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# Fractional Ostrowski and Grüss type inequalities involving several functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

**Abstract**

Using Caputo fractional left and right Taylor formulae we establish mixed fractional Ostrowski and Grüss type inequalities involving several functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

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## 1 Introduction

The following results motivate initially our work.

**Theorem 1** *(1938, Ostrowski [10]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \quad (1)$$

*for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.*

Ostrowski type inequalities have great applications to integration approximations in Numerical Analysis.

**Theorem 2** *(1882, Čebyšev [6]) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a,b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right) \left( \frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$
$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{2}$$

*The above integrals are assumed to exist.*

Grüss type inequalities have applications to Probability.

We are also biggly inspired by the beautiful work of Deepak B. Pachpatte [11]. His article contains great ideas, but it is marred by many minor errors.

So here we present mixed fractional Ostrowski and Grüss type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [1], Chapters 24-26 and [3], Chapters 2-6.

## 2 Background

Let $\nu \geq 0$; the operator $I_{a+}^{\nu}$, defined for $f \in L_1[(a,b)]$ is given by

$$I_{a+}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t) dt, \tag{3}$$

for $a \leq x \leq b$, is called the left Riemann-Liouville fractional integral operator of order $\nu$. For $\nu = 0$, we set $I_{a+}^0 := I$, the identity operator, see [1], p. 392, also [7].

Let $\nu \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^n([a,b])$ (it means $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions).

Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) dt = \left( I_{a+}^{n-\nu} f^{(n)} \right)(x), \tag{4}$$

and it exists almost everywhere for $x \in [a,b]$.

See Corollary 16.8, p. 394 of [1], and [7], pp. 49-50.

We need also the left Caputo fractional Taylor formula, see [1], p. 395 and [7], p. 54.

**Theorem 3** *Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + I_{a+}^{\nu} D_{*a}^{\nu} f(x), \quad \forall\, x \in [a,b]. \tag{5}$$

*Here $I_{a+}^{\nu} D_{*a}^{\nu} f \in AC^n([a,b])$.*

Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator ([2], [8], [9]) of order $\alpha$ is denoted by

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \quad \forall\, x \in [a,b]. \tag{6}$$

We set $I_{b-}^{0} := I$, the identity operator.

Let now $f \in AC^m([a,b])$, $m \in \mathbb{N}$, with $m := \lceil \alpha \rceil$.

We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \tag{7}$$

we set $D_{b-}^{0} f := f$, that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{8}$$

We need the right Caputo fractional Taylor formula.

**Theorem 4** *([2]) Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m := \lceil a \rceil$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^z (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z)\, dz. \tag{9}$$

We need

**Proposition 5** *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$; $x, x_0 \in [a,b] : x \geq x_0$. Then $D_{*x_0}^{\alpha} f(x)$ is continuous in $x_0$.*

**Proposition 6** *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$; $x, x_0 \in [a,b] : x \leq x_0$. Then $D_{x_0-}^{\alpha} f(x)$ is continuous in $x_0$.*

We also mention

**Theorem 7** *([4], p. 362) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in $(x, x_0)$ from $[a,b]^2$ into $\mathbb{R}$.*

**Convention 8** *([4], p. 360) We suppose that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{10}$$

*and*

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{11}$$

*for all $x, x_0 \in [a,b]$.*

Finally we are motivated by the following mixed Caputo fractional Ostrowski type inequalities.

**Theorem 9** *([3], p. 44) Let* $[a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$, *and* $\left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}$, $\left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} < \infty$, $x_0 \in [a,b]$. *Assume* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+2)} \cdot$$

$$\left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]} (x_0-a)^{\alpha+1} + \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} (b-x_0)^{\alpha+1} \right\} \leq \tag{12}$$

$$\frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{\infty,[a,x_0]}, \left\| D^{\alpha}_{*x_0} f \right\|_{\infty,[x_0,b]} \right\} (b-a)^{\alpha}. \tag{13}$$

*Inequality (12) is sharp, infact it is attained.*

**Theorem 10** *([3], p. 45) Le* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$ *and* $D^{\alpha}_{x_0-} f \in L_1([a,x_0])$, $D^{\alpha}_{*x_0} f \in L_1([x_0,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha+1)} \cdot \tag{14}$$

$$\left\{ (x_0-a)^{\alpha} \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])} + (b-x_0)^{\alpha} \left\| D^{\alpha}_{*x_0} f \right\|_{L_1([x_0,b])} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha+1)} \max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_1([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_1([x_0,b])} \right\} (b-a)^{\alpha-1}. \tag{15}$$

**Theorem 11** *([3], p. 47) Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f \in AC^m([a,b])$. *Suppose that* $f^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$. *Assume* $D^{\alpha}_{x_0-} f \in L_q([a,x_0])$, *and* $D^{\alpha}_{*x_0} f \in L_q([x_0,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(x_0) \right| \leq \frac{1}{(b-a)\,\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \cdot \tag{16}$$

$$\left\{ (x_0-a)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])} + (b-x_0)^{\alpha+\frac{1}{p}} \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha)\,(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \cdot$$

$$\max \left\{ \left\| D^{\alpha}_{x_0-} f \right\|_{L_q([a,x_0])}, \left\| D^{\alpha}_{*x_0} f \right\|_{L_q([x_0,b])} \right\} (b-a)^{\alpha-\frac{1}{q}}. \tag{17}$$

In this article we generalize Theorems 9-11 for several functions. We also produce Caputo fractional Grüss type inequalities for several functions.

## 3 Main Results

We start with Caputo fractional mixed Ostrowski type inequalities involving several functions.

**Theorem 12** *Let* $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$, *with* $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $i = 1,...,r$. *Assume that* $\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,...,r$. *Denote by*

$$\theta(f_1,...,f_r)(x_0) := r\int_a^b \left( \prod_{k=1}^r f_k(x) \right) dx - \sum_{i=1}^r \left[ f_i(x_0) \int_a^b \left( \prod_{\substack{j=1 \\ j\neq i}}^r f_j(x) \right) dx \right]. \tag{18}$$

*Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^r \left[ \left[ \left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1} \left( \prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right. \tag{19}$$

$$\left. + \left[ \left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1} \left( \prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right].$$

*Inequality (19) is sharp, infact it is attained.*

**Theorem 13** *Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. *Suppose that* $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$; $x_0 \in [a,b]$ *and* $D_{x_0-}^{\alpha} f_i \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f_i \in L_1([x_0,b])$, *for all* $i = 1,...,r$. *Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^r \left[ \left[ \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a,x_0])} I_{a+}^{\alpha} \left( \prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right. \tag{20}$$

$$\left. + \left[ \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0,b])} I_{b-}^{\alpha} \left( \prod_{\substack{j=1 \\ j\neq i}}^r |f_j(x_0)| \right) \right] \right].$$

**Theorem 14** *Let* $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. *Suppose that* $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $x_0 \in [a,b]$; $i = 1,...,r$. *Assume* $D_{x_0-}^{\alpha} f_i \in L_q([a,x_0])$, *and* $D_{*x_0}^{\alpha} f_i \in L_q([x_0,b])$, $i = 1,...,r$. *Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\alpha\right)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{21}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

**Proof.** of Theorems 12-14. Here $x_0\in[a,b]$. Since $f_i^{(k)}(x_0)=0$, $k=1,...,m-1$; $i=1,...,r$, we have by Theorem 3 that

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-z\right)^{\alpha-1}D_{*x_0}^{\alpha}f_i\left(z\right)dz,\quad\forall\ x\in\left[x_0,b\right], \tag{22}$$

and by Theorem 4 that

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{x_0}\left(z-x\right)^{\alpha-1}D_{x_0-}^{\alpha}f_i\left(z\right)dz,\quad\forall\ x\in\left[a,x_0\right]; \tag{23}$$

for all $i=1,...,r$.

Multiplying (22) and (23) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)$ we get, respectively,

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-z\right)^{\alpha-1}D_{*x_0}^{\alpha}f_i\left(z\right)dz,\quad\forall\ x\in\left[x_0,b\right], \tag{24}$$

and

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)}{\Gamma\left(\alpha\right)}\int_{x}^{x_0}\left(z-x\right)^{\alpha-1}D_{x_0-}^{\alpha}f_i\left(z\right)dz,\quad\forall\ x\in\left[a,x_0\right]; \tag{25}$$

for all $i=1,...,r$.

Adding (24) and (25), separately, we obtain

$$r\left(\prod_{k=1}^{r}f_k\left(x\right)\right)-\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right],\quad \forall\ x\in[x_0,b],\quad (26)$$

and

$$r\left(\prod_{k=1}^{r} f_k(x)\right)-\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x_0)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right],\quad \forall\ x\in[a,x_0].\quad (27)$$

Next we integrate (26) and (27) with respect to $x\in[a,b]$. We have

$$r\int_{x_0}^{b}\left(\prod_{k=1}^{r} f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right],\qquad (28)$$

and

$$r\int_{a}^{x_0}\left(\prod_{k=1}^{r} f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right].\qquad (29)$$

Finally adding (28) and (29) we obtain the useful identity

$$\theta(f_1,...,f_r)(x_0):=$$

$$r\int_{a}^{b}\left(\prod_{k=1}^{r} f_k(x)\right)dx-\sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right]\Bigg]. \tag{30}$$

Hence it holds

$$|\theta(f_1,...,f_r)(x_0)| \le$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x}^{x_0}(z-x)^{\alpha-1}\left|D_{x_0-}^{\alpha}f_i(z)\right|dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left|D_{*x_0}^{\alpha}f_i(z)\right|dz\right)dx\right]\right] =: (*). \tag{31}$$

We observe that

$$(*) \le \frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(x_0-x)^{\alpha}dx\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(x-x_0)^{\alpha}dx\right]\right] = \tag{32}$$

$$\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \tag{33}$$

Based on Proposition 15.114, p. 388, [1] we get that $I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right) \in AC([a,b])$, so at $x_0$ is finite.

Also, based on [5] we get that $I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right) \in AC([a,b])$, so at $x_0$ is finite.

Next we prove that (19) is sharp, namely it is attained.

Set

$$\overline{f_i}(x) := \begin{cases}(x-x_0)^{\alpha}, & x\in[x_0,b],\\ (x_0-x)^{\alpha}, & x\in[a,x_0],\end{cases}$$

$\alpha > 0$, $a \leq x_0 \leq b$, $i = 1, ..., r$.

Observe that $\overline{f_i} \in AC^m([x_0, b])$, and in $AC^m([a, x_0])$. See that $\overline{f_{i-}}^{(k)}(x_0) = \overline{f_{i+}}^{(k)}(x_0) = 0$, $k = 0, 1, ..., m-1$. Hence there exists $\overline{f_i}^{(m-1)}$ at $x_0$, also $\overline{f_i}^{(m-1)} \in AC([a, b])$. That is $\overline{f_i} \in AC^m([a, b])$, $i = 1, ..., r$.

We find that

$$\left\| D_{x_0-}^{\alpha} \overline{f_i} \right\|_{\infty, [a, x_0]} = \Gamma(\alpha + 1),$$

and

$$\left\| D_{*x_0}^{\alpha} \overline{f_i} \right\|_{\infty, [x_0, b]} = \Gamma(\alpha + 1).$$

Consequently it holds

$$L.H.S.(19) = r \left[ \frac{(b - x_0)^{\alpha r + 1} + (x_0 - a)^{\alpha r + 1}}{\alpha r + 1} \right] = R.H.S.(19),$$

proving optimality of (19).

Hence proving Theorem 12.

Next we notice, for $\alpha \geq 1$, that

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a, x_0])} \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (x_0 - x)^{\alpha - 1} dx \right] + \right.$$

$$\left. \left[ \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0, b])} \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) (x - x_0)^{\alpha - 1} dx \right] \right] = \tag{34}$$

$$\sum_{i=1}^{r} \left[ \left[ \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_1([a, x_0])} I_{a+}^{\alpha} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[ \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_1([x_0, b])} I_{b-}^{\alpha} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right], \tag{35}$$

proving Theorem 13.

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[ \left[ \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x)| \right) \right. \right.$$

$$\left(\int_{x}^{x_0}(z-x)^{p(\alpha-1)}dz\right)^{\frac{1}{p}}\left(\int_{x}^{x_0}\left|D_{x_0-}^{\alpha}f_i(z)\right|^{q}dz\right)^{\frac{1}{q}}dx\Bigg]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x_0}^{x}(x-z)^{p(\alpha-1)}dz\right)^{\frac{1}{p}}\left(\int_{x_0}^{x}\left|D_{*x_0}^{\alpha}f_i(z)\right|^{q}dz\right)^{\frac{1}{q}}dx\right]\Bigg]\leq \tag{36}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)\frac{(x_0-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{x_0-}^{\alpha}f_i\right\|_{q,[a,x_0]}dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)\frac{(x-x_0)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{*x_0}^{\alpha}f_i\right\|_{q,[x_0,b]}dx\right]\right]= \tag{37}$$

$$\frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\cdot$$

$$\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{q,[a,x_0]}\int_{a}^{x_0}(x_0-x)^{\alpha+\frac{1}{p}-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)dx\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{q,[x_0,b]}\int_{x_0}^{b}(x-x_0)^{\alpha+\frac{1}{p}-1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)dx\right]\right]= \tag{38}$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{q,[a,x_0]}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right. \tag{39}$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{q,[x_0,b]}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right],$$

that is proving Theorem 14. ■

Next follow Caputo fractional Grüss type inequalities for several functions.

**Theorem 15** *Let* $x_0\in[a,b]\subset\mathbb{R}$, $0<\alpha\leq1$, $f_i\in AC([a,b])$, $i=1,...,r\in\mathbb{N}-\{1\}$. *Assume that* $\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}$, $\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}<\infty$, $i=1,...,r$. *Denote by*

$$\Delta(f_1,...,f_r)(x_0):=r(b-a)\int_{a}^{b}\left(\prod_{k=1}^{r}f_k(x)\right)dx- \tag{40}$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right)\right].$$

*Then*

$$|\Delta(f_1,...,f_r)| \leq (b-a)\cdot$$

$$\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \quad (41)$$

**Theorem 16** *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{q}<\alpha\leq1$, *and* $f_i\in AC([a,b])$, $i=1,...,r\in\mathbb{N}-\{1\}$, $x_0\in[a,b]$. *Assume that* $\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}$, *and* $\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}<\infty$, $i=1,...,r$. *Then*

$$|\Delta(f_1,...,f_r)| \leq \frac{(b-a)\,\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\cdot$$

$$\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]. \quad (42)$$

**Proof.** of Theorems 15, 16. Here $0<\alpha\leq1$, i.e. $m=1$, and $f_i\in AC([a,b])$, $i=1,...,r$. Now we are not tight up to any initial conditions, i.e. (22) and (23) are valid without initial conditions. Clearly here $\theta(f_1,...,f_r)(x_0)\in AC([a,b])$. Integrating (30) over $[a,b]$ with respect to $x_0\in[a,b]$ we derive

$$\Delta(f_1,...f_r) := \int_a^b \theta(f_1,...,f_r)(x_0)\,dx_0 = \quad (43)$$

$$r(b-a)\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)dx-\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,dx\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\int_a^b \left\{\sum_{i=1}^r \left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_x^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.\right.$$

$$\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right]\right]\right\}dx_0. \quad (44)$$

Hence it holds

$$|\Delta(f_1,...,f_r)| \leq$$

$$\frac{1}{\Gamma(\alpha)}\int_a^b \left|\sum_{i=1}^r \left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_x^{x_0}(z-x)^{\alpha-1}D_{x_0-}^{\alpha}f_i(z)\,dz\right)dx\right]+\right.\right.$$

$$\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right)\left(\int_{x_0}^{x}(x-z)^{\alpha-1}D_{*x_0}^{\alpha}f_i(z)\,dz\right)dx\right]\right]\right|dx_0 =: (**)\,. \quad (45)$$

Using (19) we have

$$(**) \leq (b-a)\sum_{i=1}^r \left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^r |f_j(x_0)|\right)\right]\right.$$

$$\left.+\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^r |f_j(x_0)|\right)\right]\right], \quad (46)$$

proving Theorem 15.

When $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $\frac{1}{q}<\alpha\leq 1$, by (21) we get

$$(**) \leq \frac{(b-a)\,\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)}\cdot$$

$$\sum_{i=1}^r \left[\left[\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_i\right\|_{q,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^r |f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_i\right\|_{q,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\j\neq i}}^r |f_j(x_0)|\right)\right]\right], \quad (47)$$

proving Theorem 16. ■

## References


[1] G.A. Anastassiou, *Fractional Differentiation Inequalities*, Research Monograph, Springer, New York, 2009.

[2] G.A. Anastassiou, *On Right fractional calculus*, Chaos, Solitons and Fractals, 42 (2009), 365-376.

[3] G.A. Anastassiou, *Advances on Fractional Inequalities*, Research Monograph, Springer, New York, 2011.

[4] G.A. Anastassiou, *Intelligent Mathematics: Computational Analysis*, Research Monograph, Springer, Berlin, Heidelberg, 2011.

[5] G.A. Anastassiou, *Fractional Representation Formulae and right fractional inequalities*, Mathematical and Computer Modelling, 54 (2011), (11-12), 3098-3115.

[6] P.L. Čebyšev, *Sur les expressions approximatives des intégrales définies par les aures prises entre les mêmes limites*, Proc. Math. Soc. Charkov, 2(1882), 93-98.

[7] Kai Diethelm, T*he Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1 st edition, Springer, New York, Heidelberg, 2010.

[8] A.M.A. El-Sayed and M. Gaber, *On the finite Caputo and finite Riesz derivatives*, Electronic Journal of Theoretical Physics, Vol. 3, No. 12 (2006), 81-95.

[9] R. Gorenflo and F. Mainardi, *Essentials of Fractional Calculus*, 2000, Maphysto Center, http://www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k0a.ps.

[10] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[11] D.B. Pachpatte, *On Right Caputo fractional Ostrowski Inequalities involving three functions*, International Journal of Analysis, Vol. 2013, Article ID 127061, 5 pages, http://dx.doi.org/10.1155/2013/127061.