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Harmonic Multivariate Ostrowski and Grüss type Inequalities for several functions

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Abstract

Here we derive very general multivariate Ostrowski and Grüss type inequalities for several functions by involving harmonic polynomials. Estimates are with respect to all basic norms. We give applications.

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1 Introduction

The problem of estimating the difference of a value of a function from its average is a paramount one. The answer to it are the Ostrowski type inequalities. Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals. The problem of estimating the difference between the average of a product of functions from the product of their averages is also a very important one. The answer to it are the Grüss type inequalities. Grüss type inequalities are very useful among others in Probability for estimating expected values, etc. There exists a vast literature on Ostrowski and Grüss type inequalities to all possible directions. Mathematical community is very much interested to these inequalities due to their applications. So here we derive very general Ostrowski and Grüss type inequalities for several multivariate functions, acting to all possible directions.

We are motivated by the following results.

Theorem 1 *(1938, Ostrowski [7]) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e.,*

$\|f'\|_{\infty}^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\,\|f'\|_{\infty}^{\sup}, \tag{1}$$

for any $x\in[a,b]$. *The constant* $\frac{1}{4}$ *is the best possible.*

Theorem 2 *(1935, Grüss [6]) Let* f,g *be integrable functions from* $[a,b]$ *into* $\mathbb{R}$, *that satisfy the conditions*

$$m \le f(x) \le M,\ n \le g(x) \le N,\ \ x\in[a,b],$$

where $m,M,n,N\in\mathbb{R}$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \tag{2}$$

$$\le \frac{1}{4}(M-m)(N-n).$$

Theorem 3 *(1998, Dragomir and Wang [4]) Let* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous function with* $f'\in L_p([a,b])$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $x\in[a,b]$. *Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le$$

$$\frac{1}{(q+1)^{\frac{1}{q}}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\,\|f'\|_p. \tag{3}$$

Theorem 4 *(1882, Čebyšev [1]) Let* $f,g:[a,b]\to\mathbb{R}$ *absolutely continuous functions with* $f',g'\in L_\infty([a,b])$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \tag{4}$$

$$\le \frac{1}{12}(b-a)^2\,\|f'\|_\infty\,\|g'\|_\infty.$$

Above is also assumed that the involved integrals exist.

2 Background

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \geq 1$, $P_0 = 1$. Furthermore, let $[a,b] \subset \mathbb{R}$, $a \neq b$, and $h : [a,b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some fixed $n \geq 1$. We use the notation

$$L_n[h(x)] = \frac{1}{n}\Bigg[h(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x)\, h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(a)\, h^{(k-1)}(a) - P_k(b)\, h^{(k-1)}(b)\right]\Bigg], \tag{5}$$

$x \in [a,b]$, for convinience.

For $n = 1$ the above sums are defined to be zero, that is $L_1[h(x)] = h(x)$.

Dedic, Pečaric and Ujevic, see [2], [3], established the following identity,

$$L_n[h(x)] - \frac{1}{b-a}\int_a^b h(t)\,dt = \frac{(-1)^{n+1}}{n(b-a)}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{6}$$

where

$$q(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b]. \tag{7}$$

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \geq 0$, the identity (6) reduces to the Fink identity in [5], (see also [3], p. 177).

We rewrite (6) as follows:

$$h(x) + \sum_{k=1}^{n-1}(-1)^k P_k(x)\, h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(a)\, h^{(k-1)}(a) - P_k(b)\, h^{(k-1)}(b)\right] = \frac{n}{b-a}\int_a^b h(t)\,dt + \frac{(-1)^{n+1}}{b-a}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{8}$$

$x \in [a,b]$.

That is the generalized Fink type representation formula.

$$h(x) = \sum_{k=1}^{n-1}(-1)^{k+1} P_k(x)\, h^{(k)}(x) + \sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b-a}\left[P_k(b)\, h^{(k-1)}(b) - P_k(a)\, h^{(k-1)}(a)\right] + \frac{n}{b-a}\int_a^b h(t)\,dt + \frac{(-1)^{n+1}}{b-a}\int_a^b P_{n-1}(t)\, q(x,t)\, h^{(n)}(t)\,dt, \tag{9}$$

$x \in [a,b]$, $n \geq 1$, when $n = 1$ the above sums are zero.

3 Main Results

Here $\prod_{i=1}^{m}[a_i,b_i] \subseteq \mathbb{R}^m$, $m,n \in \mathbb{N}$.

We make

General Assumptions 5 *Let* $f : \prod_{i=1}^{m}[a_i,b_i] \to \mathbb{R}$. *We assume*

1) for $j=1,...,m$ *we have that* $\frac{\partial^{n-1}f}{\partial x_j^{n-1}}(x_1,x_2,...,x_{j-1},s_j,x_{j+1},...,x_m)$ *is absolutely continuous in* s_j *on* $[a_j,b_j]$, *for every* $(x_1,x_2,...,x_{j-1},x_{j+1},...,x_m) \in \prod_{\substack{i=1 \\ i\neq j}}^{m}[a_i,b_i]$,

2) for $j=1,...,m$ *we have that* $\frac{\partial^n f(s_1,...,s_j,x_{j+1},...,x_m)}{\partial x_j^n}$ *is continuous on* $\prod_{i=1}^{j}[a_i,b_i]$, *for every* $(x_{j+1},...,x_m) \in \prod_{i=j+1}^{m}[a_i,b_i]$,

3) for each $j=1,...,m$, *and for every* $l=1,...,n-1$, *we have that* $\frac{\partial^l f}{\partial x_j^l}(s_1,s_2,...,s_{j-1},x_j,...,x_m)$ *is continuous on* $\prod_{i=1}^{j-1}[a_i,b_i]$, *for every* $(x_j,...,x_m) \in \prod_{i=j}^{m}[a_j,b_j]$.

4) f *is continuous on* $\prod_{i=1}^{m}[a_i,b_i]$.

Brief Assumptions 6 *Let* $f : \prod_{i=1}^{m}[a_i,b_i] \to \mathbb{R}$ *with* $\frac{\partial^l f}{\partial x_i^l}$ *for* $l=0,1,...,n$; $i=1,...,m$, *are continuous on* $\prod_{i=1}^{m}[a_i,b_i]$.

Definition 7 *We put*

$$q(x_i,s_i) = \begin{cases} s_i - a_i, & \text{if } s_i \in [a_i,x_i], \\ s_i - b_i, & \text{if } s_i \in (x_i,b_i], \end{cases} \quad x_i \in [a_i,b_i], \tag{10}$$

$i=1,...,m$.

We present the following general representation result of Fink type.

Theorem 8 *Let* f *as in General Assumtions 5 or Brief Assumtions 6. Then*

$$f(x_1,...,x_n) = \frac{n^m}{\prod_{i=1}^{m}(b_i-a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} f(s_1,...,s_m)\,ds_1...ds_m \tag{11}$$

$$+\sum_{i=1}^{m} T_i\left(x_i, x_{i+1}, ..., x_m\right),$$

where

$$T_i\left(x_i, ..., x_m\right) := \frac{n^{i-1}}{\prod_{j=1}^{i-1}\left(b_j - a_j\right)}\cdot$$

$$\left[\sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_i\right)\int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f\left(s_1, ..., s_{i-1}, x_i, ..., x_m\right)}{\partial x_i^k} ds_1 ... ds_{i-1} + \right.$$

$$\sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{b_i - a_i}\cdot$$

$$\left[P_k\left(b_i\right)\int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f\left(s_1, ..., s_{i-1}, b_i, x_{i+1}, ..., x_m\right)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1} - \right.$$

$$\left. P_k\left(a_i\right)\int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f\left(s_1, ..., s_{i-1}, a_i, x_{i+1}, ..., x_m\right)}{\partial x_i^{k-1}} ds_1 ... ds_{i-1}\right] + \tag{12}$$

$$\left.\frac{(-1)^{n+1}}{\left(b_i - a_i\right)}\int_{a_1}^{b_1} ... \int_{a_i}^{b_i} P_{n-1}\left(s_i\right) q\left(x_i, s_i\right)\frac{\partial^n f\left(s_1, ..., s_i, x_{i+1}, ..., x_m\right)}{\partial x_i^n} ds_1 ... ds_i\right],$$

are continuous functions for all $i = 1, ..., m$.

Proof. We apply (9) repeatedly.

We have

$$f\left(x_1, ..., x_m\right) = \sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_1\right)\frac{\partial^k f\left(x_1, ..., x_m\right)}{\partial x_1^k} +$$

$$\sum_{k=1}^{n-1}\frac{(-1)^k (n-k)}{\left(b_1 - a_1\right)}\left[P_k\left(b_1\right)\frac{\partial^{k-1} f\left(b_1, x_2, ..., x_m\right)}{\partial x_1^{k-1}} - P_k\left(a_1\right)\frac{\partial^{k-1} f\left(a_1, x_2, ..., x_m\right)}{\partial x_1^{k-1}}\right]$$

$$+\frac{n}{\left(b_1 - a_1\right)}\int_{a_1}^{b_1} f\left(s_1, x_2, ..., x_m\right) ds_1 +$$

$$\frac{(-1)^{n+1}}{\left(b_1 - a_1\right)}\int_{a_1}^{b_1} P_{n-1}\left(s_1\right) q\left(x_1, s_1\right)\frac{\partial^n f\left(s_1, x_2, ..., x_m\right)}{\partial x_1^n} ds_1, \tag{13}$$

any $x_1 \in [a_1, b_1]$,
under the assumption that $\frac{\partial^{n-1} f(\cdot, x_2, ..., x_m)}{\partial x_1^{n-1}} \in AC\left(\left[a_1, b_1\right]\right)$.

Call

$$T_1\left(x_1, ..., x_m\right) := \sum_{k=1}^{n-1}(-1)^{k+1} P_k\left(x_1\right)\frac{\partial^k f\left(x_1, ..., x_m\right)}{\partial x_1^k} +$$

$$\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{(b_1-a_1)}\left[P_k(b_1)\frac{\partial^{k-1}f(b_1,x_2,...,x_m)}{\partial x_1^{k-1}}-P_k(a_1)\frac{\partial^{k-1}f(a_1,x_2,...,x_m)}{\partial x_1^{k-1}}\right]$$

$$+\frac{(-1)^{n+1}}{(b_1-a_1)}\int_{a_1}^{b_1}P_{n-1}(s_1)\,q(x_1,s_1)\,\frac{\partial^n f(s_1,x_2...,x_m)}{\partial x_1^n}ds. \qquad (14)$$

Hence it holds

$$f(x_1,...,x_m)=\frac{n}{(b_1-a_1)}\int_{a_1}^{b_1}f(s_1,x_2,...,x_m)\,ds_1+T_1(x_1,...,x_m). \qquad (15)$$

Next similarly we get

$$f(s_1,x_2,...,x_m)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_2)\frac{\partial^k f(s_1,x_2,...,x_m)}{\partial x_2^k}+$$

$$\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{(b_2-a_2)}\cdot$$

$$\left[P_k(b_2)\frac{\partial^{k-1}f(s_1,b_2,x_3,...,x_m)}{\partial x_2^{k-1}}-P_k(a_2)\frac{\partial^{k-1}f(s_1,a_2,x_3,...,x_m)}{\partial x_2^{k-1}}\right]+ \qquad (16)$$

$$\frac{n}{(b_2-a_2)}\int_{a_2}^{b_2}f(s_1,s_2,x_3,...,x_m)\,ds_2+$$

$$\frac{(-1)^{n+1}}{(b_2-a_2)}\int_{a_2}^{b_2}P_{n-1}(s_2)\,q(x_2,s_2)\,\frac{\partial^n f(s_1,s_2,x_3,...,x_m)}{\partial x_2^n}ds_2,$$

any $x_2\in[a_2,b_2]$.

Hence it holds

$$f(x_1,...,x_m)=\frac{n^2}{(b_1-a_1)(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(s_1,s_2,x_3,...,x_m)\,ds_1ds_2+$$

$$T_1(x_1,...,x_m)+T_2(x_2,x_3,...,x_m), \qquad (17)$$

where

$$T_2(x_2,x_3,...,x_m):=\frac{n}{(b_1-a_1)}\left\{\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_2)\int_{a_1}^{b_1}\frac{\partial^k f(s_1,x_2,...,x_m)}{\partial x_2^k}ds_1\right.$$

$$+\sum_{k=1}^{n-1}\frac{(-1)^{k}(n-k)}{(b_2-a_2)}\left[P_k(b_2)\int_{a_1}^{b_1}\frac{\partial^{k-1}f(s_1,b_2,x_3,...,x_m)}{\partial x_2^{k-1}}ds_1\right. \qquad (18)$$

$$\left.-P_k(a_2)\int_{a_1}^{b_1}\frac{\partial^{k-1}f(s_1,a_2,x_3,...,x_m)}{\partial x_2^{k-1}}ds_1\right]$$

$$+\frac{(-1)^{n+1}}{(b_2-a_2)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}P_{n-1}(s_2)\,q(x_2,s_2)\,\frac{\partial^n f(s_1,s_2,x_3,...,x_m)}{\partial x_2^n}ds_1ds_2\Bigg\}.$$

Next we see similarly that

$$\begin{aligned} f(s_1,s_2,x_3,...,x_m) &= \sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_3)\,\frac{\partial^k f(s_1,s_2,x_3,...,x_m)}{\partial x_3^k}+ \\ &\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_3-a_3)}\Bigg[P_k(b_3)\,\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,...,x_m)}{\partial x_3^{k-1}}- \\ &P_k(a_3)\,\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,...,x_m)}{\partial x_3^{k-1}}\Bigg]+ \\ &\frac{n}{(b_3-a_3)}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4,...,x_m)\,ds_3+ \\ &\frac{(-1)^{n+1}}{(b_3-a_3)}\int_{a_3}^{b_3}P_{n-1}(s_3)\,q(x_3,s_3)\,\frac{\partial^n f(s_1,s_2,s_3,x_4,...,x_m)}{\partial x_3^n}ds_3, \end{aligned} \tag{19}$$

any $x_3\in[a_3,b_3]$.

So that we get

$$\begin{aligned} f(x_1,...,x_m) &= \frac{n^3}{\prod_{j=1}^{3}(b_j-a_j)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f(s_1,s_2,s_3,x_4,...,x_m)\,ds_1ds_2ds_3+ \\ &T_1(x_1,...,x_m)+T_2(x_2,...,x_m)+T_3(x_3,...,x_m), \end{aligned} \tag{20}$$

where

$$\begin{aligned} T_3(x_3,...,x_m) &:= \frac{n^2}{(b_1-a_1)(b_2-a_2)}\cdot \\ &\Bigg[\sum_{k=1}^{n-1}(-1)^{k+1}P_k(x_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^k f(s_1,s_2,x_3,...,x_m)}{\partial x_3^k}ds_1ds_2+ \\ &\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{(b_3-a_3)}\Bigg[P_k(b_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^{k-1}f(s_1,s_2,b_3,x_4,...,x_m)}{\partial x_3^{k-1}}ds_1ds_2 \\ &-P_k(a_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^{k-1}f(s_1,s_2,a_3,x_4,...,x_m)}{\partial x_3^{k-1}}ds_1ds_2\Bigg]+ \\ &\frac{(-1)^{n+1}}{(b_3-a_3)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}P_{n-1}(s_3)\,q(x_3,s_3)\,\frac{\partial^n f(s_1,s_2,s_3,x_4,...,x_m)}{\partial x_3^n}ds_1ds_2ds_3\Bigg]. \end{aligned} \tag{21}$$

Furthermore we can write

$$f\left(s_1,s_2,s_3,x_4,...,x_m\right)=\sum_{k=1}^{n-1}\left(-1\right)^{k+1}P_k\left(x_4\right)\frac{\partial^k f\left(s_1,s_2,s_3,x_4...,x_m\right)}{\partial x_4^k}+$$

$$\sum_{k=1}^{n-1}\frac{\left(-1\right)^k\left(n-k\right)}{\left(b_4-a_4\right)}\left[P_k\left(b_4\right)\frac{\partial^{k-1}f\left(s_1,s_2,s_3,b_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}\right.$$

$$\left.-P_k\left(a_4\right)\frac{\partial^{k-1}f\left(s_1,s_2,s_3,a_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}\right]+\tag{22}$$

$$\frac{n}{\left(b_4-a_4\right)}\int_{a_4}^{b_4}f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)ds_4+$$

$$\frac{\left(-1\right)^{n+1}}{\left(b_4-a_4\right)}\int_{a_4}^{b_4}P_{n-1}\left(s_4\right)q\left(x_4,s_4\right)\frac{\partial^n f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)}{\partial x_4^n}ds_4,$$

any $x_4\in\left[a_4,b_4\right]$.

Therefore it holds

$$f\left(x_1,...,x_m\right)=\frac{n^4}{\prod_{j=1}^{4}\left(b_j-a_j\right)}\cdot$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4}f\left(s_1,s_2,s_3,s_4,x_5,...,x_m\right)ds_1ds_2ds_3ds_4+$$

$$T_1\left(x_1,...,x_m\right)+T_2\left(x_2,...,x_m\right)+T_3\left(x_3,...,x_m\right)+T_4\left(x_4,...,x_m\right),\tag{23}$$

where

$$T_4\left(x_4,...,x_m\right):=\frac{n^3}{\prod_{j=1}^{3}\left(b_j-a_j\right)}\left[\sum_{k=1}^{n-1}\left(-1\right)^{k+1}P_k\left(x_4\right)\cdot\right.$$

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^k f\left(s_1,s_2,s_3,x_4,...,x_m\right)}{\partial x_4^k}ds_1ds_2ds_3+$$

$$\sum_{k=1}^{n-1}\frac{\left(-1\right)^k\left(n-k\right)}{\left(b_4-a_4\right)}\left[P_k\left(b_4\right)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^{k-1}f\left(s_1,s_2,s_3,b_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}ds_1ds_2ds_3\right.$$

$$\left.-P_k\left(a_4\right)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\frac{\partial^{k-1}f\left(s_1,s_2,s_3,a_4,x_5,...,x_m\right)}{\partial x_4^{k-1}}ds_1ds_2ds_3\right]\tag{24}$$

$$+\frac{\left(-1\right)^{n+1}}{\left(b_4-a_4\right)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4}P_{n-1}\left(s_4\right)q\left(x_4,s_4\right)\cdot$$

$$\frac{\partial^n f(s_1,s_2,s_3,s_4,x_5,...,x_m)}{\partial x_4^n} ds_1 ds_2 ds_3 ds_4 \Bigg],$$

etc.

The theorem is proved. ■

We make

Remark 9 *Let f_λ, $\lambda = 1,...,r \in \mathbb{N} - \{1\}$, as in Assumtions 5 or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ. Here $x = (x_1,...,x_m)$, $s = (s_1,...,s_m) \in \prod_{i=1}^{m} [a_i, b_i]$. Then*

$$f_\lambda(x) = \frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i - a_i)} \int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds + \sum_{i=1}^{m} T_{i\lambda}(x_i, x_{i+1}, ..., x_m). \tag{25}$$

Here we have

$$\begin{gathered} T_{i\lambda}(x_i,...,x_m) := \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1}(b_j - a_j)} \left[\sum_{k=1}^{n_\lambda - 1} (-1)^{k+1} P_k(x_i) \cdot \right. \\ \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^k f_\lambda(s_1,...,s_{i-1},x_i,...,x_m)}{\partial x_i^k} ds_1...ds_{i-1} + \\ \sum_{k=1}^{n_\lambda - 1} \frac{(-1)^k (n_\lambda - k)}{b_i - a_i} \cdot \\ \left[P_k(b_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1,...,s_{i-1},b_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}} ds_1...ds_{i-1} \right. \\ \left. -P_k(a_i) \int_{a_1}^{b_1} ... \int_{a_{i-1}}^{b_{i-1}} \frac{\partial^{k-1} f_\lambda(s_1,...,s_{i-1},a_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}} ds_1...ds_{i-1} \right] + \\ \left. \frac{(-1)^{n_\lambda + 1}}{(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} P_{n_\lambda - 1}(s_i)\, q(x_i, s_i) \frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}} ds_1...ds_i \right], \end{gathered} \tag{26}$$

are continuous functions, $i = 1,...,m$; $\lambda = 1,...,r$.

Hence it holds

$$\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x)\right) f_\lambda(x) = \left(\frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i - a_i)}\right) \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho(x)\right) \left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) + \tag{27}$$

$$\left(\prod_{\substack{\rho=1\\\rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\sum_{i=1}^{m} T_{i\lambda}(x_1,...,x_m)\right).$$

Therefore we derive

$$\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\\rho\neq\lambda}}^{r} f_{\rho}(x)\right) f_{\lambda}(x)\right)-$$

$$\frac{1}{\prod_{i=1}^{m}(b_i-a_i)}\left\{\sum_{\lambda=1}^{r}\left(n_{\lambda}^{m}\left(\prod_{\substack{\rho=1\\\rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_{\lambda}(s)\,ds\right)\right)\right\} \tag{28}$$

$$=\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\\rho\neq\lambda}}^{r} f_{\rho}(x)\right)\left(\sum_{i=1}^{m} T_{i\lambda}(x_i,...,x_m)\right)\right).$$

We notice that

$$T_{i\lambda}(x_i,...,x_m)=A_{i\lambda}(x_i,...,x_m)+B_{i\lambda}(x_i,...,x_m), \tag{29}$$

$i=1,...,m;$ *where*

$$A_{i\lambda}(x_i,...,x_m):=\frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i-1}(b_j-a_j)}\left[\sum_{k=1}^{n_{\lambda}-1}(-1)^{k+1}P_k(x_i)\cdot\right. \tag{30}$$

$$\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^k f_{\lambda}(s_1,...,s_{i-1},x_i,...,x_m)}{\partial x_i^k}ds_1...ds_{i-1}$$

$$+\sum_{k=1}^{n_{\lambda}-1}\frac{(-1)^k(n_{\lambda}-k)}{b_i-a_i}\cdot$$

$$\left[P_k(b_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1} f_{\lambda}(s_1,...,s_{i-1},b_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}-\right.$$

$$\left.\left.P_k(a_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1} f_{\lambda}(s_1,...,s_{i-1},a_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}\right]\right],$$

and

$$B_{i\lambda}(x_i,...,x_m):=\frac{n_{\lambda}^{i-1}(-1)^{n_{\lambda}+1}}{\prod_{j=1}^{i}(b_j-a_j)}\cdot \tag{31}$$

$$\left[\int_{a_1}^{b_1}...\int_{a_i}^{b_i} P_{n_\lambda-1}(s_i)\, q(x_i,s_i)\, \frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}} ds_1...ds_i\right],$$

for all $i=1,...,m;\ \lambda=1,...,r.$

We call and have the identity

$$S(f_1,...,f_r)(x) :=$$

$$\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left[f_\lambda(x) - \frac{n_\lambda^m}{\prod_{i=1}^{m}(b_i-a_i)}\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) - \right.\right.$$

$$\left.\left.\sum_{i=1}^{m} A_{i\lambda}(x_i,...,x_m)\right]\right\} = \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left(\sum_{i=1}^{m} B_{i\lambda}(x_i,...,x_m)\right)\right), \quad (32)$$

true for any fixed $x \in \prod_{i=1}^{m}[a_i,b_i]$.

Then we have

$$|S(f_1,...,f_r)(x)| \leq \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} |f_\rho(x)|\right)\left(\sum_{i=1}^{m} |B_{i\lambda}(x_i,...,x_m)|\right)\right). \quad (33)$$

We estimate the right hand side of (33).

We also make

Remark 10 *We observe that*

$$|B_{i\lambda}(x_i,...,x_m)| \leq \frac{n_\lambda^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\left[\int_{a_1}^{b_1}...\int_{a_i}^{b_i} |P_{n_\lambda-1}(s_i)|\,|q(x_i,s_i)| \cdot\right.$$

$$\left.\left|\frac{\partial^{n_\lambda} f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right| ds_1...ds_i\right] =: (\xi), \quad (34)$$

for all $i=1,...,m;\ \lambda=1,...,r.$

We know that

$$|q(x_i,s_i)| \leq \max(x_i-a_i, b_i-x_i) = \frac{(b_i-a_i)+|a_i+b_i-2x_i|}{2}. \quad (35)$$

We have

$$(\xi) \overset{(35)}{\leq} \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \left[\|P_{n_\lambda - 1}\|_{\infty,[a_i,b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \cdot \right.$$

$$\left. \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right]. \tag{36}$$

Thus

$$|S(f_1, \dots, f_r)(x)| \leq \sum_{\lambda=1}^{r} \left(\left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} |f_\rho(x)| \right) \left(\sum_{i=1}^{m} \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \cdot \right. \right.$$

$$\left[\|P_{n_\lambda - 1}\|_{\infty,[a_i,b_i]} \left(\frac{(b_i - a_i) + |a_i + b_i - 2x_i|}{2} \right) \cdot \right.$$

$$\left. \left. \left. \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right] \right) \right) =: \theta_1(x). \tag{37}$$

Next let $p_{li} > 1 : \sum_{li=1}^{3} \frac{1}{p_{li}} = 1$. *Then*

$$|B_{i\lambda}(x_i, \dots, x_m)| \overset{(34)}{\leq} (\xi) \leq \frac{n_{\lambda}^{i-1}}{\prod_{j=1}^{i}(b_j - a_j)} \cdot$$

$$\left[\left(\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} |P_{n_\lambda - 1}(s_i)|^{p_{1i}} ds_1 \dots ds_i \right)^{\frac{1}{p_{1i}}} \left(\int_{a_1}^{b_1} \dots \int_{a_i}^{b_i} |q(x_i, s_i)|^{p_{2i}} ds_1 \dots ds_i \right)^{\frac{1}{p_{2i}}} \right. \tag{38}$$

$$\left. \cdot \left\| \frac{\partial^{n_\lambda} f_\lambda(\dots, x_{i+1}, \dots, x_m)}{\partial x_i^{n_\lambda}} \right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)} \right] =$$

$$\frac{n_\lambda^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i,b_i])}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\cdot\right.$$

$$\left.\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]. \tag{39}$$

Therefore we get

$$|S(f_1,...,f_r)(x)|\overset{(39)}{\leq}\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\frac{n_\lambda^{i-1}}{\prod_{j=1}^{i}(b_j-a_j)}\cdot\right.\right.$$

$$\left[\|P_{n_\lambda-1}\|_{L_{p_{1i}}([a_i,b_i])}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\cdot\right.$$

$$\left.\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right])) \tag{40}$$

$$=:\theta_2(x).$$

We also have

$$|B_{i\lambda}(x_i,...,x_m)|\leq(\xi)\leq n_\lambda^{i-1}\left[\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\cdot\right.$$

$$\left.\left\|\frac{\partial^{n_\lambda}f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{i}[a_j,b_j]}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\right], \tag{41}$$

$i=1,...,m;\ \lambda=1,...,r.$

Consequently we find

$$|S(f_1,...,f_r)(x)|\leq\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\left[n_\lambda^{i-1}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\cdot\right.\right.\right.$$

$$\left\| P_{n_\lambda - 1} \right\|_{\infty,[a_i,b_i]} \left\| \frac{\partial^{n_\lambda} f_\lambda (..., x_{i+1}, ..., x_m)}{\partial x_i^{n_\lambda}} \right\|_{\infty, \prod_{j=1}^{i} [a_j,b_j]} \Bigg])) =: \theta_3 (x) . \quad (42)$$

Finally we derive that

$$|S (f_1, ..., f_r) (x)| \leq \min \{\theta_1 (x) , \theta_2 (x) , \theta_3 (x)\} . \quad (43)$$

We have proved the following general multivariate Ostrowski type inequality for several functions.

Theorem 11 *Let f_λ, $\lambda = 1, ..., r \in \mathbb{N} - \{1\}$, as in Assumptions 5 or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ, $x = (x_1, ..., x_m)$, $s = (s_1, ..., s_m) \in \prod_{i=1}^{m} [a_i, b_i]$. Here $A_{i\lambda} (x_i, ..., x_m)$ as in (30), $i = 1, ..., m$. We put*

$$S (f_1, ..., f_r) (x) := \sum_{\lambda=1}^{r} \left\{ \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho (x) \right) \left[f_\lambda (x) - \frac{n_\lambda^m}{\prod_{i=1}^{m} (b_i - a_i)} \cdot \right.\right.$$

$$\left.\left. \left(\int_{\prod_{i=1}^{m} [a_i,b_i]} f_\lambda (s) \, ds \right) - \sum_{i=1}^{m} A_{i\lambda} (x_i, ..., x_m) \right] \right\} . \quad (44)$$

Here $\theta_1 (x)$ is as in (37). Let $p_{li} > 1 : \sum_{li=1}^{3} \frac{1}{p_{li}} = 1$, $i = 1, ..., m$, and $\theta_2 (x)$ as in (40). And $\theta_3 (x)$ as in (42). Then

$$|S (f_1, ..., f_r) (x)| \leq \min \{\theta_1 (x) , \theta_2 (x) , \theta_3 (x)\} . \quad (45)$$

We continue with

Remark 12 *Additionally assume that $\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}$ are continuous on $\prod_{j=1}^{m} [a_j, b_j]$ for all $i = 1, ..., m$; $\lambda = 1, ..., r$.*

We define and observe

$$W := \int_{\prod_{j=1}^{m} [a_j,b_j]} S (f_1, ..., f_r) (x) \, dx = r \int_{\prod_{j=1}^{m} [a_j,b_j]} \left(\prod_{\substack{\rho=1 \\ \rho \neq \lambda}}^{r} f_\rho (x) \right) dx - \quad (46)$$

$$\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\sum_{\lambda=1}^{r}n_{\lambda}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)dx\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]}f_{\lambda}(s)\,ds\right)-$$

$$\sum_{\lambda=1}^{r}\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)\left(\sum_{i=1}^{m}A_{i\lambda}(x_i,...,x_m)\right)\right)dx=$$

$$\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}(x)\right)\left(\sum_{i=1}^{m}B_{i\lambda}(x_i,...,x_m)\right)\right)dx\right\}. \qquad (47)$$

Clearly here $B_{i\lambda}(x_i,...,x_m)$ is a continuous function for all $i=1,...,m$; $\lambda=1,...,r$.

Hence

$$|W|\overset{(47)}{\leq}\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_{\rho}(x)|\right)\left(\sum_{i=1}^{m}|B_{i\lambda}(x_i,...,x_m)|\right)\right\}dx\right\}=:(\omega_1). \qquad (48)$$

That is

$$|W|\leq\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_{\rho}\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\left(\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx\right)\right)\right\}. \qquad (49)$$

From (41) we obtain

$$|B_{i\lambda}(x_i,...,x_m)|\leq$$

$$n_{\lambda}^{i-1}\|P_{n_{\lambda}-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_{\lambda}}f_{\lambda}}{\partial x_i^{n_{\lambda}}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right), \qquad (50)$$

all $i=1,...,m$; $\lambda=1,...,r$.

Then

$$\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx\leq$$

$$\frac{n_{\lambda}^{i-1}\|P_{n_{\lambda}-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_{\lambda}}f_{\lambda}}{\partial x_i^{n_{\lambda}}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}}{2(b_i-a_i)}.$$

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{m}(b_j-a_j)\right)\left(\int_{a_i}^{b_i}\left((b_i-x_i)^2+(x_i-a_i)^2\right)dx_i\right), \tag{51}$$

and consequently it holds,

$$\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx \leq$$

$$\frac{(b_i-a_i)\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}, \tag{52}$$

for all $i=1,...,m$; $\lambda=1,...,r$.

Hence

$$\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}(x_i,...,x_m)|\,dx\right)\leq\left(\frac{\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}\right)\cdot$$

$$\left(\sum_{i=1}^{m}\left[(b_i-a_i)\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}n_\lambda^{i-1}\right]\right), \tag{53}$$

for $\lambda=1,...,r$.

Using (49) and (53) we obtain

$$|W|\leq\left(\frac{\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}\right)\left[\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\cdot\right.\right.$$

$$\left.\left.\left(\sum_{i=1}^{m}\left[(b_i-a_i)n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right)\right\}\right]=:A_1. \tag{54}$$

Notice next that

$$(\omega_1) = \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)|B_{i\lambda}(x_i,...,x_m)|\,dx\right) =: (\omega_2)\,. \quad (55)$$

Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. Then

$$(\omega_2) \leq \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}\|_{L_q\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}}. \quad (56)$$

We have also proved that

$$|W| \leq \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}\|_{L_q\left(\prod_{j=i}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}} =: A_2. \quad (57)$$

From (50) we get

$$|B_{i\lambda}(x_i,...,x_m)| \leq n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\left(\frac{b_i-a_i}{2}\right), \quad (58)$$

all $i=1,...,m$; $\lambda=1,...,r$.

Thus it holds

$$\sum_{i=1}^{m}|B_{i\lambda}(x_i,...,x_m)| \leq$$

$$\frac{1}{2}\left\{\sum_{i=1}^{m}\left[(b_i-a_i)\,n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right\}, \quad (59)$$

$\lambda=1,...,r$.

Using (48) and (59) we finally derive

$$|W| \leq \frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}[\sum_{i=1}^{m}\left[(b_i-a_i)\,n_\lambda^{i-1}\|P_{n_\lambda-1}\|_{\infty,[a_i,b_i]}\cdot\right.\right.\right.$$

$$\left\|\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\Bigg]\Bigg]\Bigg\}\Bigg\} =: A_3. \tag{60}$$

We have proved the following general multivariate Grüss inequality.

Theorem 13 *Let f_λ, $\lambda = 1, ..., r \in \mathbb{N}-\{1\}$, as in Assumotions 5 plus $\frac{\partial^{n_\lambda} f_\lambda}{\partial x_i^{n_\lambda}}$ are continuous on $\prod_{j=1}^{m}[a_j,b_j]$ for all $i = 1, ..., m$; $\lambda = 1, ..., r$, or Brief Assumptions 6; $n_\lambda \in \mathbb{N}$ associated with f_λ. Here $A_{i\lambda}(x_i, ..., x_m)$ as in (30), and $B_{i\lambda}(x_i, ..., x_m)$ as in (31), $i = 1, ..., m$. We set*

$$W := r\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)dx -$$

$$\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\sum_{\lambda=1}^{r} n_\lambda^m\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)dx\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) -$$

$$\sum_{\lambda=1}^{r}\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left(\sum_{i=1}^{m} A_{i\lambda}(x_i, ..., x_m)\right)\right)dx. \tag{61}$$

Let A_1 as in (54); $p, q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, A_2 as in (57), and A_3 as in (60).

Then

$$|W| \leq \min\{A_1, A_2, A_3\}. \tag{62}$$

4 Applications

We apply Theorems 8, 11 and 13 for the case of $n_1 = n_2 = ... = n_r = 1$.

We simplify General Assumptions 5 and Brief Assumptions 6, respectively, as follows:

General Assumptions 14 *Let $f : \prod_{j=1}^{m}[a_j,b_j] \to \mathbb{R}$ satisfying:*

1) for $j = 1, ..., m$ we have that $f(x_1, x_2, ..., x_{j-1}, s_j, x_{j+1}, ..., x_m)$ is absolutely continuous in $s_j \in [a_j, b_j]$, for every $(x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_m) \in \prod_{\substack{i=1\\ i\neq j}}^{m}[a_i,b_i]$, *2) for* $j=1,...,m$ *we have that* $\frac{\partial f(s_1,...,s_j,x_{j+1},...,x_m)}{\partial x_j}$ *is continuous on* $\prod_{i=1}^{j}[a_i,b_i]$, *for every* $(x_{j+1},...,x_m)\in\prod_{i=j+1}^{m}[a_i,b_i]$,

3) f *is continuous on* $\prod_{i=1}^{m}[a_i,b_i]$.

Brief Assumptions 15 *Let* $f:\prod_{j=1}^{m}[a_j,b_j]\to\mathbb{R}$ *with* $\frac{\partial^l f}{\partial x_i^l}$ *for* $l=0,1;\ j=1,...,m$, *are continuous on* $\prod_{j=1}^{m}[a_j,b_j]$.

We give the following multivariate representation result which is an application of Theorem 8

Theorem 16 *Let* f *as in General Assumtions 14 or Brief Assumtions 15. Then*

$$f(x_1,...,x_n)=\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\int_{\prod_{j=1}^{m}[a_j,b_j]}f(s_1,...,s_m)\,ds_1...ds_m+\sum_{i=1}^{m}T_i^*(x_i,...,x_m), \tag{63}$$

where

$$T_i^*(x_i,...,x_m)=\frac{1}{\prod_{j=1}^{i}(b_j-a_j)}\cdot$$

$$\int_{a_1}^{b_1}...\int_{a_i}^{b_i}q(x_i,s_i)\frac{\partial f(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i}ds_1...ds_i, \tag{64}$$

are continuous functions for all $i=1,...,m$.

Next we make

Remark 17 *Let* f_λ *as in Assumptions 14 or 15,* $\lambda=1,...,r$. *Then*

$$f_\lambda(x)=\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\int_{\prod_{j=1}^{m}[a_j,b_j]}f_\lambda(s)\,ds+\sum_{i=1}^{m}T_{i\lambda}^*(x_i,...,x_m). \tag{65}$$

Here the corresponding

$$A_{i\lambda}^*(x_i,...,x_m)=0,$$

and

$$B_{i\lambda}^{*}\left(x_{i},...,x_{m}\right)=\frac{1}{\prod_{j=1}^{i}\left(b_{j}-a_{j}\right)}\cdot$$

$$\int_{a_{1}}^{b_{1}}...\int_{a_{i}}^{b_{i}}q\left(x_{i},s_{i}\right)\frac{\partial f_{\lambda}\left(s_{1},...,s_{i},x_{i+1},...,x_{m}\right)}{\partial x_{i}}ds_{1}...ds_{i}, \tag{66}$$

for all $i=1,...,m;\ \lambda=1,...,r.$

That is

$$T_{i\lambda}^{*}\left(x_{i},...,x_{m}\right)=B_{i\lambda}^{*}\left(x_{i},...,x_{m}\right), \tag{67}$$

$i=1,...,m;\ \lambda=1,...,r.$

We call and have the identity

$$S^{*}\left(f_{1},...,f_{r}\right)(x):=$$

$$\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}\left(x\right)\right)\left[f_{\lambda}\left(x\right)-\frac{1}{\prod_{j=1}^{m}\left(b_{j}-a_{j}\right)}\left(\int_{\prod_{j=1}^{m}[a_{j},b_{j}]}f_{\lambda}\left(s\right)ds\right)\right]\right\}$$

$$=\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_{\rho}\left(x\right)\right)\left(\sum_{i=1}^{m}B_{i\lambda}^{*}\left(x_{i},...,x_{m}\right)\right)\right), \tag{68}$$

true for any fixed $x\in\prod_{j=1}^{m}\left[a_{j},b_{j}\right].$

Hence it holds

$$\left|S^{*}\left(f_{1},...,f_{r}\right)(x)\right|\leq\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\left|f_{\rho}\left(x\right)\right|\right)\left(\sum_{i=1}^{m}\left|B_{i\lambda}^{*}\left(x_{i},...,x_{m}\right)\right|\right)\right). \tag{69}$$

We obtain:

1) From (37) we derive

$$\left|S^{*}\left(f_{1},...,f_{r}\right)(x)\right|\leq\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\left|f_{\rho}\left(x\right)\right|\right)\left(\sum_{i=1}^{m}\frac{1}{\prod_{j=1}^{i}\left(b_{j}-a_{j}\right)}\cdot\right.\right.$$

$$\left[\left(\frac{(b_i-a_i)+|a_i+b_i-2x_i|}{2}\right)\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{L_1\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]\Bigg)\Bigg) \tag{70}$$

$$=: \theta_1^*(x).$$

2) Let $p_{li}>1:\sum_{li=1}^{3}\frac{1}{p_{li}}=1$. *Then by (40) we derive*

$$|S^*(f_1,...,f_r)(x)| \leq$$

$$\sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\frac{1}{\prod_{j=1}^{i}(b_j-a_j)}\left[(b_i-a_i)^{\frac{1}{p_{1i}}}\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{p_{1i}}+\frac{1}{p_{2i}}}\cdot\right.\right.\right.$$

$$\left(\frac{(b_i-x_i)^{p_{2i}+1}+(x_i-a_i)^{p_{2i}+1}}{p_{2i}+1}\right)^{\frac{1}{p_{2i}}}\cdot$$

$$\left.\left.\left.\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{L_{p_{3i}}\left(\prod_{j=1}^{i}[a_j,b_j]\right)}\right]\right)\right) =: \theta_2^*(x). \tag{71}$$

3) We get by (42) that

$$|S^*(f_1,...,f_r)(x)| \leq \sum_{\lambda=1}^{r}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}|f_\rho(x)|\right)\left(\sum_{i=1}^{m}\left(\frac{(b_i-x_i)^2+(x_i-a_i)^2}{2(b_i-a_i)}\right)\cdot\right.\right.$$

$$\left.\left\|\frac{\partial f_\lambda(...,x_{i+1},...,x_m)}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{i}[a_j,b_j]}\right] =: \theta_3^*(x). \tag{72}$$

So as an applications of Theorem 11 we give the following multivariate Ostrowski type inequality.

Theorem 18 *All as in Remark 17. Then*

$$|S^*(f_1,...,f_r)(x)| \leq \min\{\theta_1^*(x),\theta_2^*(x),\theta_3^*(x)\}. \tag{73}$$

We also make

Remark 19 *Here Assumptions 14 hold and* $\frac{\partial f_\lambda}{\partial x_i}$ *are continuous on* $\prod_{j=1}^{m}[a_j,b_j]$ *for all* $i=1,...,m;\ \lambda=1,...,r,$ *or Assumptions 15 are valid.*

We set

$$W^* = \int_{\prod_{j=1}^{m}[a_j,b_j]} S^*(f_1,...,f_r)(x)\,dx = r\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\rho=1}^{r} f_\rho(x)\right)dx - \tag{74}$$

$$\frac{1}{\prod_{j=1}^{m}(b_j-a_j)}\sum_{\lambda=1}^{r}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)dx\right)\left(\int_{\prod_{i=1}^{m}[a_i,b_i]} f_\lambda(s)\,ds\right) =$$

$$\sum_{\lambda=1}^{r}\left\{\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho(x)\right)\left(\sum_{i=1}^{m}B_{i\lambda}^*(x_i,...,x_m)\right)\right)dx\right\}. \tag{75}$$

Here $B_{i\lambda}^*$ is a continuous for any $i=1,...,m;\ \lambda=1,...,r.$

Then

1) following (49) we find

$$|W^*| \le \sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\left(\sum_{i=1}^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}|B_{i\lambda}^*(x_i,...,x_m)|\,dx\right)\right)\right\}. \tag{76}$$

We also get that

$$|W^*| \overset{(54)}{\le} \left(\frac{\prod_{j=1}^{m}(b_j-a_j)}{3}\right)\left[\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\right.\right.$$

$$\left.\left.\left(\sum_{i=1}^{m}(b_i-a_i)\left\|\frac{\partial f_\lambda}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\right\}\right] =: A_1^*. \tag{77}$$

2) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1.$ Then

$$|W^*| \overset{(57)}{\le} \sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\|B_{i\lambda}^*\|_{L_q\left(\prod_{j=1}^{m}[a_j,b_j]\right)}. \tag{78}$$

$$\left(\prod_{j=1}^{i-1}(b_j-a_j)\right)^{\frac{1}{q}} =: A_2^*.$$

3) From (60) we obtain

$$|W^*| \leq \frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r} f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left[\sum_{i=1}^{m}\left[(b_i-a_i)\left\|\frac{\partial f_\lambda}{\partial x_i}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right]\right\}\right\} \tag{79}$$

$$=: A_3^*.$$

We have proved the following multivariate Grüss type inequality as an application of Theorem 13.

Theorem 20 *Here all as in Remark 19. We derive*

$$|W^*| \leq \min\{A_1^*, A_2^*, A_3^*\}. \tag{80}$$

References

[1] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2 (1882), 93-98.

[2] Lj. Dedic, M. Matic and J. Pečaric, *On some generalizations of Ostrowski's inequality for Lipschitz functions and functions of bounded variation*, Math. Inequal. Appl., 3(1)(2000), 1-14.

[3] Lj. Dedic, J. Pečaric and N. Ujevic, *On generalizations of Ostrowski inequality and some related results*, Czechoslovak Math. J., 53(128)(2003), 173-189.

[4] S.S. Dragomir and S. Wang, *A new inequality of Ostrowski type in L_p norm*, Indian Jour. Math., 40(1998), 299-304.

[5] A.M. Fink, *Bounds of the deviation of a function from its averages*, Czechoslovak. Math. J., 42(117)(1992), 289-310.

[6] G. Grüss, *Über das Maximum des absoluten Betrages von* $\left[\left(\frac{1}{b-a}\right)\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right)\right]$, Math. Z. 39 (1935), 215-226.

[7] A. Ostrowski, *Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.