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INEQUALITIES OF JENSEN TYPE FOR φ-CONVEX FUNCTIONS

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Abstract. Some inequalities of Jensen type for φ-convex functions defined on real intervals are given.

1. Introduction

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([38])**.** *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [29], [30], [32], [44], [47] and [48]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([32])**.** *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [32] and [45] while for quasi convex functions, the reader can consult [31].

Definition 3 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

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For some properties of this class of functions see [1], [2], [7], [8], [27], [28], [39], [41] and [50].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([53]). *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [53], [6], [42], [51], [49] and [52].

We can introduce now another class of functions.

Definition 5. *We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \le s_1 \le s_2 \le 1$.

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a) f\left(\frac{a+b}{2}\right) < \int_a^b f(x)dx < (b-a)\frac{f(a)+f(b)}{2}, \quad a, b \in \mathbb{R}. \tag{1.6}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [43]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [43]. Since (1.6) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [22]-[26], [33]-[36] and [46].

The following inequality of Hermite-Hadamard type for h-convex function holds [49].

Theorem 1. *Assume that the function $f : I \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in I$ with $y \ne x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le [f(x) + f(y)] \int_0^1 h(t)\,dt. \tag{1.7}$$

If we write (1.7) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{2}. \tag{1.8}$$

If we write (1.7) for the case of P-type functions $f : I \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2}f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le f(x)+f(y), \tag{1.9}$$

that has been obtained for functions of real variable in [32].

If f is Breckner s-convex on I, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (1.7) we get

$$2^{s-1}f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{s+1}, \tag{1.10}$$

that was obtained for functions of a real variable in [27].

If $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}}f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(u)\,du \le \frac{f(x)+f(y)}{1-s}. \tag{1.11}$$

We notice that for $s = 1$ the first inequality in (1.11) still holds, i.e.

$$\frac{1}{4}f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x+ty]\,dt. \tag{1.12}$$

The case for functions of real variables was obtained for the first time in [32].

2. φ-Convex Functions

We introduce the following class of h-convex functions.

Definition 6. *Let $\varphi : (0,1) \to (0,\infty)$ a measurable function. We say that the function $f : I \to [0,\infty)$ is a φ-convex function on the interval I if for all $x, y \in I$ we have*

$$f(tx+(1-t)y) \le t\varphi(t)f(x) + (1-t)\varphi(1-t)f(y) \tag{2.1}$$

for all $t \in (0,1)$.

If we denote $\ell(t) = t$, the identity function, then it is obvious that f is h-convex with $h = \ell\varphi$. Also, all the examples from the introduction can be seen as φ-convex functions with appropriate choices of φ.

If we take $\varphi(t) = \frac{1}{t^{s+1}}$ with $s \in [0,1]$, then we get the class of s-Godunova-Levin functions. Also, if we put $\varphi(t) = t^{s-1}$ with $s \in (0,1)$, then we get the concept of Breckner s-convexity. We notice that for all these examples we have

$$\varphi_+(0) := \lim_{t\to 0+}\varphi(t) = \infty.$$

The case of convex functions, i.e. when $\varphi(t) = 1$ is the only example from above for which $\varphi_+(0)$ is finite, namely $\varphi_+(0) = 1$.

Consider the family of functions, for $p > 1$ and $k > 0$

$$\delta(p,k) : [0,1] \to \mathbb{R}_+,\ \delta(p,k)(t) = k(1-t)^p + 1. \tag{2.2}$$

We observe that $\delta_+ (p,k)(0) = \delta(p,k)(0) = k+1$, $\delta(p,k)$ is strictly decreasing on $[0,1]$ and $\delta(p,k)(t) \geq \delta(p,k)(1) = 1$.

Definition 7. *We say that the function $f : I \to [0,\infty)$ is a $\delta(p,k)$-convex function on the interval I if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq t[k(1-t)^p + 1] f(x) + (1-t)(kt^p + 1) f(y) \tag{2.3}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\delta^{(p,k)}$-convex function for any $p > 1$ and $k > 0$.

For $m > 0$ we consider the family of functions

$$\eta(m) : [0,1] \to \mathbb{R}_+, \eta(m)(t) := \exp[m(1-t)].$$

We observe that $\eta_+(m)(0) = \eta(m)(0) = \exp(m)$, $\eta(m)$ is strictly decreasing on $[0,1]$ and $\eta(m)(t) \geq \eta(m)(1) = 1$.

Definition 8. *We say that the function $f : I \to [0,\infty)$ is a $\eta(m)$-convex function on the interval I if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq t \exp[m(1-t)] f(x) + (1-t)\exp(mt) f(y) \tag{2.4}$$

for all $t \in (0,1)$.

It is obvious that any nonnegative convex function is a $\eta(m)$-convex function for any $m > 0$.

There are many other examples one can consider. In fact any continuos function $\varphi : [0,1] \to [1,\infty)$ can generate a class of φ-convex function that contains the class of nonnegative convex functions.

Utilising Theorem 1 we can state the following result.

Theorem 2. *Assume that the function $f : I \to [0,\infty)$ is a φ-convex function with $\ell\varphi \in L[0,1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x + ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{\varphi\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq [f(x) + f(y)] \int_0^1 t\varphi(t)\,dt. \tag{2.5}$$

The proof follows from (1.7) by taking $h(t) = t\varphi(t)$, $t \in (0,1)$.

Remark 1. *We notice that, since $\int_0^1 t\varphi(t)\,dt$ can be seen as the expectation of a random variable X with the density function φ, the inequality (2.5) provides a connection to Probability Theory and motivates the introduction of φ-convex function as a natural concept, having available many examples of density functions φ that arise in applications.*

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[37], [40]-[42] and [45]-[52].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t)h(s) \text{ for any } t, s \in J. \tag{2.6}$$

If the inequality (2.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (2.6) then h is said to be a multiplicative function on J.

In [53] it has been noted that if $h:[0,\infty)\to[0,\infty)$ with $h(t)=(x+c)^{p-1}$, then for $c=0$ the function h is multiplicative. If $c\geq 1$, then for $p\in(0,1)$ the function h is supermultiplicative and for $p>1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t)=t^r$ is also supermultiplicative.

The case of h-convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [53] for functions of a real variable.

Theorem 3. *Let $h:J\to[0,\infty)$ be a supermultiplicative function on J. If the function $f:I\to[0,\infty)$ is h-convex on the interval I, then for any $w_i\geq 0$, $x_i\in I$, $i\in\{1,...,n\}$, $n\geq 2$ with $W_n:=\sum_{i=1}^n w_i>0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right)\leq\sum_{i=1}^n h\left(\frac{w_i}{W_n}\right)f(x_i). \tag{2.7}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\leq h\left(\frac{1}{n}\right)\sum_{i=1}^n f(x_i). \tag{2.8}$$

Let $h(z)=\sum_{n=0}^\infty a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. We have the following examples

$$\begin{aligned} h(z)&=\sum_{n=1}^\infty\frac{1}{n}z^n=\ln\frac{1}{1-z},\ \ z\in D(0,1); \\ h(z)&=\sum_{n=0}^\infty\frac{1}{(2n)!}z^{2n}=\cosh z,\ \ z\in\mathbb{C}; \\ h(z)&=\sum_{n=0}^\infty\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ \ z\in\mathbb{C}; \\ h(z)&=\sum_{n=0}^\infty z^n=\frac{1}{1-z},\ \ z\in D(0,1). \end{aligned} \tag{2.9}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\begin{aligned} h(z)&=\sum_{n=0}^\infty\frac{1}{n!}z^n=\exp(z) \qquad z\in\mathbb{C}, \\ h(z)&=\sum_{n=1}^\infty\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad z\in D(0,1); \\ h(z)&=\sum_{n=0}^\infty\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z), \qquad z\in D(0,1); \end{aligned} \tag{2.10}$$

and

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z), \qquad z\in D(0,1) \tag{2.11}$$

$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n, \alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

The following result may provide many examples of supemultiplicative functions.

Lemma 1. *Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. Assume that $0<r<R$ and define $h_r:[0,1]\to[0,\infty)$, $h_r(t):=\frac{h(rt)}{h(r)}$. Then h_r is supemultiplicative on $[0,1]$.*

Proof. We use the Čebyšev inequality for synchronous (the same monotonicity) sequences $(c_i)_{i\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}}$ and nonnegative weights $(p_i)_{i\in\mathbb{N}}$:

$$\sum_{i=0}^{n}p_i\sum_{i=0}^{n}p_ic_ib_i\geq\sum_{i=0}^{n}p_ic_i\sum_{i=0}^{n}p_ib_i, \tag{2.12}$$

for any $n\in\mathbb{N}$.

Let $t,s\in(0,1)$ and define the sequences $c_i:=t^i$, $b_i:=s^i$. These sequences are decreasing and if we apply Čebyšev's inequality for these sequences and the weights $p_i:=a_ir^i\geq 0$ we get

$$\sum_{i=0}^{n}a_ir^i\sum_{i=0}^{n}a_i(rts)^i\geq\sum_{i=0}^{n}a_i(rt)^i\sum_{i=0}^{n}a_i(rs)^i \tag{2.13}$$

for any $n\in\mathbb{N}$.

Since the series

$$\sum_{i=0}^{\infty}a_ir^i,\ \sum_{i=0}^{\infty}a_i(rts)^i,\ \sum_{i=0}^{\infty}a_i(rt)^i \text{ and } \sum_{i=0}^{\infty}a_i(rs)^i$$

are convergent, then by letting $n\to\infty$ in (2.13) we get

$$h(r)h(rts)\geq h(rt)h(rs)$$

i.e.

$$h_r(ts)\geq h_r(t)h_r(s).$$

This inequality is also obviously satisfied at the end points of the interval $[0,1]$ and the proof is completed. □

Remark 2. *Utilising the above theorem, we then conclude that the functions*

$$h_r:[0,1]\to[0,\infty),\ h_r(t):=\frac{1-r}{1-rt},\ r\in(0,1)$$

and

$$h_r:[0,1]\to[0,\infty),\ h_r(t):=\exp[-r(1-t)],\ r>0$$

are supermultiplicative.

We say that the function $f : I \to [0,\infty)$ is r-resolvent convex with r fixed in $(0,1)$, if f is h-convex with $h(t) = \frac{1-r}{1-rt}$, i.e.

$$f(tx+(1-t)y) \le (1-r)\left[\frac{1}{1-rt}f(x) + \frac{1}{1-r+rt}f(y)\right] \tag{2.14}$$

for any $x,y \in I$ and $t \in [0,1]$.

In particular, for $r = \frac{1}{2}$ we have $\frac{1}{2}$-resolvent convex functions defined by the condition

$$f(tx+(1-t)y) \le \frac{1}{2-t}f(x) + \frac{1}{1+t}f(y) \tag{2.15}$$

for any $t \in [0,1]$ and $x,y \in I$.

Since

$$t < \frac{1}{2-t} < \frac{1}{t} \text{ and } 1-t < \frac{1}{1+t} < \frac{1}{1-t} \text{ for } t \in (0,1)$$

it follows that any nonnegative convex function is $\frac{1}{2}$-resolvent convex which, in its turn, is of Godunova-Levin type.

We say that the function $f : I \to [0,\infty)$ is r-exponential convex with r fixed in $(0,\infty)$, if f is h-convex with $h(t) = \exp[-r(1-t)]$, i.e.

$$f(tx+(1-t)y) \le \exp[-r(1-t)]f(x) + \exp(-rt)f(y) \tag{2.16}$$

for any $t \in [0,1]$ and $x,y \in C$.

Since

$$t \le \exp[-r(1-t)] \text{ and } 1-t \le \exp(-rt) \text{ for } t \in [0,1]$$

it follows that any nonnegative convex function is r-exponential convex with $r \in (0,\infty)$.

Corollary 1. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. Assume that $0 < r < R$ and define $h_r : [0,1] \to [0,\infty)$, $h_r(t) := \frac{h(rt)}{h(r)}$. If the function $f : I \to [0,\infty)$ is h_r-convex on the on the interval I, namely*

$$f(tx+(1-t)y) \le \frac{1}{h(r)}[h(rt)f(x) + h(r(1-t))f(y)] \tag{2.17}$$

for any $t \in [0,1]$ and $x,y \in I$, then for any $x_i \in I$, $w_i \ge 0$, $i \in \{1,...,n\}$, $n \ge 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le \frac{1}{h(r)}\sum_{i=1}^{n} h\left(r\frac{w_i}{W_n}\right) f(x_i). \tag{2.18}$$

Remark 3. *If the function $f : I \to [0,\infty)$ is $\frac{1}{2}$-resolvent convex on I, then for any $x_i \in I$, $w_i \ge 0$, $i \in \{1,...,n\}$, $n \ge 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le W_n \sum_{i=1}^{n} \frac{1}{2W_n - w_i} f(x_i).$$

If the function $f : I \to [0,\infty)$ is r-exponential convex with r fixed in $(0,\infty)$, then for any $x_i \in I$, $w_i \ge 0$, $i \in \{1,...,n\}$, $n \ge 2$ with $W_n := \sum_{i=1}^{n} w_i > 0$ we have

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le \sum_{i=1}^{n} \exp\left[-r\left(1 - \frac{w_i}{W_n}\right)\right] f(x_i).$$

We have the following Jensen type inequality for φ-convex functions.

Corollary 2. *Let $\varphi : J \to [0,\infty)$ be a supermultiplicative function on J. If the function $f : I \to [0,\infty)$ is φ-convex on the interval I, then for any $w_i \geq 0$, $x_i \in I$, $i \in \{1,...,n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^n w_i \varphi\left(\frac{w_i}{W_n}\right) f(x_i). \tag{2.19}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq \varphi\left(\frac{1}{n}\right)\frac{1}{n}\sum_{i=1}^n f(x_i). \tag{2.20}$$

The proof follows by Theorem 3 for the supermultiplicative function $h(t) = t\varphi(t)$, $t \in J$.

The inequality (2.19) will be used further to obtain an integral Jensen type inequality.

3. Some Results for Differentiable Functions

If we assume that the function $f : I \to [0,\infty)$ is differentiable on the interior of I, denoted by $\mathring{I}$, then we have the following *"gradient inequality"* that will play an essential role in the following.

Lemma 2. *Let $\varphi : (0,1) \to (0,\infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1) = 1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\varphi_+(0) f(x) - \left[\varphi'_-(1) + 1\right] f(y) \geq f'(y)(x-y) \tag{3.1}$$

for any $x, y \in \mathring{I}$ with $x \neq y$.

Proof. Since f is φ-convex on I, then

$$t\varphi(t) f(x) + (1-t)\varphi(1-t) f(y) \geq f(tx + (1-t)y)$$

for any $t \in (0,1)$ and for any $x, y \in \mathring{I}$, which is equivalent to

$$t\varphi(t) f(x) + [(1-t)\varphi(1-t) - 1] f(y) \geq f(tx + (1-t)y) - f(y)$$

and by dividing by $t > 0$ we get

$$\varphi(t) f(x) + \left[\frac{(1-t)\varphi(1-t) - 1}{t}\right] f(y) \geq \frac{f(tx + (1-t)y) - f(y)}{t} \tag{3.2}$$

for any $t \in (0,1)$.

Now, since f is differentiable on $y \in \mathring{I}$, then we have

$$\begin{aligned}\lim_{t\to 0+}\frac{f(tx + (1-t)y) - f(y)}{t} &= \lim_{t\to 0+}\frac{f(y + t(x-y)) - f(y)}{t} \\ &= (x-y)\lim_{t\to 0+}\frac{f(y + t(x-y)) - f(y)}{t(x-y)} \\ &= (x-y) f'(y)\end{aligned} \tag{3.3}$$

for any $x \in \mathring{I}$ with $x \neq y$.

Also since $\varphi_{-}(1)=1$ and $\varphi_{-}'(1)$ exists and is finite, we have

$$\begin{aligned}\lim_{t\to 0+}\frac{(1-t)\varphi(1-t)-1}{t} &= \lim_{s\to 1-}\frac{s\varphi(s)-1}{1-s}=-\lim_{s\to 1-}\frac{s\varphi(s)-1}{s-1} \\ &= -\lim_{s\to 1-}\frac{s(\varphi(s)-\varphi(1))+s-1}{s-1} \\ &= -\varphi_{-}'(1)-1.\end{aligned} \tag{3.4}$$

Taking the limit over $t\to 0+$ in (3.2) and utilizing (3.3) and (3.4) we get the desired result (3.1). □

Remark 4. *If we assume that*

$$\varphi_{+}(0)\geq \varphi_{-}'(1)+1, \tag{3.5}$$

then the inequality (3.1) also holds for $x=y$.

There are numerous examples of such functions, for instance, if, as above we take $\varphi(t)=k(1-t)^{p}+1$, $t\in[0,1]$ $(p>1,k>0)$ then $\varphi_{+}(0)=k+1$, $\varphi_{-}(1)=1$ and $\varphi_{-}'(1)=0$, which satisfy the condition (3.5).

If we take $\varphi(t)=\exp[m(1-t)]$ $(m>0)$, then $\varphi_{+}(0)=\exp m$, $\varphi_{-}(1)=1$ and $\varphi_{-}'(1)=-m$. We have

$$\varphi_{+}(0)-\varphi_{-}(1)-\varphi_{-}'(1)=e^{m}-1+m>0$$

for $m>0$.

The following result holds:

Theorem 4. *Let $\varphi:(0,1)\to(0,\infty)$ a measurable function and such that the right limit $\varphi_{+}(0)$ exists and is finite, the left limit $\varphi_{-}(1)=1$ and the left derivative in 1 denoted $\varphi_{-}'(1)$ exists and is finite. Assume also that $\varphi_{-}'(1)>-1$. If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then*

$$\frac{\varphi_{+}(0)}{\varphi_{-}'(1)+1}\cdot\frac{f(x)+f(y)}{2}\geq\frac{1}{y-x}\int_{x}^{y}f(u)\,du\geq\frac{\varphi_{-}'(1)+1}{\varphi_{+}(0)}f\left(\frac{x+y}{2}\right) \tag{3.6}$$

for any $x,y\in I$.

Remark 5. *It has been shown in* [25] *that the inequalities (2.5) and (3.6) are not comparable, meaning that some time one is better then the other, depending on the φ-convex function involved.*

The following discrete Jensen type inequality holds:

Theorem 5. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function and such that the right limit $\varphi_{+}(0)$ exists and is finite, the left limit $\varphi_{-}(1)=1$ and the left derivative in 1 denoted $\varphi_{-}'(1)$ exists and is finite. Assume also that*

$$\varphi_{+}(0)\geq\varphi_{-}'(1)+1>0. \tag{3.7}$$

If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then for any $w_i\geq 0$, $x_i\in\mathring{I}$, $i\in\{1,...,n\}$, $n\geq 2$ with $W_n:=\sum_{i=1}^{n}w_i>0$ we have

$$\frac{\varphi_{+}(0)}{\varphi_{-}'(1)+1}\cdot\frac{1}{W_n}\sum_{i=1}^{n}w_i f(x_j)\geq f\left(\frac{1}{W_n}\sum_{i=1}^{n}w_i x_i\right). \tag{3.8}$$

If $\frac{1}{W_n}\sum_{i=1}^{n}w_i x_i\neq x_j$ for any $j\in\{1,...,n\}$, then the first condition in 3.7 can be dropped.

Proof. From (3.1) we have

$$\varphi_{+}(0) f(x_j) - \left[\varphi'_{-}(1) + 1\right] f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \geq f'\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(x_j - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \tag{3.9}$$

for any $j \in \{1, ..., n\}$.

If we multiply (3.9) by $w_i \geq 0$ and sum over j from 1 to n we get

$$\varphi_{+}(0)\sum_{j=1}^{n} w_j f(x_j) - \left[\varphi'_{-}(1) + 1\right]\sum_{j=1}^{n} w_j f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)$$
$$\geq f'\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\sum_{j=1}^{n} w_j\left(x_j - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) = 0,$$

which proves the desired result (3.8). □

4. Integral Inequalities

We have the following Jensen inequality for the Riemann integral:

Theorem 6. *Let $u : [a, b] \to [m, M]$ be a Riemann integrable function. Suppose that $\varphi : J \to [0, \infty)$ is a supermultiplicative function on J and the function $f : [m, M] \to [0, \infty)$ is φ-convex and continuous on the interval $[m, M]$. If the right limit $\varphi_{+}(0)$ exists and is finite, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \varphi_{+}(0)\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{4.1}$$

Proof. Consider the sequence of divisions

$$d_n : \ x_i^{(n)} = a + \frac{i}{n}(b-a), \ i \in \{0, ..., n\}$$

and the intermediate points

$$\xi_i^{(n)} = a + \frac{i}{n}(b-a), \ i \in \{0, ..., n\}.$$

We observe that the norm of the division $\Delta_n := \max_{i\in\{0,...,n-1\}}\left(x_{i+1}^{(n)} - x_i^{(n)}\right) = \frac{b-a}{n} \to 0$ as $n \to \infty$ and since u is Riemann integrable on $[a, b]$, then

$$\int_a^b u(t)\,dt = \lim_{n\to\infty}\sum_{i=0}^{n-1} u\left(\xi_i^{(n)}\right)\left[x_{i+1}^{(n)} - x_i^{(n)}\right] = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a + \frac{i}{n}(b-a)\right).$$

Also, since $f : [m, M] \to [0, \infty)$ is Riemann integrable, then $f \circ u$ is Riemann integrable and

$$\int_a^b f(u(t))\,dt == \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1} f\left[u\left(a + \frac{i}{n}(b-a)\right)\right].$$

Utilising the inequality (2.19) for $w_i := \frac{b-a}{n}$ and $x_i := u\left(a + \frac{i}{n}(b-a)\right)$ we have

$$f\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a+\frac{i}{n}(b-a)\right)\right) \tag{4.2}$$
$$\leq \frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1}\varphi\left(\frac{1}{n}\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)$$
$$= \frac{1}{b-a}\varphi\left(\frac{1}{n}\right)\frac{b-a}{n}\sum_{i=0}^{n-1} f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)$$

for any $n \geq 1$.

Since f is continuous, then

$$\lim_{n\to\infty} f\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1} u\left(a+\frac{i}{n}(b-a)\right)\right)$$
$$= f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right).$$

Also

$$\lim_{n\to\infty}\varphi\left(\frac{1}{n}\right) = \varphi_+(0) < \infty.$$

Therefore, taking the limit over $n \to \infty$ in the inequality (4.2) we deduce the desired result (4.1). □

We have the following Hermite-Hadamard type inequality:

Corollary 3. *Suppose that $\varphi : J \to [0,\infty)$ is a supermultiplicative function on J and the function $f : I \to [0,\infty)$ is φ-convex and continuous on the interval I. If the right limit $\varphi_+(0)$ exists and is finite with $\varphi_+(0) > 0$, then for any $x, y \in I$ with $x \neq y$ we have*

$$\frac{1}{\varphi_+(0)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u(t))\,dt. \tag{4.3}$$

Remark 6. *If the function $f : [m,M] \to [0,\infty)$ is a $\delta(p,k)$-convex and continuous function on the interval $[m,M]$ ($p > 1$ and $k > 0$, see Definition 7) then for any $u : [a,b] \to [m,M]$ a Riemann integrable function on $[a,b]$ we have*

$$\frac{1}{k+1} f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{4.4}$$

If the function $f : [m,M] \to [0,\infty)$ is a $\eta(s)$-convex and continuous function on the interval $[m,M]$ ($s > 0$, see Definition 8) then for any $u : [a,b] \to [m,M]$ a Riemann integrable function on $[a,b]$ we have

$$\frac{1}{e^s} f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{4.5}$$

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma -$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a $\mu-$measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for $\mu-$ a.e.(almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x)\left|f(x)\right|d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

Theorem 7. *Let $\varphi:(0,1)\to(0,\infty)$ be a measurable function and such that the right limit $\varphi_+(0)$ exists and is finite, the left limit $\varphi_-(1)=1$ and the left derivative in 1 denoted $\varphi'_-(1)$ exists and is finite. Assume also that*

$$\varphi_+(0)\geq\varphi'_-(1)+1>0. \tag{4.6}$$

If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and φ-convex, then for any $u:\Omega\to[m,M]\subset\mathring{I}$ so that $f\circ u,\ u\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. (almost everywhere) on Ω with $\int_\Omega wd\mu=1$ we have

$$\frac{\varphi_+(0)}{\varphi'_-(1)+1}\cdot\int_\Omega w(f\circ u)\,d\mu\geq f\left(\int_\Omega wud\mu\right). \tag{4.7}$$

If $\int_\Omega wud\mu\neq u(x)$ for μ-a.e. $x\in\Omega$, then we can drop the first condition in (4.6).

Proof. From (3.1) and since $\int_\Omega wud\mu\in[m,M]\subset\mathring{I}$ we have

$$\begin{aligned}\varphi_+(0)\,f(u(x))-\left[\varphi'_-(1)+1\right]f\left(\int_\Omega wud\mu\right)\\ \geq f'\left(\int_\Omega wud\mu\right)\left(u(x)-\int_\Omega wud\mu\right)\end{aligned} \tag{4.8}$$

for any $x\in\Omega$.

If we multiply (4.8) by $w\geq 0$ μ-a.e. on Ω and integrate over the positive measure μ we get

$$\begin{aligned}&\varphi_+(0)\int_\Omega w(x)\,f(u(x))\,d\mu(x)-\left[\varphi'_-(1)+1\right]f\left(\int_\Omega wud\mu\right)\int_\Omega w(x)\,d\mu(x)\\ &\geq f'\left(\int_\Omega wud\mu\right)\int_\Omega w(x)\left(u(x)-\int_\Omega wud\mu\right)d\mu(x)=0,\end{aligned}$$

which produces the desired result (4.7). □

Remark 7. *If the function $f:[m,M]\to[0,\infty)$ is a $\delta(p,k)$-convex and continuous function on the interval $[m,M]$, then for any $u:\Omega\to[m,M]\subset\mathring{I}$ so that $f\circ u$, $u\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$ we have*

$$\int_\Omega w(f\circ u)\,d\mu\geq\frac{1}{k+1}f\left(\int_\Omega wud\mu\right). \tag{4.9}$$

If the function $f:[m,M]\to[0,\infty)$ is a $\eta(s)$-convex and continuous function on the interval $[m,M]$ then for any $u:\Omega\to[m,M]\subset\mathring{I}$ so that $f\circ u$, $u\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$ we have

$$\int_\Omega w(f\circ u)\,d\mu\geq\frac{1}{e^s}f\left(\int_\Omega wud\mu\right). \tag{4.10}$$

These results generalize the inequalities (4.4) and (4.5).

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