous function with* $f' \in L_p([a,b])$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $x \in [a,b]$. *Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq$$

$$\frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p. \quad (3)$$

Ostrowski type inequalities are very useful in Numerical Analysis.

Theorem 4 *(1882, Čebyšev [3]) Let* $f, g : [a,b] \to \mathbb{R}$ *absolutely continuous functions with* $f', g' \in L_\infty([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \quad (4)$$

Above is also assumed that the involved integrals exist.

Grüss type inequalities are very useful in Probability.

We are also biggly inspired by the great work of B.G. Pachpatte, see [12], [13], [14].

So here we produce Ostrowski and Grüss type inequalities for several functions, acting to all possible directions, including s-convexity and s-concavity in the second sense complete study. Our results are univariate.

2 Background

Let $(P_n)_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geq 1$, $P_0 = 1$. Furthermore, let $[a,b] \subset \mathbb{R}$, $a \neq b$, and $h : [a,b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some fixed $n \geq 1$. We use the notation

$$L_n[h(x)] = \frac{1}{n} \left[h(x) + \sum_{k=1}^{n-1} (-1)^k P_k(x) h^{(k)}(x) + \right.$$

$$\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(a) h^{(k-1)}(a) - P_k(b) h^{(k-1)}(b) \right] \Bigg],$$

$x \in [a, b]$, for convinience.

For $n = 1$ the above sums are defined to be zero, that is $L_1 [h(x)] = h(x)$.

Dedic, Pečaric and Ujevic, see [5], [4], established the following identity,

$$L_n [h(x)] - \frac{1}{b-a} \int_a^b h(t)\, dt = \frac{(-1)^{n+1}}{n(b-a)} \int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\, dt, \quad (6)$$

where

$$q(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b]. \quad (7)$$

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \geq 0$, the identity (6) reduces to the Fink identity in [8], (see also [5], p. 177).

3 Main Results

We present our first main result, a set of very general Ostrowski type inequalities involving several functions.

Theorem 5 *Let $n_j \in \mathbb{N}$, $j = 1, ..., r \in \mathbb{N} - \{1\}$, $n_1 \leq n_2 \leq ... \leq n_r$ and $f_j : [a,b] \to \mathbb{R}$ be such that $f_j^{(n_j-1)}$ is absolutely continuous function. Denote*

$$S_1(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[L_{n_j} [f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right], \quad (8)$$

$$S_2(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} L_{n_i} [f_i(x)] \right) \left[L_{n_j} [f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right], \quad (9)$$

$x \in [a,b]$. Then

1)

$$|S_1(f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\frac{1}{2} + \frac{|a+b-2x|}{2(b-a)} \right] \cdot$$
$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \left\| f_j^{(n_j)} \right\|_{1,[a,b]} \right] \right], \quad (10)$$

and

$$|S_2(f_1, ..., f_r)| \leq \frac{1}{n_1} \left[\frac{1}{2} + \frac{|a+b-2x|}{2(b-a)} \right] \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right], \tag{11}$$

2) let $p_{lj}>1:\sum_{lj=1}^{3}\frac{1}{p_{lj}}=1$*, with* $f_j^{(n_j)}\in L_{p_{3j}}\left([a,b]\right)$*,* $j=1,...,r$*, it holds*

$$\left|S_1\left(f_1,...,f_r\right)\right|\leq\frac{1}{n_1\left(b-a\right)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\left(\frac{\left(b-x\right)^{p_{2j}+1}+\left(x-a\right)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{12}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right|\leq\frac{1}{n_1\left(b-a\right)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]}\left(\frac{\left(b-x\right)^{p_{2j}+1}+\left(x-a\right)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{13}$$

3) assuming $f_j^{(n_j)}\in L_\infty\left([a,b]\right)$*,* $j=1,...,r$*, we get*

$$\left|S_1\left(f_1,...,f_r\right)\right|\leq\left(\frac{\left(b-x\right)^2+\left(x-a\right)^2}{2n_1\left(b-a\right)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right], \tag{14}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right|\leq\left(\frac{\left(b-x\right)^2+\left(x-a\right)^2}{2n_1\left(b-a\right)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right]. \tag{15}$$

Proof. For $j=1,...,r$, $r\in\mathbb{N}-\{1\}$, we have

$$L_{n_j}\left[f_j\left(x\right)\right]=\frac{1}{n_j}\left[f_j\left(x\right)+\sum_{k=1}^{n_j-1}\left(-1\right)^kP_k\left(x\right)f_j^{(k)}\left(x\right)+\right.$$

$$\sum_{k=1}^{n_j-1} \frac{(-1)^k (n_j - k)}{b-a} \left[P_k(a) f_j^{(k-1)}(a) - P_k(b) f_j^{(k-1)}(b) \right] \Bigg], \tag{16}$$

and

$$L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \overset{(6)}{=} \frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\, dt. \tag{17}$$

Hence it holds

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] = \tag{18}$$

$$\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\, dt \right], \quad \text{for all } j = 1, ..., r.$$

Therefore by addition of (18), we derive the identity

$$S_1(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right] \tag{19}$$

$$= \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\, dt \right] \right].$$

Similarly we produce the identity

$$S_2(f_1, ..., f_r) := \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} L_{n_i}[f_i(x)] \right) \left[L_{n_j}[f_j(x)] - \frac{1}{b-a} \int_a^b f_j(t)\, dt \right] \right] \tag{20}$$

$$= \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} L_{n_i}[f_i(x)] \right) \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\, dt \right] \right].$$

Consequently we have

$$|S_1(f_1, ..., f_r)| \leq$$

$$\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left[\frac{1}{n_j(b-a)} \int_a^b \left| P_{n_j-1}(t) \right| |q(x,t)| \left| f_j^{(n_j)}(t) \right| dt \right] \right], \tag{21}$$

and

$$|S_2(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{1}{n_j(b-a)}\int_a^b\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right]. \tag{22}$$

Furthermore it holds

$$|S_1(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right], \tag{23}$$

and

$$|S_2(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right]\right]. \tag{24}$$

Since

$$|q(x,t)| \leq \max(x-a,b-x) = \frac{(b-a)+|a+b-2x|}{2}, \tag{25}$$

we get

$$|S_1(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right], \tag{26}$$

and

$$|S_2(f_1,...,f_r)| \leq$$

$$\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j(b-a)}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]. \tag{27}$$

Let now $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$, with $f_j^{(n_j)} \in L_{p_{3j}}([a,b])$, $j = 1,...,r$.

Hence, by Hölder inequality for three functions, it holds

$$\int_a^b\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dt \leq$$

$$\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\left(\int_a^b|q(x,t)|^{p_{2j}}dt\right)^{\frac{1}{p_{2j}}}\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} =$$

$$\left\| P_{n_j-1} \right\|_{p_{1j},[a,b]} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \left\| f_j^{(n_j)} \right\|_{p_{3j},[a,b]}. \tag{28}$$

Consequently we derive

$$|S_1(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left[\frac{1}{n_j(b-a)} \left\| P_{n_j-1} \right\|_{p_{1j},[a,b]} \right. \right. \tag{29}$$

$$\left. \left. \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \left\| f_j^{(n_j)} \right\|_{p_{3j},[a,b]} \right] \right],$$

and

$$|S_2(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |L_{n_i}[f_i(x)]| \right) \left[\frac{1}{n_j(b-a)} \left\| P_{n_j-1} \right\|_{p_{1j},[a,b]} \right. \right. \tag{30}$$

$$\left. \left. \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \left\| f_j^{(n_j)} \right\|_{p_{3j},[a,b]} \right] \right].$$

Assuming that $f_j^{(n_j)} \in L_\infty([a,b])$, $j = 1, ..., r$, we find

$$|S_1(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left[\frac{1}{n_j(b-a)} \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \right. \right. \tag{31}$$

$$\left. \left. \left\| f_j^{(n_j)} \right\|_{\infty,[a,b]} \left(\frac{(b-x)^2 + (x-a)^2}{2} \right) \right] \right],$$

and

$$|S_2(f_1, ..., f_r)| \leq \sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |L_{n_i}[f_i(x)]| \right) \left[\frac{1}{n_j(b-a)} \left\| P_{n_j-1} \right\|_{\infty,[a,b]} \right. \right. \tag{32}$$

$$\left. \left. \left\| f_j^{(n_j)} \right\|_{\infty,[a,b]} \left(\frac{(b-x)^2 + (x-a)^2}{2} \right) \right] \right].$$

The proof of the theorem is now complete. ∎

We need

Definition 6 *([10]) A function $f : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1 - \lambda) y) \le \lambda^s f(x) + (1 - \lambda)^s f(y), \tag{33}$$

for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

When $s = 1$, s-convexity in the second sense reduces to ordinary convexity.
If "$\ge$" holds in (33), we talk about s-concavity in the second sense.
We also need

Definition 7 *(see also [1]) Let I be a subinterval of $\mathbb{R}_+$ and $f : I \to (0, \infty)$. We call f s-logarithmically convex (s-log-convex) in the second sense, iff logf(x) is s-convex in the second sense, iff*

$$f(\lambda x + (1 - \lambda) y) \le (f(x))^{\lambda^s} (f(y))^{(1-\lambda)^s}, \tag{34}$$

for all $x, y \in I$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

When $s = 1$, s-log-convexity in the second sense reduces to usual log-convexity.
If "$\ge$" holds in (34), we talk about s-log-concavity in the second sense.
We also need the s-convex Hadamard's inequality.

Theorem 8 *([6]) Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1([a, b])$, then*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a) + f(b)}{s+1}. \tag{35}$$

The constant $K = \frac{1}{s+1}$ is the best possible in the second inequality (35). The above inequalities are sharp.

Next we present general Ostrowski type inequalities for several functions under s-convexity and s-concavity in the second sense.

Theorem 9 *Same terms and assumptions as in Theorem 5. Assume that $a \ge 0$.*

1) Suppose $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}(x)\right| \le M_j$, $x \in [a, b]$, $j = 1, ..., r$. Then

$$|S_1(f_1, ..., f_r)| \le \frac{(b-a)}{n_1} \left[\frac{2}{s+2} \left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}} \right) - \right.$$

$$\left. \frac{1}{s+1} \left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}} \right) + \frac{2}{(s+1)(s+2)} \right]$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right], \tag{36}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right| \leq \frac{(b-a)}{n_1}\left[\frac{2}{s+2}\left(\frac{(b-x)^{s+2}+(x-a)^{s+2}}{(b-a)^{s+2}}\right)-\right.$$

$$\left.\frac{1}{s+1}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\right]$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right], \tag{37}$$

$x \in [a,b]$.

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$, *with* $f_j^{(n_j)} \in L_{p_{3j}}([a,b])$, $j = 1,...,r$.

2i) Assume again $\left|f_j^{(n_j)}\right|$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$\left|S_1\left(f_1,...,f_r\right)\right| \leq \frac{2}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{38}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right| \leq \frac{2}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right]. \tag{39}$$

2ii) Assume that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $j = 1,...,r$, $x \in [a,b]$. *Then*

$$\left|S_1\left(f_1,...,f_r\right)\right| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\frac{2^{\frac{1}{p_{3j}}} M_j (b-a)^{\frac{1}{p_{3j}}-1}}{(s+1)^{\frac{1}{p_{3j}}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \Bigg] \Bigg], \tag{40}$$

and

$$|S_2 (f_1, ..., f_r)| \le \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{r} |L_{n_i} [f_i (x)]| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right. \right.$$

$$\frac{2^{\frac{1}{p_{3j}}} M_j (b-a)^{\frac{1}{p_{3j}}-1}}{(s+1)^{\frac{1}{p_{3j}}}} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \Bigg] \Bigg]. \tag{41}$$

2iii) Assume that $\left| f_j^{(n_j)} \right|^{p_{3j}}$ *is s-concave in the second sense. Then*

$$|S_1 (f_1, ..., f_r)| \le \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{r} |f_i (x)| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right. \right.$$

$$2^{\frac{s-1}{p_{3j}}} \left| f_j^{(n_j)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_{3j}}-1} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \Bigg] \Bigg], \tag{42}$$

and

$$|S_2 (f_1, ..., f_r)| \le \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{r} |L_{n_i} [f_i (x)]| \right) \left\| P_{n_j - 1} \right\|_{p_{1j}, [a,b]} \right. \right.$$

$$2^{\frac{s-1}{p_{3j}}} \left| f_j^{(n_j)} \left(\frac{a+b}{2} \right) \right| (b-a)^{\frac{1}{p_{3j}}-1} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j}+1} \right)^{\frac{1}{p_{2j}}} \Bigg] \Bigg]. \tag{43}$$

Proof. As in (23) and (24) we have that

$$|S_1 (f_1, ..., f_r)| \le \frac{1}{n_1 (b-a)} \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \ne j}}^{r} |f_i (x)| \right) \left\| P_{n_j - 1} \right\|_{\infty, [a,b]} \int_a^b |q (x,t)| \left| f_j^{(n_j)} (t) \right| dt \right] \right], \tag{44}$$

and

$$|S_2 (f_1, ..., f_r)| \le \frac{1}{n_1 (b-a)} \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt\right]\right]. \tag{45}$$

Set

$$p\left(x,t\right):=\begin{cases} t, & t\in\left[0,\frac{b-x}{b-a}\right), \\ t-1, & t\in\left[\frac{b-x}{b-a},1\right]. \end{cases} \tag{46}$$

In [2], for $\lambda\in[0,1]$, we proved that

$$q\left(x,\lambda a+\left(1-\lambda\right)b\right)=\left(a-b\right)p\left(x,\lambda\right), \tag{47}$$

that is

$$\left|q\left(x,\lambda a+\left(1-\lambda\right)b\right)\right|=\left(b-a\right)\left|p\left(x,\lambda\right)\right|. \tag{48}$$

One can write

$$\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt= \tag{49}$$

$$\left(b-a\right)\int_0^1\left|q\left(x,\lambda a+\left(1-\lambda\right)b\right)\right|\left|f_j^{(n_j)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|d\lambda\overset{(48)}{=}$$

$$\left(b-a\right)^2\int_0^1\left|p\left(x,\lambda\right)\right|\left|f_j^{(n_j)}\left(\lambda a+\left(1-\lambda\right)b\right)\right|d\lambda=:\left(*\right). \tag{50}$$

We notice under the assumption that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}\left(x\right)\right|\leq M_j$, $x\in[a,b]$, that

$$\left(*\right)\leq\left(b-a\right)^2\int_0^1\left|p\left(x,t\right)\right|\left(\lambda^s\left|f_j^{(n_j)}\left(a\right)\right|+\left(1-\lambda\right)^s\left|f_j^{(n_j)}\left(b\right)\right|\right)d\lambda\leq \tag{51}$$

$$M_j\left(b-a\right)^2\int_0^1\left|p\left(x,\lambda\right)\right|\left(\lambda^s+\left(1-\lambda\right)^s\right)d\lambda=$$

(as in [2])

$$M_j\left(b-a\right)^2\left[\frac{2}{s+2}\left(\frac{\left(b-x\right)^{s+2}+\left(x-a\right)^{s+2}}{\left(b-a\right)^{s+2}}\right)-\right.$$

$$\left.\frac{1}{s+1}\left(\frac{\left(b-x\right)^{s+1}+\left(x-a\right)^{s+1}}{\left(b-a\right)^{s+1}}\right)+\frac{2}{\left(s+1\right)\left(s+2\right)}\right]. \tag{52}$$

So we got that

$$\int_a^b\left|q\left(x,t\right)\right|\left|f_j^{(n_j)}\left(t\right)\right|dt\leq M_j\left(b-a\right)^2\left[\frac{2}{s+2}\left(\frac{\left(b-x\right)^{s+2}+\left(x-a\right)^{s+2}}{\left(b-a\right)^{s+2}}\right)-\right.$$

$$\frac{1}{s+1}\left(\frac{(b-x)^{s+1}+(x-a)^{s+1}}{(b-a)^{s+1}}\right)+\frac{2}{(s+1)(s+2)}\Bigg], \quad j=1,...,r. \tag{53}$$

Using (53) into (44) and (45) we derive (36) and (37).

Next we elaborate on (12) and (13).

Assume that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, acting as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq 2M_j\left(\frac{b-a}{p_{3j}s+1}\right)^{\frac{1}{p_{3j}}}, \tag{54}$$

$j=1,...,r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x\in[a,b]$.

Next suppose that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ is s-convex in the second sense. As in [2] we get

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq \frac{2^{\frac{1}{p_{3j}}}M_j(b-a)^{\frac{1}{p_{3j}}}}{(s+1)^{\frac{1}{p_{3j}}}}, \tag{55}$$

$j=1,...,r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x\in[a,b]$.

Finally assume that $\left|f_j^{(n_j)}\right|^{p_{3j}}$ is s-concave in the second sense. Based on Theorem 8 and acting as in [2], we derive

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq 2^{\frac{s-1}{p_{3j}}}\left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right|(b-a)^{\frac{1}{p_{3j}}}. \tag{56}$$

The proof is completed. ■

Ostrowski type inequalities for several functions under s-log-convexity in the second sense follow.

Theorem 10 *Same terms and assumptions as in Theorem 5. Assume that* $a\geq 0$. *We further suppose that* $\left|f_j^{(n_j)}\right| \neq 0$ *is* s-log-*convex in the second sense, and* $\left|f_j^{(n_j)}(a)\right|, \left|f_j^{(n_j)}(b)\right| \in (0,1]$, $j=1,...,r$. *Call* $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $s\in(0,1]$, $j=1,...,r$, *and*

$$\psi_s(z) := \begin{cases} \frac{z^s-1}{s\ln z}, & \text{if } z\in(0,\infty)-\{1\}, \\ 1, & \text{if } z=1. \end{cases} \tag{57}$$

1) It holds

$$|S_1(f_1,...,f_r)| \leq \left[\frac{(b-a)+|a+b-2x|}{2n_1}\right]\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}|f_i(x)|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^s\psi_s(A_j)\right]\right], \tag{58}$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right| \leq \left[\frac{(b-a)+|a+b-2x|}{2n_1}\right] \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}\left(b\right)\right|^s\psi_s\left(A_j\right)\right]\right]. \quad (59)$$

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3}\frac{1}{p_{lj}} = 1$, *with* $f_j^{(n_j)} \in L_{p_{3j}}\left([a,b]\right)$, *and set* $B_j := A_j^{p_{3j}}$, $j = 1,...,r$. *Then*

$$\left|S_1\left(f_1,...,f_r\right)\right| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.(b-a)^{\frac{1}{p_{3j}}-1}\left|f_j^{(n_j)}\left(b\right)\right|^s\left(\psi_s\left(B_j\right)\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \quad (60)$$

and

$$\left|S_2\left(f_1,...,f_r\right)\right| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|L_{n_i}\left[f_i\left(x\right)\right]\right|\right)\left\|P_{n_j-1}\right\|_{p_{1j},[a,b]}\right.\right.$$

$$\left.\left.(b-a)^{\frac{1}{p_{3j}}-1}\left|f_j^{(n_j)}\left(b\right)\right|^s\left(\psi_s\left(B_j\right)\right)^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right]. \quad (61)$$

Proof. 1) See also [1]. Here we assume that $0 < \left|f_j^{(n_j)}\left(a\right)\right|, \left|f_j^{(n_j)}\left(b\right)\right| \leq 1$, set $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $\left|f_j^{(n_j)}\right|$ is s-logarithmically convex in the second sense, $s \in (0,1]$, $\lambda \in [0,1]$, $j = 1,...,r$. From (26) we get

$$\left|S_1\left(f_1,...,f_r\right)\right| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\left|f_i\left(x\right)\right|\right)\right.$$

$$\left.\left[\frac{\left\|P_{n_j-1}\right\|_{\infty,[a,b]}}{n_j}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\int_0^1\left|f_j^{(n_j)}\left(\lambda a+(1-\lambda)b\right)\right|d\lambda\right]\right], \quad (62)$$

and from (27) we obtain

$$|S_2(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|L_{n_i}[f_i(x)]|\right)\right.$$

$$\left.\left[\frac{\|P_{n_j-1}\|_{\infty,[a,b]}}{n_j}\left[\frac{(b-a)+|a+b-2x|}{2}\right]\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda\right]\right]. \tag{63}$$

We study separately the integral

$$\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda \overset{(34)}{\leq} \int_0^1\left|f_j^{(n_j)}(a)\right|^{\lambda^s}\left|f_j^{(n_j)}(b)\right|^{(1-\lambda)^s}d\lambda \leq \tag{64}$$

$$\int_0^1\left|f_j^{(n_j)}(a)\right|^{\lambda s}\left|f_j^{(n_j)}(b)\right|^{(1-\lambda)s}d\lambda = \left|f_j^{(n_j)}(b)\right|^{s}\int_0^1\left(\frac{\left|f_j^{(n_j)}(a)\right|}{\left|f_j^{(n_j)}(b)\right|}\right)^{\lambda s}d\lambda =$$

$$\left|f_j^{(n_j)}(b)\right|^{s}\int_0^1 A_j^{\lambda s}d\lambda =: (**). \tag{65}$$

If $A_j = 1$, then

$$(**) = \left|f_j^{(n_j)}(b)\right|^{s}. \tag{66}$$

If $A_j \neq 1$, we get

$$(**) = \left|f_j^{(n_j)}(b)\right|^{s}\int_0^1 e^{s(\ln A_j)\lambda}d\lambda = \left|f_j^{(n_j)}(b)\right|^{s}\left(\frac{A_j^s-1}{s\ln A_j}\right). \tag{67}$$

Therefore we derive

$$\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda \leq \left|f_j^{(n_j)}(b)\right|^{s}\psi_s(A_j). \tag{68}$$

Hence by (68) used in (62) and (63) we derive (58) and (59).

2) As before and as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{3j},[a,b]} \leq (b-a)^{\frac{1}{p_{3j}}}\left|f_j^{(n_j)}(b)\right|^{s}(\psi_s(B_j))^{\frac{1}{p_{3j}}}. \tag{69}$$

Using (69) into (12), (13) we derive (60), (61). ■

Next we give applications when $n_1 = n_2 = ... = n_r = 1$.

Corollary 11 *(to Theorem 5) Let $f_j : [a,b] \to \mathbb{R}$ is absolutely continuous function, $j = 1,...,r \in \mathbb{N} - \{1\}$. Denote*

$$S^*(f_1,...,f_r) := \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i(x)\right)\left[f_j(x)-\frac{1}{b-a}\int_a^b f_j(t)\,dt\right]\right], \tag{70}$$

$x \in [a, b]$.

Then

1)

$$|S^* (f_1, ..., f_r)| \leq \left[\frac{1}{2} + \frac{|a + b - 2x|}{2(b-a)} \right] \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) \left\| f_j' \right\|_{1,[a,b]} \right] \right], \tag{71}$$

2) let $p_{lj} > 1 : \sum_{lj=1}^{3} \frac{1}{p_{lj}} = 1$, *with* $f_j' \in L_{p_{3j}} ([a, b])$, $j = 1, ..., r$, *it holds*

$$|S^* (f_1, ..., f_r)| \leq$$

$$\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) (b-a)^{\frac{1}{p_{1j}} - 1} \left\| f_j' \right\|_{p_{3j},[a,b]} \left(\frac{(b-x)^{p_{2j}+1} + (x-a)^{p_{2j}+1}}{p_{2j} + 1} \right)^{\frac{1}{p_{2j}}} \right], \tag{72}$$

3) assuming $f_j' \in L_\infty ([a, b])$, $j = 1, ..., r$, *we get*

$$|S^* (f_1, ..., f_r)| \leq \left(\frac{(b-x)^2 + (x-a)^2}{2(b-a)} \right) \cdot$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) \left\| f_j' \right\|_{\infty,[a,b]} \right] \right]. \tag{73}$$

We continue with

Corollary 12 *(to Theorem 9) Same terms and assumptions as in Corollary 11. Assume that* $a \geq 0$.

1) Suppose $\left| f_j' \right|$ *is s-convex in the second sense and* $\left| f_j' (x) \right| \leq M_{1j}$, $x \in [a, b]$, $j = 1, ..., r$. *Then*

$$|S^* (f_1, ..., f_r)| \leq (b-a) \left[\frac{2}{s+2} \left(\frac{(b-x)^{s+2} + (x-a)^{s+2}}{(b-a)^{s+2}} \right) - \right.$$

$$\left. \frac{1}{s+1} \left(\frac{(b-x)^{s+1} + (x-a)^{s+1}}{(b-a)^{s+1}} \right) + \frac{2}{(s+1)(s+2)} \right]$$

$$\left[\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i (x)| \right) M_{1j} \right] \right], \tag{74}$$

2) Let $p_{lj} > 1 : \sum_{lj=1}^{3} \frac{1}{p_{lj}} = 1$*, with* $f_j' \in L_{p_{3j}}([a,b])$*,* $j = 1, ..., r$*.*

2i) Assume again $\left|f_j'\right|$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$*,* $j = 1, ..., r$*,* $x \in [a,b]$*. Then*

$$\left|S^*(f_1, ..., f_r)\right| \leq 2\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i(x)\right|\right)(b-a)^{-\frac{1}{p_{2j}}}\right.\right.$$

$$\left.\left.M_{1j}\left(p_{3j}s+1\right)^{-\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]\right], \tag{75}$$

2ii) Assume that $\left|f_j'\right|^{p_{3j}}$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$*,* $j = 1, ..., r$*,* $x \in [a,b]$*. Then*

$$\left|S^*(f_1, ..., f_r)\right| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i(x)\right|\right)\right.$$

$$\left.\frac{2^{\frac{1}{p_{3j}}}M_{1j}(b-a)^{-\frac{1}{p_{2j}}}}{(s+1)^{\frac{1}{p_{3j}}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right], \tag{76}$$

2iii) Assume that $\left|f_j'\right|^{p_{3j}}$ *is s-concave in the second sense. Then*

$$\left|S^*(f_1, ..., f_r)\right| \leq \sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left|f_i(x)\right|\right)\right.$$

$$\left.2^{\frac{s-1}{p_{3j}}}\left|f_j'\left(\frac{a+b}{2}\right)\right|(b-a)^{-\frac{1}{p_{2j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]. \tag{77}$$

We also give

Corollary 13 *(to Theorem 10) Same terms and assumptions as in Corollary 11. Assume that* $a \geq 0$*. We further suppose that* $\left|f_j'\right| \neq 0$ *is* s-log-convex *in the second sense, and* $\left|f_j'(a)\right|, \left|f_j'(b)\right| \in (0,1]$*,* $j = 1, ..., r$*. Call* $A_{1j} := \left|\frac{f_j'(a)}{f_j'(b)}\right|$*,* $s \in (0,1]$*,* $j = 1, ..., r$*, and* ψ_s *as in (57).*

1) It holds

$$\left|S^*(f_1, ..., f_r)\right| \leq \left[\frac{(b-a)+|a+b-2x|}{2}\right].$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left|f_j'(b)\right|^s\psi_s(A_{1j})\right]\right]. \tag{78}$$

2) Let $p_{lj}>1:\sum_{lj=1}^{3}\frac{1}{p_{lj}}=1$, *with* $f_j'\in L_{p_{3j}}([a,b])$, $B_{1j}:=A_{1j}^{p_{3j}}$, $j=1,...,r$. *Then*

$$|S^*(f_1,...,f_r)|\leq\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\right.$$

$$\left.(b-a)^{-\frac{1}{p_{2j}}}\left|f_j'(b)\right|^s(\psi_s(B_{1j}))^{\frac{1}{p_{3j}}}\left(\frac{(b-x)^{p_{2j}+1}+(x-a)^{p_{2j}+1}}{p_{2j}+1}\right)^{\frac{1}{p_{2j}}}\right]. \tag{79}$$

Next we present a set of very general Grüss type inequalities involving several functions.

Theorem 14 *Let* $n_j\in\mathbb{N}$, $j=1,...,r\in\mathbb{N}-\{1\}$, $n_1\leq n_2\leq...\leq n_r$ *and* $f_j:[a,b]\to\mathbb{R}$ *be such that* $f_j^{(n_j-1)}$ *is absolutely continuous function. Denote*

$$\Delta(f_1,...,f_r):=\sum_{j=1}^{r}\left[\left(\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i(x)\right)L_{n_j}[f_j(x)]\,dx\right)-\right.$$

$$\left.\frac{1}{b-a}\left(\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i(x)\right)dx\right)\left(\int_a^b f_j(x)\,dx\right)\right]. \tag{80}$$

Then

1)

$$|\Delta(f_1,...,f_r)|\leq\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right], \tag{81}$$

2)

$$|\Delta(f_1,...,f_r)|\leq(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right], \tag{82}$$

3) let $p_{i,j} > 1 : \sum_{i=1}^{r+2} \frac{1}{p_{i,j}} = 1$, and $f_j^{(n_j)} \in L_{(r+2),j}([a,b])$, $j = 1,...,r$, it holds

$$|\Delta(f_1,...,f_r)| \leq \frac{1}{n_1} \left[\sum_{j=1}^{r} \left[\left(\frac{2^{\frac{1}{p_{r+1,j}}} (b-a)^{1+\frac{1}{p_{r+1,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}} \right) \right. \right.$$

$$\left. \left. \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} \|f_i\|_{p_{i,j},[a,b]} \right) \left\| P_{n_j-1} \right\|_{p_{j,j},[a,b]} \left\| f_j^{(n_j)} \right\|_{p_{r+2,j},[a,b]} \right] \right]. \quad (83)$$

Proof. From (19) we obtain

$$\sum_{j=1}^{r} \left[\left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) L_{n_j}[f_j(x)] - \frac{1}{b-a} \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \int_a^b f_j(t)\,dt \right] =$$

$$= \sum_{j=1}^{r} \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) \int_a^b P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dt \right]. \quad (84)$$

Hence we get

$$\Delta(f_1,...,f_r) := \sum_{j=1}^{r} \left[\left(\int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) L_{n_j}[f_j(x)]\,dx \right) - \right.$$

$$\left. \frac{1}{b-a} \left(\int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) dx \right) \left(\int_a^b f_j(t)\,dt \right) \right] =$$

$$= \sum_{j=1}^{r} \left[\frac{(-1)^{n_j+1}}{n_j(b-a)} \int_a^b \int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} f_i(x) \right) P_{n_j-1}(t)\, q(x,t)\, f_j^{(n_j)}(t)\,dtdx \right]. \quad (85)$$

Therefore it holds

$$|\Delta(f_1,...,f_r)| \leq$$

$$\frac{1}{n_1(b-a)} \sum_{j=1}^{r} \left[\int_a^b \int_a^b \left(\prod_{\substack{i=1 \\ i \neq j}}^{r} |f_i(x)| \right) \left| P_{n_j-1}(t) \right| |q(x,t)| \left| f_j^{(n_j)}(t) \right| dtdx \right]. \quad (86)$$

We first find

$$|\Delta(f_1,...,f_r)| \leq \left(\frac{(b-a)+|a+b-2x|}{2n_1(b-a)} \right).$$

$$\left[\sum_{j=1}^{r}\left[\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}(b-a)\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]=$$

$$\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\left[\sum_{j=1}^{r}\left[\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{1,[a,b]}\right]\right]. \tag{87}$$

Also it holds

$$|\Delta(f_1,...,f_r)|\leq\left(\frac{(b-a)+|a+b-2x|}{2n_1(b-a)}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}(b-a)^2\right]\right]=$$

$$(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left\|f_j^{(n_j)}\right\|_{\infty,[a,b]}\right]\right]. \tag{88}$$

Let $p_{i,j}>1:\sum_{i=1}^{r+2}\frac{1}{p_{i,j}}=1$, and $f_j^{(n_j)}\in L_{(r+2),j}([a,b])$, $j=1,...,r$. Hence, by Hölder's inequality we find

$$\int_a^b\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dtdx\leq$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\left(\int_a^b\int_a^b|f_i(x)|^{p_{i,j}}dtdx\right)^{\frac{1}{p_{i,j}}}\right)\left(\int_a^b\int_a^b\left|P_{n_j-1}(t)\right|^{p_{j,j}}dtdx\right)^{\frac{1}{p_{j,j}}}\cdot \tag{89}$$

$$\left(\int_a^b\int_a^b|q(x,t)|^{p_{r+1,j}}dtdx\right)^{\frac{1}{p_{r+1,j}}}\left(\int_a^b\int_a^b\left|f_j^{(n_j)}(t)\right|^{p_{r+2,j}}dtdx\right)^{\frac{1}{p_{r+2,j}}}=$$

$$=\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}(b-a)^{\frac{1}{p_{i,j}}}\right)\left(\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}(b-a)^{\frac{1}{p_{j,j}}}\right)$$

$$\left(\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}(b-a)^{\frac{1}{p_{r+2,j}}}\right)\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{2}{p_{r+1,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}=$$

$$\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{2+\frac{1}{p_{r+1,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\cdot$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}. \tag{90}$$

That is we found

$$\int_a^b\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}|f_i(x)|\right)\left|P_{n_j-1}(t)\right||q(x,t)|\left|f_j^{(n_j)}(t)\right|dtdx\leq$$

$$\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{2+\frac{1}{p_{r+1,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\cdot$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]}. \tag{91}$$

Using (91) into (86) we obtain (83).

The proof of the theorem now is complete. ■

Next we produce Grüss type inequalities for several functions under s-convexity and s-concavity in the second sense.

Theorem 15 *Here all as in Theorem 14, with $a\geq 0$.*

1) Suppose $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}(x)\right|\leq M_j$, $x\in[a,b]$, $j=1,...,r$. Then

$$|\Delta(f_1,...,f_r)|\leq\frac{4(b-a)^2}{(s+2)(s+3)n_1}\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}M_j\right]\right]. \tag{92}$$

2) Let $p_{i,j}>1:\sum_{i=1}^{r+2}\frac{1}{p_{i,j}}=1$, with $f_j^{(n_j)}\in L_{p_{r+2.j}}([a,b])$, $j=1,...,r$.

2i) Assume again $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, and $\left|f_j^{(n_j)}(x)\right|\leq M_j$, $j=1,...,r$, $x\in[a,b]$. Then

$$|\Delta(f_1,...,f_r)|\leq\frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{1+\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]} M_j\left(p_{r+2,j}s+1\right)^{-\frac{1}{p_{r+2,j}}}\Bigg]\Bigg]. \qquad (93)$$

2ii) Assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ *is s-convex in the second sense, and* $\left|f_j^{(n_j)}(x)\right| \leq M_j,\ j=1,...,r,\ x\in[a,b]$. *Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{\left((p_{r+1,j}+1)(p_{r+1,j}+2)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left.\left.\frac{M_j}{(s+1)^{\frac{1}{p_{r+2,j}}}}\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\right]\right]. \qquad (94)$$

2iii) Assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ *is s-concave in the second sense. Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{s-1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{\left((p_{r+1,j}+1)(p_{r+1,j}+2)\right)^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\left.\left.\left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right|\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\right]\right]. \qquad (95)$$

Proof. From (86) we get

$$|\Delta(f_1,...,f_r)| \leq \frac{1}{n_1(b-a)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\right.\right. \qquad (96)$$

$$\left.\left.\int_a^b\left(\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right)dx\right]\right].$$

Here $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense and $\left|f_j^{(n_j)}(x)\right| \leq M_j,\ x\in[a,b]$, $j=1,...,r$.

Using (53) we obtain

$$\int_a^b\left(\int_a^b|q(x,t)|\left|f_j^{(n_j)}(t)\right|dt\right)dx \leq \frac{4M_j(b-a)^3}{(s+2)(s+3)}, \qquad (97)$$

$j=1,...,r$.

Consequently by (97) and (96) we derive (92).

Next we elaborate on (83).

Assume that $\left|f_j^{(n_j)}\right|$ is s-convex in the second sense, acting as in [2], we obtain

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq 2M_j \left(\frac{b-a}{p_{r+2,j}s+1}\right)^{\frac{1}{p_{r+2,j}}}, \tag{98}$$

$j = 1, ..., r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Next suppose that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ is s-convex in the second sense. As in [2] we get

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq \frac{2^{\frac{1}{p_{r+2,j}}} M_j (b-a)^{\frac{1}{p_{r+2,j}}}}{(s+1)^{\frac{1}{p_{r+2,j}}}}, \tag{99}$$

$j = 1, ..., r$, with $\left|f_j^{(n_j)}(x)\right| \leq M_j$, $x \in [a,b]$.

Finally assume that $\left|f_j^{(n_j)}\right|^{p_{r+2,j}}$ is s-concave in the second sense. Based on Theorem 8 and acting as in [2], we derive

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq 2^{\frac{s-1}{p_{r+2,j}}} \left|f_j^{(n_j)}\left(\frac{a+b}{2}\right)\right| (b-a)^{\frac{1}{p_{r+2,j}}}. \tag{100}$$

The proof is done. ■

Grüss type inequalities for several functions under s-log-convexity in the second sense follow.

Theorem 16 *Same terms and assumptions as in Theorem 14, $a \geq 0$. We further suppose that $\left|f_j^{(n_j)}\right| \neq 0$ is s-log-convex in the second sense, and $\left|f_j^{(n_j)}(a)\right|$, $\left|f_j^{(n_j)}(b)\right| \in (0,1]$, $j = 1, ..., r$. Call $A_j := \left|\frac{f_j^{(n_j)}(a)}{f_j^{(n_j)}(b)}\right|$, $s \in (0,1]$, $j = 1, ..., r$, and $\psi_s(z)$ as in (57).*

1) It holds

$$|\Delta(f_1, ..., f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right) \cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\ i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^{s}\psi_s(A_j)\right]\right]. \tag{101}$$

2) Let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$, and $f_j^{(n_j)} \in L_{(r+2),j}([a,b])$, $B_j^* := A_j^{p_{r+2,j}}$, $j = 1, ..., r$. Then

$$|\Delta(f_1, ..., f_r)| \leq \frac{1}{n_1}\left[\sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.\right.$$

$$\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\left\|P_{n_j-1}\right\|_{p_{j,j},[a,b]}\left|f_j^{(n_j)}(b)\right|^s\left(\psi_s\left(B_j^*\right)\right)^{\frac{1}{p_{r+2,j}}}\Bigg]\Bigg]. \tag{102}$$

Proof. 1) From (81) we get

$$|\Delta(f_1,...,f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\int_0^1\left|f_j^{(n_j)}(\lambda a+(1-\lambda)b)\right|d\lambda\right]\right] \overset{\text{(by (68))}}{\leq} \tag{103}$$

$$(b-a)\left(\frac{(b-a)+|a+b-2x|}{2n_1}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|P_{n_j-1}\right\|_{\infty,[a,b]}\left|f_j^{(n_j)}(b)\right|^s\psi_s(A_j)\right]\right]. \tag{104}$$

That is proving (101).

2) As in (69) we get

$$\left\|f_j^{(n_j)}\right\|_{p_{r+2,j},[a,b]} \leq (b-a)^{\frac{1}{p_{r+2,j}}}\left|f_j^{(n_j)}(b)\right|^s\left(\psi_s\left(B_j^*\right)\right)^{\frac{1}{p_{r+2,j}}}. \tag{105}$$

Using (105) into (83), we derive (102). ■

Finally we give applications to Grüss type inequalities for several functions when $n_1=n_2=...=n_r=1$.

Corollary 17 *(to Theorem 14) Let $f_j:[a,b]\to\mathbb{R}$ be absolutely continuous, $j=1,...,r\in\mathbb{N}-\{1\}$. Denote*

$$\Delta^*(f_1,...,f_r) := r\int_a^b\left(\prod_{i=1}^{r}f_i(x)\right)dx-$$

$$\frac{1}{b-a}\left[\sum_{j=1}^{r}\left[\left(\int_a^b\left(\prod_{\substack{i=1\\i\neq j}}^{r}f_i(x)\right)dx\right)\left(\int_a^b f_j(x)\,dx\right)\right]\right]. \tag{106}$$

Then

1)

$$|\Delta^*(f_1,...,f_r)| \leq \left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|f_j'\right\|_{1,[a,b]}\right]\right], \tag{107}$$

2)

$$|\Delta^*(f_1,...,f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left\|f_j'\right\|_{\infty,[a,b]}\right]\right], \tag{108}$$

3) let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$*, and* $f_j' \in L_{(r+2),j}([a,b])$*,* $j = 1,...,r$*, it holds*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\left\|f_j'\right\|_{p_{r+2,j},[a,b]}\right]. \tag{109}$$

Corollary 18 *(to Theorem 15) Here all as in Corollary 17, with* $a \geq 0$*.*

1) Suppose $\left|f_j'\right|$ *is s-convex in the second sense and* $\left|f_j'(x)\right| \leq M_{1j}$*,* $x \in [a,b]$*,* $j = 1,...,r$*. Then*

$$|\Delta^*(f_1,...,f_r)| \leq \frac{4(b-a)^2}{(s+2)(s+3)}\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)M_{1j}\right]\right]. \tag{110}$$

2) Let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$*, with* $f_j' \in L_{p_{r+2.j}}([a,b])$*,* $j = 1,...,r$*.*

2i) Assume again $\left|f_j'\right|$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$*,* $j = 1,...,r$*,* $x \in [a,b]$*. Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{1+\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)M_{1j}(p_{r+2,j}s+1)^{-\frac{1}{p_{r+2,j}}}\right]. \tag{111}$$

2ii) Assume that $\left|f_j'\right|^{p_{r+2,j}}$ *is s-convex in the second sense, and* $\left|f_j'(x)\right| \leq M_{1j}$*,* $j = 1,...,r$*,* $x \in [a,b]$*. Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\frac{M_{1j}}{(s+1)^{\frac{1}{p_{r+2,j}}}}\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\Bigg]. \tag{112}$$

2iii) Assume that $\left|f_j'\right|^{p_{r+2,j}}$ *is s-concave in the second sense. Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}+\frac{s-1}{p_{r+2,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\left|f_j'\left(\frac{a+b}{2}\right)\right|\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\right)\right]. \tag{113}$$

Corollary 19 *(to Theorem 16) Here all as in Corollary 17, with* $a \geq 0$*. We further suppose that* $\left|f_j'\right| \neq 0$ *is* s-log-*convex in the second sense, and* $\left|f_j'(a)\right|, \left|f_j'(b)\right| \in (0,1]$*,* $j = 1,...,r$*. Call* $A_{1j} := \left|\frac{f_j'(a)}{f_j'(b)}\right|$*,* $s \in (0,1]$*,* $j = 1,...,r$*, and* $\psi_s(z)$ *as in (57).*

1) It holds

$$|\Delta^*(f_1,...,f_r)| \leq (b-a)\left(\frac{(b-a)+|a+b-2x|}{2}\right)\cdot$$

$$\left[\sum_{j=1}^{r}\left[\left(\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{\infty,[a,b]}\right)\left|f_j'(b)\right|^s\psi_s(A_{1j})\right]\right]. \tag{114}$$

2) Let $p_{i,j} > 1 : \sum_{i=1}^{r+2}\frac{1}{p_{i,j}} = 1$*, and* $f_j' \in L_{(r+2),j}([a,b])$*,* $B_{1j}^* := A_{1j}^{p_{r+2,j}}$*,* $j = 1,...,r$*. Then*

$$|\Delta^*(f_1,...,f_r)| \leq \sum_{j=1}^{r}\left[\left(\frac{2^{\frac{1}{p_{r+1,j}}}(b-a)^{1+\frac{1}{p_{r+1,j}}+\frac{1}{p_{r+2,j}}+\frac{1}{p_{j,j}}}}{((p_{r+1,j}+1)(p_{r+1,j}+2))^{\frac{1}{p_{r+1,j}}}}\right)\right.$$

$$\left.\prod_{\substack{i=1\\i\neq j}}^{r}\|f_i\|_{p_{i,j},[a,b]}\left|f_j'(b)\right|^s\left(\psi_s\left(B_{1j}^*\right)\right)^{\frac{1}{p_{r+2,j}}}\right]. \tag{115}$$

Remark 20 *From (20) one can work out the analogous Grüss type inequalities general theory involving the functions* $L_{n_j}[f_j(x)]$*, for* $j = 1,...,r \in \mathbb{N} - \{1\}$*. The results will be very similar to the results of Theorems 14-16, and when* $n_1 = n_2 = ... = n_r = 1$ *their applications will be identical to Corollaries 17-19. We choose to omit this study.*

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