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GENERALIZATION OF INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE FUNCTIONS THROUGH PREINVEXITY

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Abstract. In this paper, we establish a new integral identity for n-times differentiable functions defined on an invex subset of $\mathbb{R}$. Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions are then established by using this identity and the Hölder's inequality.

1. Introduction

It is well-known in mathematical literature that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I if the inequality

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. The above inequality holds in reversed direction if the function f is concave.

A number of papers have been written containing inequalities for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows(see [10]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

Recently, Hermite-Hadamard type inequality has been the subject of intensive research. Various refinements of the Hermite-Hadamard inequalities for the convex functions and its variant forms are being obtained in the literature by many researchers see for instance [5, 6, 7, 9, 11, 12, 14, 15, 17, 28, 29, 30, 34, 37, 40].

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of preinvex functions introduced by Weir and Mond [39]. Many researchers have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems, for example Mohn and Neogy [20], Noor [23] and Yang et al. [42].

Let us recall some known results concerning invexity and preinvexity.

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A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a function $\eta : K \times K \to \mathbb{R}^n$ such that

$$x + t\eta(y, x) \in K, \ \forall x, y \in K, t \in [0, 1].$$

The invex set K is also called an η-connected set.

Definition 1. [32] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1 - t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [39].

Noor [22] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1. [22] *Let $f : [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.2}$$

For several new results on inequalities connected with the right and left part of the inequalities (1.2) for preinvex functions, we refer the interested reader to [3, 18, 21, 36], [43] and closely related articles references therein.

Most recently, Wei-Dong Jiang et al. [7], Shu-Hong Wang et al. [9, 40], Dah-Yang Hwang [11] and Latif [18] obtained a number of inequalities for n-times differentiable functions which are s-convex, m-convex, convex and preinvex. The main source of inspiration of the present paper is [40] in which more general inequalities for n-times differentiable functions convex functions are presented. In section 2, a more general identity for n-times differentiable functions defined on an invex subset of $\mathbb{R}$ is established and by using this identity and the Hölder's integral inequality, several new integral inequalities for n-times preinvex functions are established, which are more general than those proved in [11] and extend those given in [40].

2. Main Results

The following Lemmas are essential in establishing our main results in this section:

Lemma 1. *Let $n \in \mathbb{N}$ and $K \subseteq \mathbb{R}$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $f : K \to \mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a + \eta(b, a)]$, a, $b \in K$ with $\eta(b, a) > 0$. If $f^{(n)}(x)$ exists on $[a, a + \eta(b, a)]$, then* *for* λ, $\mu \in \mathbb{R}$ *and* $t \in [0,1]$, *we have the following identity:*

$$\begin{aligned} S(t;\lambda,\mu) &\triangleq -\lambda f(a) - (1-\mu) f(a+\eta(b,a)) \\ &+ \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=0}^{n-1} \frac{(-1)^k (\eta(b,a))^k}{(k+1)!} \left\{ t^k [t-(k+1)\lambda] \right. \\ &\left. - (t-1)^k [t-1+(k+1)(1-\mu)] \right\} f^{(k)}(a+t\eta(b,a)) \\ &= \frac{(-1)^{n-1}(\eta(b,a))^n}{n!} \left\{ \int_0^t z^{n-1}(n\lambda - z) f^{(n)}(a+z\eta(b,a))\,dz \right. \\ &\left. + \int_t^1 (z-1)^{n-1}(1-z-n(1-\mu)) f^{(n)}(a+z\eta(b,a))\,dz \right\}. \end{aligned} \tag{2.1}$$

Proof. When $n=1$, we have by integrating by parts that

$$\begin{aligned} \eta(b,a) &\left\{ \int_0^t (\lambda - z) f'(a+z\eta(b,a))\,dz \right. \\ &\left. + \int_t^1 (\mu - z) f'(a+z\eta(b,a))\,dz \right\} \\ &= -\lambda f(a) - (1-\mu) f(a+\eta(b,a)) \\ &- (\mu-\lambda) f(a+t\eta(b,a)) + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx. \end{aligned} \tag{2.2}$$

Suppose (2.1) is valid for $n = m-1$, that is

$$\begin{aligned} &-\lambda f(a) - (1-\mu) f(a+\eta(b,a)) \\ &+ \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=0}^{m-2} \frac{(-1)^k (\eta(b,a))^k}{(k+1)!} \left\{ t^k [t-(k+1)\lambda] \right. \\ &\left. - (t-1)^k [t-1+(k+1)(1-\mu)] \right\} f^{(k)}(a+t\eta(b,a)) \\ &= \frac{(-1)^{m-2}(\eta(b,a))^{m-1}}{(m-1)!} \left\{ \int_0^t z^{m-2}((m-1)\lambda - z) f^{(m-1)}(a+z\eta(b,a))\,dz \right. \\ &\left. + \int_t^1 (z-1)^{m-2}(1-z-(m-1)(1-\mu)) f^{(m-1)}(a+z\eta(b,a))\,dz \right\}. \end{aligned} \tag{2.3}$$

Now for $n=m$, we have by integrating by parts that

$$\begin{aligned}(2.4)\quad &\frac{(-1)^{m-1}(\eta(b,a))^{m}}{m!}\left\{\int_0^t z^{m-1}(m\lambda-z)f'(a+z\eta(b,a))\,dz\right.\\ &\left.+\int_t^1 (z-1)^{m-1}(1-z-m(1-\mu))f'(a+z\eta(b,a))\,dz\right\}\\ =&-\frac{(-1)^{m-1}(\eta(b,a))^{m-1}}{m!}\left\{\left[t^{m-1}(t-m\lambda)-(t-1)^{m-1}(t-1+m(1-\mu))\right]\right.\\ &\times f(a+t\eta(b,a))+\frac{(-1)^{m-2}(\eta(b,a))^{m-1}}{(m-1)!}\\ &\times\left\{\int_0^t z^{m-2}((m-1)\lambda-z)f'(a+z\eta(b,a))\,dz\right.\\ &\left.+\int_t^1 (z-1)^{m-2}(1-z-(m-1)(1-\mu))f'(a+z\eta(b,a))\,dz\right\}\end{aligned}$$

Using (2.4) in (2.3) and simplifying, we get (2.1). This completes the proof of the Lemma. □

Remark 1. *If $\eta(b,a)=b-a$ in Lemma 1. Then*

$$\begin{aligned}(2.5)\quad &-\lambda f(a)-(1-\mu)f(b)+\frac{1}{b-a}\int_a^b f(x)\,dx\\ &-\sum_{k=0}^{n-1}\frac{(-1)^k(b-a)^k}{(k+1)!}\left\{t^k[t-(k+1)\lambda]\right.\\ &\left.-(t-1)^k[t-1+(k+1)(1-\mu)]\right\}f^{(k)}(tb+(1-t)a)\\ =&\frac{(-1)^{n-1}(b-a)^n}{n!}\left\{\int_0^t z^{n-1}(n\lambda-z)f^{(n)}(zb+(1-z)a)\,dz\right.\\ &\left.+\int_t^1 (z-1)^{n-1}(1-z-n(1-\mu))f^{(n)}(zb+(1-z)a)\,dz\right\}.\end{aligned}$$

Lemma 2. [40] *Let $\alpha,\ \beta\in\mathbb{R}$, $\xi,\ c\geq 0$ and $r>-1$. Then*

$$\int_0^c u^r|\xi-u|\,du =\frac{1}{(r+1)(r+2)}\begin{cases}[(r+2)\xi-(r+1)c]\,c^{r+1}, & \xi\geq c\\ (r+1)c^{r+2}-(r+2)c^{r+1}\xi+2\xi^{r+2}, & 0\leq\xi\leq c\end{cases}$$

and

$$\int_0^c(\alpha u+\beta)|\xi-u|^r\,du=\frac{1}{(r+1)(r+2)}$$
$$\times\begin{cases}[(r+2)\beta+\alpha\xi]\xi^{r+1}-[\alpha c(r+1)+\beta(r+2)+\alpha\xi](\xi-c)^{r+1}, & \xi\geq c\\ [(r+2)\beta+\alpha\xi]\xi^{r+1}+[\beta(r+2)+\alpha(c+cr+\xi)+\alpha\xi](c-\xi)^{r+1}, & 0\leq\xi\leq c.\end{cases}$$

Theorem 2. *Let $n\in\mathbb{N}$ and $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $f:K\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely* *continuous on* $[a, a+\eta(b,a)]$, $a,\ b \in K$ *with* $\eta(b,a) > 0$. *If* $\left|f^{(n)}\right|^q$ *is preinvex function on* K *for* $q \geq 1$, *then for all* $t \in [0,1]$ *and* $\lambda,\ \mu \in [0,1]$, *we have the inequality*

$$
\begin{aligned}
(2.6)\qquad |S(t;\lambda,\mu)| &\leq \frac{(\eta(b,a))^n}{n!}\Big\{[A(\lambda,t;n)]^{1-1/q} \\
&\times \left[A(\lambda,t;n)\left|f^{(n)}(a)\right|^q + A(\lambda,t;n+1)\left(\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q\right)\right]^{1/q} \\
&+ [A(1-\mu,1-t;n)]^{1-1/q}\Big[A(1-\mu,1-t;n)\left|f^{(n)}(b)\right|^q \\
&+ A(1-\mu,1-t;n+1)\left(\left|f^{(n)}(a)\right|^q - \left|f^{(n)}(b)\right|^q\right)\Big]^{1/q}\Big\},
\end{aligned}
$$

where for $c \geq 0$ *and* $r > -1$

$$
\begin{aligned}
A(\xi,c;r+1) &= \int_0^c u^r\,|n\xi - u|\,du \\
&= \frac{1}{(r+1)(r+2)}\begin{cases} [n\xi(r+2) - (r+1)c]\,c^{r+1}, & n\xi \geq c \\[2ex] (r+1)c^{r+2} - n\xi(r+2)c^{r+1} + 2(n\xi)^{r+2}, & 0 \leq n\xi \leq c. \end{cases}
\end{aligned}
$$

Proof. By Lemma 1, the Hölder's inequality and the preinvexity of $\left|f^{(n)}\right|^q$ on K, $q \geq 1$, $n \in \mathbb{N}$, we have

$$
\begin{aligned}
(2.7)\qquad |S(t;\lambda,\mu)| &\leq \frac{(\eta(b,a))^n}{n!}\Big\{\int_0^t z^{n-1}|n\lambda - z|\left|f^{(n)}(a+z\eta(b,a))\right|dz \\
&+ \int_t^1 (z-1)^{n-1}|1-z-n(1-\mu)|\left|f^{(n)}(a+z\eta(b,a))\right|dz\Big\} \\
&\leq \frac{(\eta(b,a))^n}{n!}\left\{\left(\int_0^t z^{n-1}|n\lambda - z|\,dz\right)^{1-1/q}\right. \\
&\times \left[\int_0^t z^{n-1}|n\lambda - z|\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q} \\
&+ \left(\int_t^1 (z-1)^{n-1}|1-z-n(1-\mu)|\right)^{1-1/q} \\
&\left.\times \left[\int_t^1 (z-1)^{n-1}|1-z-n(1-\mu)|\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}.
\end{aligned}
$$

Using Lemma 2, we observe that

$$
\int_0^t z^{n-1}|n\lambda - z|\,dz = A(\lambda,t;n),
$$

$$
\int_t^1 (z-1)^{n-1}|1-z-n(1-\mu)| = A(1-\mu,1-t;n),
$$

$$
\begin{aligned}
&\int_0^t z^{n-1}|n\lambda - z|\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz \\
&= A(\lambda,t;n)\left|f^{(n)}(a)\right|^q + A(\lambda,t;n+1)\left(\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q\right)
\end{aligned}
$$

and

$$\int_t^1 (z-1)^{n-1} |1-z-n(1-\mu)| \left((1-z) \left| f^{(n)}(a) \right|^q + z \left| f^{(n)}(b) \right|^q \right) dz$$
$$= A(1-\mu, 1-t; n) \left| f^{(n)}(b) \right|^q + A(1-\mu, 1-t; n+1) \left(\left| f^{(n)}(a) \right|^q - \left| f^{(n)}(b) \right|^q \right).$$

Substituting the above inequalities into (2.7), gives us the desired inequality (2.6). □

Remark 2. *If we take $n=1$, $t=\frac{1}{2}$ and $0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1$ in Theorem 2, we get the following inequality:*

$$\begin{aligned} &\left| \lambda f(a) + (1-\mu) f(a+\eta(b,a)) + (\mu-\lambda) f\left(a + \frac{1}{2}\eta(b,a)\right) \right. \\ &\qquad \left. - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right| \\ &\leq \frac{\eta(b,a)}{24} \left[\left(8 - 9\lambda + 24\lambda^2 - 8\lambda^3 - 21\mu + 24\mu^2 - 8\mu^3\right) \left| f^{(n)}(a) \right| \right. \\ &\qquad \left. + \left(10 - 3\lambda + 8\lambda^3 - 15\mu + 8\mu^3\right) \left| f^{(n)}(b) \right| \right]. \end{aligned} \tag{2.8}$$

Theorem 3. *Let $n \in \mathbb{N}$ and $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $f : K \to \mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a+\eta(b,a)]$, a, $b \in K$ with $\eta(b,a) > 0$. If $\left| f^{(n)} \right|^q$ is preinvex function on K for $q > 1$ and $q(n-1) \geq r \geq 0$, then for all $t \in [0,1]$ and λ, $\mu \in [0,1]$, we have the inequality*

$$\begin{aligned} |S(t;\lambda,\mu)| &\leq \frac{(\eta(b,a))^n}{n!} \left\{ \left[A\left(\lambda, t; \frac{nq-r-1}{q-1}\right) \right]^{1-1/q} \right. \\ &\times \left[A(\lambda,t;r+1) \left| f^{(n)}(a) \right|^q + A(\lambda,t;r+2) \left(\left| f^{(n)}(b) \right|^q - \left| f^{(n)}(a) \right|^q \right) \right]^{1/q} \\ &+ \left[A\left(1-\mu, 1-t; \frac{nq-r-1}{q-1}\right) \right]^{1-1/q} \left[A(1-\mu,1-t;r+1) \left| f^{(n)}(b) \right|^q \right. \\ &\left. \left. + A(1-\mu,1-t;r+2) \left(\left| f^{(n)}(a) \right|^q - \left| f^{(n)}(b) \right|^q \right) \right]^{1/q} \right\}, \end{aligned} \tag{2.9}$$

where $A(\xi, c; r+1)$ is defined as in Theorem 2, $c \geq 0$, $r > 1$.

Proof. From Lemma 1, Hölder's inequality and the preinvexity of $\left|f^{(n)}\right|^{q}$ on K, $q>1$ and $q(n-1)\geq r\geq 0$, $n\in\mathbb{N}$, we have

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq{}& \frac{(\eta(b,a))^{n}}{n!}\Bigg\{\left(\int_{0}^{t} z^{[(n-1)q-r]/(q-1)}\,|n\lambda - z|\,dz\right)^{1-1/q} \\
&\times\left[\int_{0}^{t} z^{r}\,|n\lambda - z|\left((1-z)\left|f^{(n)}(a)\right|^{q} + z\left|f^{(n)}(b)\right|^{q}\right)dz\right]^{1/q} \\
&+\left(\int_{t}^{1}(z-1)^{[(n-1)q-r]/(q-1)}\,|1-z-n(1-\mu)|\right)^{1-1/q} \\
&\times\left[\int_{t}^{1}(z-1)^{r}\,|1-z-n(1-\mu)|\left((1-z)\left|f^{(n)}(a)\right|^{q} + z\left|f^{(n)}(b)\right|^{q}\right)dz\right]^{1/q}\Bigg\}.
\end{aligned}
\tag{2.10}
$$

The rest of the proof is similar to that of the proof of Theorem 2 □

Corollary 1. *Under the assumptions of Theorem 3*

(1) *If $r=0$, we have*

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq{}& \frac{(\eta(b,a))^{n}}{n!}\Bigg\{\left[A\left(\lambda,t;\frac{nq-1}{q-1}\right)\right]^{1-1/q} \\
&\times\left[A(\lambda,t;1)\left|f^{(n)}(a)\right|^{q} + A(\lambda,t;2)\left(\left|f^{(n)}(b)\right|^{q} - \left|f^{(n)}(a)\right|^{q}\right)\right]^{1/q} \\
&+\left[A\left(1-\mu,1-t;\frac{nq-1}{q-1}\right)\right]^{1-1/q}\Big[A(1-\mu,1-t;1)\left|f^{(n)}(b)\right|^{q} \\
&+A(1-\mu,1-t;2)\left(\left|f^{(n)}(a)\right|^{q} - \left|f^{(n)}(b)\right|^{q}\right)\Big]^{1/q}\Bigg\}.
\end{aligned}
\tag{2.11}
$$

(2) *If $r=(n-1)q$, we get*

$$
\begin{aligned}
&|S(t;\lambda,\mu)| \\
&\leq \frac{(\eta(b,a))^{n}}{n!}\Big\{[A(\lambda,t;1)]^{1-1/q}\Big[A(\lambda,t;(n-1)q+1)\left|f^{(n)}(a)\right|^{q} \\
&\quad +A(\lambda,t;(n-1)q+2)\left(\left|f^{(n)}(b)\right|^{q} - \left|f^{(n)}(a)\right|^{q}\right)^{1/q}\Big] \\
&\quad +[A(1-\mu,1-t;1)]^{1-1/q}\Big[A(1-\mu,1-t;(n-1)q+1)\left|f^{(n)}(b)\right|^{q} \\
&\quad +A(1-\mu,1-t;(n-1)q+2)\left(\left|f^{(n)}(a)\right|^{q} - \left|f^{(n)}(b)\right|^{q}\right)\Big]^{1/q}\Big\}.
\end{aligned}
\tag{2.12}
$$

Theorem 4. *Let $n\in\mathbb{N}$ and $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $f: K\to\mathbb{R}$ be a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on $[a,a+\eta(b,a)]$, a, $b\in K$ with $\eta(b,a)>0$. If $\left|f^{(n)}\right|^{q}$ is preinvex function on K for $q>1$, then for all $t\in[0,1]$ and λ, $\mu\in[0,1]$, we have the* *inequality*

$$(2.13)\quad |S(t;\lambda,\mu)| \le \frac{(\eta(b,a))^n}{n!}\left\{\left[B\left(\lambda,0,1,t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\right.$$
$$\times\left[A(0,t;(n-1)q+1)\left|f^{(n)}(b)\right|^q + B(0,-1,1;(n-1)q+1)\left|f^{(n)}(a)\right|^q\right]^{1/q}$$
$$+\left[B\left(1-\mu,0,1,1-t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}$$
$$\times\left[B(0,-1,1,1-t;(n-1)q+1)\left|f^{(n)}(b)\right|^q\right.$$
$$\left.\left.+A(0,1-t;(n-1)q+1)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\},$$

where $A(\xi,c;r+1)$ *is defined as in Theorem 2 and*

$$B(\xi,\alpha,\beta,c;r+1) = \frac{1}{(r+1)(r+2)}$$
$$\times\begin{cases}[n\xi\beta+\alpha n\xi](n\xi)^{r+1} \\ -[\alpha c(r+1)+\beta(r+2)+\alpha n\xi](n\xi-c)^{r+1}, & n\xi\ge c \\ [(r+2)\beta+\alpha n\xi](n\xi)^{r+1} \\ +[\beta(r+2)+\alpha(c+cr+n\xi)](c-n\xi)^{r+1}, & 0\le n\xi\le c,\end{cases}$$

$c\ge 0,\ r>1,\ \alpha,\ \beta\in\mathbb{R}$.

Proof. Applying Lemma 1, Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, we have

$$(2.14)\quad |S(t;\lambda,\mu)| \le \frac{(\eta(b,a))^n}{n!}\left\{\left(\int_0^t |n\lambda-z|^{q/(q-1)}\,dz\right)^{1-1/q}\right.$$
$$\times\left[\int_0^t z^{(n-1)q}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}$$
$$+\left(\int_t^1 |1-z-n(1-\mu)|^{q/(q-1)}\right)^{1-1/q}$$
$$\left.\times\left[\int_t^1 (z-1)^{(n-1)q}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}.$$

Using the similar arguments as that of the proof of Theorem 2, we get the inequality (2.13). □

Theorem 5. *Let* $n\in\mathbb{N}$ *and* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta: K\times K\to\mathbb{R}$ *and* $f:K\to\mathbb{R}$ *be a mapping such that* $f^{(n-1)}(x)$ *is absolutely continuous on* $[a,a+\eta(b,a)]$, $a,\ b\in K$ *with* $\eta(b,a)>0$. *If* $\left|f^{(n)}\right|^q$ *is preinvex function on* K *for* $q>1$, *then for all* $t\in[0,1]$ *and* $\lambda,\ \mu\in[0,1]$, *we have the* *inequality*

$$(2.15)\quad |S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}\left\{\left[B\left(\lambda,0,1,t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\right.$$
$$\times\left[A(0,t;(n-2)q+2)\left|f^{(n)}(b)\right|^q + B(0,-1,1,t;(n-2)q+2)\left|f^{(n)}(a)\right|^q\right]^{1/q}$$
$$+\left[B\left(1-\mu,1,0,1-t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}$$
$$\left.\times\left[B(0,-1,1,1-t;(n-2)q+2)\left|f^{(n)}(b)\right|^q + A(0,1-t;(n-2)q+2)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\},$$

where $A(\xi,c;r+1)$ *and* $B(\xi,\alpha,\beta,c;r+1)$ *are defined as in Theorem 2 and Theorem 4 respectively,* $c\geq 0$*,* $r>1$*,* α*,* $\beta\in\mathbb{R}$*.*

Proof. Applying Lemma 1, using Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, results in

$$(2.16)\quad |S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}\left\{\left(\int_0^t z\,|n\lambda - z|^{q/(q-1)}\,dz\right)^{1-1/q}\right.$$
$$\times\left[\int_0^t z^{(n-2)q+1}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}$$
$$+\left(\int_t^1 (1-z)\,|1-z-n(1-\mu)|^{q/(q-1)}\right)^{1-1/q}$$
$$\left.\times\left[\int_t^1 (z-1)^{(n-2)q+1}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}.$$

The rest of the proof is similar to that of the proof of Theorem 2. □

Theorem 6. *Let* $n\in\mathbb{N}$ *and* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta: K\times K\to\mathbb{R}$ *and* $f:K\to\mathbb{R}$ *be a mapping such that* $f^{(n-1)}(x)$ *is absolutely continuous on* $[a,a+\eta(b,a)]$*,* a*,* $b\in K$ *with* $\eta(b,a)>0$*. If* $\left|f^{(n)}\right|^q$ *is preinvex function on* K *for* $q>1$*, then for all* $t\in[0,1]$ *and* λ*,* $\mu\in[0,1]$*, we have the inequality*

$$(2.17)\quad |S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q}$$
$$\times\left\{t^{(nq-1)/q}\left[B(\lambda,1,0,t;q+1)\left|f^{(n)}(b)\right|^q + B(\lambda,-1,1,t;q+1)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right.$$
$$+(1-t)^{(nq-1)/q}\left[B(1-\mu,-1,1,1-t;q+1)\left|f^{(n)}(b)\right|^q\right.$$
$$\left.\left.+B(1-\mu,1,0,1-t;q+1)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\},$$

where $B(\xi,\alpha,\beta,c;r+1)$ *is defined as in Theorem 4,* $c\geq 0$*,* $r>1$*,* α*,* $\beta\in\mathbb{R}$*.*

Proof. Utilizing Lemma 1, using Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, results in

$$|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}\left\{\left(\int_0^t z^{(n-1)q/(q-1)}dz\right)^{1-1/q}\right.$$
$$\times\left[\int_0^t |n\lambda - z|^q\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}$$
$$+\left(\int_t^1 (1-z)^{(n-1)q/(q-1)}\right)^{1-1/q}$$
$$\left.\times\left[\int_t^1 |1-z-n(1-\mu)|^q\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}. \tag{2.18}$$

The rest of the proof is similar to that of the Theorem 2. □

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