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Further interpretation of some fractional Ostrowski and Grüss type inequalities

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Abstract

We further interpret and simplify earlier produced fractional Ostrowski and Grüss type inequalities involving several functions.

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1 Background

Let $\nu \geq 0$; the operator I_{a+}^{ν}, defined for $f \in L_1[(a,b)]$ is given by

$$I_{a+}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} f(t)\, dt, \tag{1}$$

for $a \leq x \leq b$, is called the left Riemann-Liouville fractional integral operator of order ν. For $\nu = 0$, we set $I_{a+}^{0} := I$, the identity operator, see [1], p. 392, also [7].

Let $\nu \geq 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f \in AC^n([a,b])$ (it means $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions).

Then the left Caputo fractional derivative is given by

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt = \left(I_{a+}^{n-\nu} f^{(n)}\right)(x), \tag{2}$$

and it exists almost everywhere for $x \in [a,b]$.

Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator ([2], [8], [9]) of order α is denoted by

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \quad \forall x \in [a,b]. \tag{3}$$

We set $I_{b-}^{0} := I$, the identity operator.

Let now $f \in AC^{m}([a,b])$, $m \in \mathbb{N}$, with $m := \lceil \alpha \rceil$.

We define the right Caputo fractional derivative of order $\alpha \geq 0$, by

$$D_{b-}^{\alpha} f(x) := (-1)^{m} I_{b-}^{m-\alpha} f^{(m)}(x), \tag{4}$$

we set $D_{b-}^{0} f := f$, that is

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)} \int_{x}^{b} (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz. \tag{5}$$

We need

Proposition 1 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \geq x_0$. Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.*

Proposition 2 *([4], p. 361) Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$; $x, x_0 \in [a,b] : x \leq x_0$. Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.*

We also mention

Theorem 3 *([4], p. 362) Let $f \in C^{m}([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

Convention 4 *([4], p. 360) We suppose that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{6}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{7}$$

for all $x, x_0 \in [a,b]$.

2 Motivation

We mention some Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 5 *([6]) Let $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_i \in AC^{m}([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$, with $f_i^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $i = 1, ..., r$. Assume that $\left\| D_{x_0-}^{\alpha} f_i \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1, ..., r$. Denote by*

$$\theta(f_1, ..., f_r)(x_0) := r \int_{a}^{b} \left(\prod_{k=1}^{r} f_k(x) \right) dx - \sum_{i=1}^{r} \left[f_i(x_0) \int_{a}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right]. \tag{8}$$

Then

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right. \tag{9}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

Inequality (9) is sharp, infact it is attained.

Theorem 6 *([6]) Let $\alpha \geq 1$, $m = \lceil \alpha \rceil$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$; $x_0 \in [a,b]$ and $D_{x_0-}^{\alpha}f_i \in L_1([a,x_0])$, $D_{*x_0}^{\alpha}f_i \in L_1([x_0,b])$, for all $i = 1,...,r$. Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right. \tag{10}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

Theorem 7 *([6]) Let $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, and $f_i \in AC^m([a,b])$, $i = 1,...,r \in \mathbb{N} - \{1\}$. Suppose that $f_i^{(k)}(x_0) = 0$, $k = 1,...,m-1$, $x_0 \in [a,b]$; $i = 1,...,r$. Assume $D_{x_0-}^{\alpha}f_i \in L_q([a,x_0])$, and $D_{*x_0}^{\alpha}f_i \in L_q([x_0,b])$, $i = 1,...,r$. Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_q([a,x_0])} I_{a+}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right. \tag{11}$$

$$\left.+\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_q([x_0,b])} I_{b-}^{\alpha+\frac{1}{p}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

Next we mention some Caputo fractional Grüss type inequalities for several functions.

Theorem 8 *([6]) Let $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_i \in AC([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$. Assume that $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{x_0 -} f_i \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{*x_0} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1, ..., r$. Denote by*

$$\Delta(f_1, ..., f_r) := r(b-a) \int_a^b \left(\prod_{k=1}^{r} f_k(x) \right) dx - \tag{12}$$

$$\sum_{i=1}^{r} \left[\left(\int_a^b f_i(x)\, dx \right) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) \right].$$

Then

$$|\Delta(f_1, ..., f_r)| \leq (b-a) \cdot$$

$$\sum_{i=1}^{r} \left[\left[\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{x_0 -} f_i \right\|_{\infty,[a,x_0]} \sup_{x_0 \in [a,b]} I^{\alpha+1}_{a+} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{*x_0} f_i \right\|_{\infty,[x_0,b]} \sup_{x_0 \in [a,b]} I^{\alpha+1}_{b-} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right]. \tag{13}$$

Theorem 9 *([6]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, and $f_i \in AC([a,b])$, $i = 1, ..., r \in \mathbb{N} - \{1\}$, $x_0 \in [a,b]$. Assume that $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{x_0 -} f_i \right\|_{L_q([a,x_0])}$, and $\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{*x_0} f_i \right\|_{L_q([x_0,b])} < \infty$, $i = 1, ..., r$. Then*

$$|\Delta(f_1, ..., f_r)| \leq \frac{(b-a)\, \Gamma\left(\alpha + \frac{1}{p}\right)}{(p(\alpha-1)+1)^{\frac{1}{p}}\, \Gamma(\alpha)} \cdot$$

$$\sum_{i=1}^{r} \left[\left[\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{x_0 -} f_i \right\|_{L_q([a,x_0])} \sup_{x_0 \in [a,b]} I^{\alpha+\frac{1}{p}}_{a+} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] + \right.$$

$$\left. \left[\sup_{x_0 \in [a,b]} \left\| D^{\alpha}_{*x_0} f_i \right\|_{L_q([x_0,b])} \sup_{x_0 \in [a,b]} I^{\alpha+\frac{1}{p}}_{b-} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \right] \right]. \tag{14}$$

3 Main Results

We make

Remark 10 *Let $g \in C([a,b])$, $\alpha > 0$, $x_0 \in [a,b] \subset \mathbb{R}$. Notice that*

$$I_{a+}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} g(z) \, dz. \tag{15}$$

Hence

$$\left| I_{a+}^{\alpha+1}(g)(x_0) \right| \leq \frac{1}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} |g(z)| \, dz \leq$$

$$\frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \int_a^{x_0} (x_0 - z)^{\alpha} \, dz = \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)} \frac{(x_0 - a)^{\alpha+1}}{(\alpha+1)}$$

$$= \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{16}$$

That is

$$\left| I_{a+}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}. \tag{17}$$

Similarly we have

$$I_{b-}^{\alpha+1}(g)(x_0) = \frac{1}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha} g(z) \, dz, \tag{18}$$

and

$$\left| I_{b-}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \int_{x_0}^{b} (z - x_0)^{\alpha} \, dz$$

$$= \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)} \frac{(b - x_0)^{\alpha+1}}{(\alpha+1)} = \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{19}$$

That is

$$\left| I_{b-}^{\alpha+1}(g)(x_0) \right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)} (b - x_0)^{\alpha+1}. \tag{20}$$

Consequently we derive

$$I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} |f_j(x_0)| \right) \overset{(17)}{\leq} \frac{\left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+2)} (x_0 - a)^{\alpha+1}, \tag{21}$$

and

$$I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\overset{(20)}{\leq}\frac{\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+2)}(b-x_0)^{\alpha+1}. \tag{22}$$

Therefore it holds

$$|\theta(f_1,...,f_r)(x_0)|\overset{(9)}{\leq}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}I_{a+}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}I_{b-}^{\alpha+1}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right]\overset{((21),(22))}{\leq}$$

$$\frac{1}{\Gamma(\alpha+2)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right](x_0-a)^{\alpha+1}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right](b-x_0)^{\alpha+1}\right]=:(\xi_1). \tag{24}$$

Call

$$M_1(f_1,...,f_r)(x_0):=\max_{i=1,\ldots,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{\infty,[a,x_0]},\left\|D_{*x_0}^{\alpha}f_i\right\|_{\infty,[x_0,b]}\right\}. \tag{25}$$

Then

$$(\xi_1)\leq\frac{M_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}(x_0-a)^{\alpha+1}+\right.$$

$$\left.\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}(b-x_0)^{\alpha+1}\right]=:(\xi_2). \tag{26}$$

Call

$$\psi_1(f_1,...,f_r)(x_0):=\max\left\{\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]},\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \tag{27}$$

So that

$$(\xi_2) \leq \frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq \frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{28}$$

We have proved simpler interpretations of Caputo fractional mixed Ostrowski type inequalities involving several functions.

Theorem 11 *Here all as in Theorem 5, $M_1(f_1,...,f_r)(x_0)$ as in (25) and $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then*

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}\left[(b-x_0)^{\alpha+1}+(x_0-a)^{\alpha+1}\right] \leq \tag{29}$$

$$\frac{M_1(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{30}$$

We make

Remark 12 *Let $g \in C([a,b])$, $\alpha \geq 1$, $x_0 \in [a,b] \subset \mathbb{R}$. We have that*

$$\left|I_{a+}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{31}$$

and

$$\left|I_{b-}^{\alpha}(g)(x_0)\right| \leq \frac{\|g\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{32}$$

Consequently we derive

$$I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right) \overset{(31)}{\leq} \frac{\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(x_0-a)^{\alpha}, \tag{33}$$

$$I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right) \overset{(32)}{\leq} \frac{\left\|\prod_{\substack{j=1\\ j\neq i}}^{r} f_j\right\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(b-x_0)^{\alpha}. \tag{34}$$

Therefore it holds

$$\left|\theta\left(f_1,...,f_r\right)(x_0)\right| \overset{(10)}{\leq} \sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])} I_{a+}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])} I_{b-}^{\alpha}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right] \overset{((33),(34))}{\leq}$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\right]\left(x_0-a\right)^{\alpha}+\right.$$

$$\left.\left[\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right]\left(b-x_0\right)^{\alpha}\right] =: (\eta)\,. \tag{35}$$

Call

$$M_2\left(f_1,...,f_r\right)(x_0) := \max_{i=1,...,r}\left\{\left\|D_{x_0-}^{\alpha}f_i\right\|_{L_1([a,x_0])},\left\|D_{*x_0}^{\alpha}f_i\right\|_{L_1([x_0,b])}\right\}. \tag{36}$$

Then

$$(\eta) \leq \frac{M_2\left(f_1,...,f_r\right)(x_0)}{\Gamma(\alpha+1)}\cdot$$

$$\sum_{i=1}^{r}\left[\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}\left(x_0-a\right)^{\alpha}+\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\left(b-x_0\right)^{\alpha}\right] \tag{37}$$

$$\leq \frac{M_2\left(f_1,...,f_r\right)(x_0)\,\psi_1\left(f_1,...,f_r\right)(x_0)}{\Gamma(\alpha+1)}\left[\left(b-x_0\right)^{\alpha}+\left(x_0-a\right)^{\alpha}\right] \tag{38}$$

$$\leq \frac{M_2\left(f_1,...,f_r\right)(x_0)\,\psi_1\left(f_1,...,f_r\right)(x_0)}{\Gamma(\alpha+1)}\left(b-a\right)^{\alpha}. \tag{39}$$

We have proved

Theorem 13 *Let all as in Theorem 6, $M_2\left(f_1,...,f_r\right)(x_0)$ as in (36) and $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then*

$$\left|\theta\left(f_1,...,f_r\right)(x_0)\right| \leq \frac{M_2\left(f_1,...,f_r\right)(x_0)\,\psi_1\left(f_1,...,f_r\right)(x_0)}{\Gamma(\alpha+1)}\left[\left(b-x_0\right)^{\alpha}+\left(x_0-a\right)^{\alpha}\right] \tag{40}$$

$$\leq \frac{M_2\left(f_1,...,f_r\right)(x_0)\,\psi_1\left(f_1,...,f_r\right)(x_0)}{\Gamma(\alpha+1)}\left(b-a\right)^{\alpha}. \tag{41}$$

Similarly we obtain

Theorem 14 *Let all as in Theorem 7. Call*

$$M_3(f_1,...,f_r)(x_0) := \max_{i=1,\dots,r}\left\{\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\}. \quad (42)$$

Here $\psi_1(f_1,...,f_r)(x_0)$ as in (27). Then

$$|\theta(f_1,...,f_r)(x_0)| \leq$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\left[(b-x_0)^{\alpha+\frac{1}{p}}+(x_0-a)^{\alpha+\frac{1}{p}}\right] \leq \quad (43)$$

$$\frac{M_3(f_1,...,f_r)(x_0)\,\psi_1(f_1,...,f_r)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}}. \quad (44)$$

Finally we give a simpler interpretation of Caputo fractional Grüss type inequalities (13), (14).

Theorem 15 *All as in Theorem 8. We define*

$$M_4(f_1,...,f_r) := \max_{i=1,\dots,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{\infty,[a,x_0]}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{\infty,[x_0,b]}\right\} \quad (45)$$

and

$$\psi_2(f_1,...,f_r)(x_0) :=$$

$$\max\left\{\sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}, \sum_{i=1}^{r}\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}\right\}. \quad (46)$$

Then

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_4(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\Gamma(\alpha+2)}(b-a)^{\alpha+2}. \quad (47)$$

Theorem 16 *All as in Theorem 9. We define*

$$M_5(f_1,...,f_r) := \max_{i=1,\dots,r}\left\{\sup_{x_0\in[a,b]}\left\|D^{\alpha}_{x_0-}f_i\right\|_{L_q([a,x_0])}, \sup_{x_0\in[a,b]}\left\|D^{\alpha}_{*x_0}f_i\right\|_{L_q([x_0,b])}\right\} \quad (48)$$

Here ψ_2 is as in (46). Then

$$|\Delta(f_1,...,f_r)| \leq \frac{2M_5(f_1,...,f_r)\,\psi_2(f_1,...,f_r)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}(b-a)^{\alpha+\frac{1}{p}+1}. \quad (49)$$

We finish with applications.

4 Applications

We apply above theory for $r=2$. In that case

$$\theta\left(f_1,f_2\right)\left(x_0\right)=2\int_a^b f_1\left(x\right)f_2\left(x\right)dx-f_1\left(x_0\right)\int_a^b f_2\left(x\right)dx-f_2\left(x_0\right)\int_a^b f_1\left(x\right)dx, \tag{50}$$

$x_0\in[a,b]$,

$$M_1\left(f_1,f_2\right)\left(x_0\right)=$$
$$\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{\infty,[a,x_0]},\left\|D_{x_0-}^{\alpha}f_2\right\|_{\infty,[a,x_0]},\left\|D_{*x_0}^{\alpha}f_1\right\|_{\infty,[x_0,b]},\left\|D_{*x_0}^{\alpha}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{51}$$

$$\psi_1\left(f_1,f_2\right)\left(x_0\right)=\max\left\{\|f_1\|_{\infty,[a,x_0]}+\|f_2\|_{\infty,[a,x_0]},\|f_1\|_{\infty,[x_0,b]}+\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{52}$$

$$M_2\left(f_1,f_2\right)\left(x_0\right)=\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_1([a,x_0])},\right.$$
$$\left.\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_1([a,x_0])},\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_1([x_0,b])},\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_1([x_0,b])}\right\}, \tag{53}$$

$$M_3\left(f_1,f_2\right)\left(x_0\right):=\max\left\{\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_q([a,x_0])},\right.$$
$$\left.\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_q([a,x_0])},\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_q([x_0,b])},\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{54}$$

$$\Delta\left(f_1,f_2\right)=2\left[(b-a)\int_a^b f_1\left(x\right)f_2\left(x\right)dx-\left(\int_a^b f_1\left(x\right)dx\right)\left(\int_a^b f_2\left(x\right)dx\right)\right], \tag{55}$$

$$M_4\left(f_1,f_2\right)=\max\left\{\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_1\right\|_{\infty,[a,x_0]},\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_2\right\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_1\right\|_{\infty,[x_0,b]},\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_2\right\|_{\infty,[x_0,b]}\right\}, \tag{56}$$

$$\psi_2\left(f_1,f_2\right)=\max\left\{\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[a,x_0]},\right.$$
$$\left.\sup_{x_0\in[a,b]}\|f_1\|_{\infty,[x_0,b]}+\sup_{x_0\in[a,b]}\|f_2\|_{\infty,[x_0,b]}\right\}, \tag{57}$$

and

$$M_5\left(f_1,f_2\right)=\max\left\{\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_1\right\|_{L_q([a,x_0])},\sup_{x_0\in[a,b]}\left\|D_{x_0-}^{\alpha}f_2\right\|_{L_q([a,x_0])},\right.$$
$$\left.\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_1\right\|_{L_q([x_0,b])},\sup_{x_0\in[a,b]}\left\|D_{*x_0}^{\alpha}f_2\right\|_{L_q([x_0,b])}\right\}, \tag{58}$$

above $p,q>1:\frac{1}{p}+\frac{1}{q}=1$.

Proposition 17 *Let* $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f_1, f_2 \in AC^m([a,b])$, *with* $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$. *Assume that* $\left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. *Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} \left[(b-x_0)^{\alpha+1} + (x_0-a)^{\alpha+1}\right] \tag{59}$$

$$\leq \frac{M_1(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+2)} (b-a)^{\alpha+1}. \tag{60}$$

Proof. By Theorem 11. ■

Proposition 18 *Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, *and* $f_1, f_2 \in AC^m([a,b])$. *Suppose that* $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$; $x_0 \in [a,b]$ *and* $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_1([a,x_0])$, $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_1([x_0,b])$. *Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} \left[(b-x_0)^{\alpha} + (x_0-a)^{\alpha}\right] \tag{61}$$

$$\leq \frac{M_2(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\Gamma(\alpha+1)} (b-a)^{\alpha}. \tag{62}$$

Proof. By Theorem 13. ■

Proposition 19 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$, $m = \lceil \alpha \rceil$, $\alpha > 0$, *and* $f_1, f_2 \in AC^m([a,b])$. *Suppose that* $f_1^{(k)}(x_0) = f_2^{(k)}(x_0) = 0$, $k = 1, ..., m-1$, $x_0 \in [a,b]$. *Assume* $D_{x_0-}^{\alpha} f_1, D_{x_0-}^{\alpha} f_2 \in L_q([a,x_0])$, *and* $D_{*x_0}^{\alpha} f_1, D_{*x_0}^{\alpha} f_2 \in L_q([x_0,b])$. *Then*

$$|\theta(f_1,f_2)(x_0)| \leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \left[(b-x_0)^{\alpha+\frac{1}{p}} + (x_0-a)^{\alpha+\frac{1}{p}}\right] \tag{63}$$

$$\leq \frac{M_3(f_1,f_2)(x_0)\,\psi_1(f_1,f_2)(x_0)}{\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}\,\Gamma(\alpha)} (b-a)^{\alpha+\frac{1}{p}}. \tag{64}$$

Proof. By Theorem 14. ■

Proposition 20 *Let* $x_0 \in [a,b] \subset \mathbb{R}$, $0 < \alpha \leq 1$, $f_1, f_2 \in AC([a,b])$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_1 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_2 \right\|_{\infty,[a,x_0]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_1 \right\|_{\infty,[x_0,b]}$, $\sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_2 \right\|_{\infty,[x_0,b]} < \infty$. *Then*

$$|\Delta(f_1,f_2)| \leq \frac{2M_4(f_1,f_2)\,\psi_2(f_1,f_2)}{\Gamma(\alpha+2)} (b-a)^{\alpha+2}. \tag{65}$$

Proof. By Theorem 15. ■

Proposition 21 *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, *and* $f_1, f_2 \in AC([a,b])$, $x_0 \in [a,b]$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| D_{x_0-}^{\alpha} f_i \right\|_{L_q([a,x_0])}, \sup_{x_0 \in [a,b]} \left\| D_{*x_0}^{\alpha} f_i \right\|_{L_q([x_0,b])} < \infty$, $i = 1, 2$. *Then*

$$|\Delta (f_1, f_2)| \leq \frac{2M_5 (f_1, f_2) \psi_2 (f_1, f_2)}{\left(\alpha + \frac{1}{p}\right) (p (\alpha - 1) + 1)^{\frac{1}{p}} \Gamma (\alpha)} (b - a)^{\alpha + \frac{1}{p} + 1}. \tag{66}$$

Proof. By Theorem 16. ■

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