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GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE WITH APPLICATIONS FOR UNITARY OPERATORS IN HILBERT SPACES

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Abstract. Some generalized trapezoid type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of integrators of bounded variation are given. Natural applications for functions of unitary operators in Hilbert spaces are provided.

1. Introduction

In [13], in order to approximate the *Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ by the *generalised trapezoid formula*

$$[u(b)-u(x)]\,f(b)+[u(x)-u(a)]\,f(a), \qquad x\in[a,b] \tag{1.1}$$

the authors considered the error functional

$$T(f,u;a,b;x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]\,f(b)-[u(x)-u(a)]\,f(a) \tag{1.2}$$

and proved that

$$|T(f,u;a,b;x)|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f), \qquad x\in[a,b], \tag{1.3}$$

provided that $f:[a,b]\to\mathbb{R}$ is of *bounded variation* on $[a,b]$ and u is of $r-H$*-Hölder type*, that is, $u:[a,b]\to\mathbb{R}$ satisfies the condition $|u(t)-u(s)|\le H\,|t-s|^r$ for any $t,s\in[a,b]$, where $r\in(0,1]$ and $H>0$ are given.

The dual case, namely, when f is of $q-K-$Hölder type and u is of bounded variation has been considered by the authors in [7] in which they obtained the bound:

$$\begin{aligned}&|T(f,u;a,b;x)|\\&\le K\left[(x-a)^q\bigvee_a^x(u)+(b-x)^q\bigvee_x^b(u)\right]\end{aligned} \tag{1.4}$$

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$$\leq \begin{cases} K\left[(x-a)^{q}+(b-x)^{q}\right]\left[\frac{1}{2}\bigvee_{a}^{b}(u)+\frac{1}{2}\left|\bigvee_{a}^{x}(u)-\bigvee_{x}^{b}(u)\right|\right]; \\ K\left[(x-a)^{q\alpha}+(b-x)^{q\alpha}\right]^{\frac{1}{\alpha}}\left[\left[\bigvee_{a}^{x}(u)\right]^{\beta}-\left[\bigvee_{x}^{b}(u)\right]^{\beta}\right]^{\frac{1}{\beta}} \\ \qquad\qquad\qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{q}\bigvee_{a}^{b}(u), \end{cases}$$

for any $x\in[a,b]$.

The case where f is monotonic and u is of $r-H-$Hölder type, which provides a refinement for (1.3), respectively the case where u is monotonic and f of $q-K-$Hölder type were considered by Cheung and Dragomir in [10], while the case where one function was of Hölder type and the other was Lipschitzian were considered in [6]. For other recent results in estimating the error $T(f,u;a,b,x)$ for absolutely continuous integrands f and integrators u of bounded variation, see [8] and [9].

For other inequalities for Riemann-Stieltjes integral, see [1]-[5], [6]-[10], [11]-[15] and [17].

Motivated by the above facts, we consider in the present paper the problem of approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{is}\right)du(s)$ by the *generalised trapezoidal rule*

$$f\left(e^{ib}\right)[u(b)-u(t)]+f\left(e^{ia}\right)[u(t)-u(a)]$$

for continuous complex valued function $f:\mathcal{C}(0,1)\to\mathbb{C}$ defined on the complex unit circle $\mathcal{C}(0,1)$ and various subclasses of functions $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation.

We denote the *error functional* by

$$\begin{aligned} (1.5)\qquad & T_{\mathcal{C}}(f,u;a,b;t) \\ & := f\left(e^{ib}\right)[u(b)-u(t)]+f\left(e^{ia}\right)[u(t)-u(a)]-\int_a^b f\left(e^{is}\right)du(s), \end{aligned}$$

where $t\in[a,b]$ and will provide some bounds for its magnitude for f of $r-H-$Hölder type and u belonging to different subclasses of functions of bounded variation.

The Riemann-Stieltjes integral $\int_0^{2\pi} f\left(e^{is}\right)du(s)$ is related with functions of unitary operators U defined on complex Hilbert spaces as follows.

We recall here some basic facts on unitary operators and spectral families that will be used in the sequel.

We say that the bounded linear operator $U:H\to H$ on the Hilbert space H is *unitary* iff $U^*=U^{-1}$.

It is well known that (see for instance [16, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda\leq E_\mu$ for $0\leq\lambda\leq\mu\leq2\pi$;
b) $E_0=0$ and $E_{2\pi}=1_H$ (the *identity operator* on H);
c) $E_{\lambda+0}=E_\lambda$ for $0\leq\lambda<2\pi$;
d) $U=\int_0^{2\pi}e^{i\lambda}dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $f : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$f(U) = \int_0^{2\pi} f\left(e^{i\lambda}\right) dE_\lambda \tag{1.6}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle f(U)x, y\rangle = \int_0^{2\pi} f\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{1.7}$$

and

$$\|f(U)x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|f\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{1.8}$$

for any $x, y \in H$.

From the above properties it follows that the function $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[0, 2\pi]$ for any $x \in H$.

Some examples of such functions of unitary operators are

$$\exp(U) = \int_0^{2\pi} \exp\left(e^{i\lambda}\right) dE_\lambda$$

and

$$U^n = \int_0^{2\pi} e^{in\lambda} dE_\lambda$$

for n an integer.

We can also define the *trigonometric functions* for a unitary operator U by

$$\sin(U) = \int_0^{2\pi} \sin\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cos(U) = \int_0^{2\pi} \cos\left(e^{i\lambda}\right) dE_\lambda$$

and the *hyperbolic functions* by

$$\sinh(U) = \int_0^{2\pi} \sinh\left(e^{i\lambda}\right) dE_\lambda \text{ and } \cosh(U) = \int_0^{2\pi} \cosh\left(e^{i\lambda}\right) dE_\lambda$$

where

$$\sinh(z) := \frac{1}{2}\left[\exp z - \exp(-z)\right] \text{ and } \cosh(z) := \frac{1}{2}\left[\exp z + \exp(-z)\right], z \in \mathbb{C}.$$

2. Inequalities for the Riemann-Stieltjes Integral

We have the following result.

Theorem 1. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z) - f(w)| \le H|z - w|^r \tag{2.1}$$

for any $w, z \in \mathcal{C}(0,1)$, where $H > 0$ and $r \in (0,1]$ are given.

If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then

$$|T_{\mathcal{C}}(f, u; a, b; t)| \le 2^r H B_r(u; a, b, t) \tag{2.2}$$

for any $t \in [a,b]$*, where the bound* $B_r(u;a,b;t)$ *is given by*

$$B_r(u;a,b;t) := \max_{s\in[a,t]}\left\{\sin^r\left(\frac{s-a}{2}\right)\right\}\bigvee_a^t(u) + \max_{s\in[t,b]}\left\{\sin^r\left(\frac{b-s}{2}\right)\right\}\bigvee_t^b(u). \tag{2.3}$$

Moreover, if we denote

$$A_r(t) := \max_{s\in[a,t]}\left\{\sin^r\left(\frac{s-a}{2}\right)\right\} \text{ and } B_r(t) := \max_{s\in[t,b]}\left\{\sin^r\left(\frac{b-s}{2}\right)\right\},$$

then we have the inequalities:

$$B_r(u;a,b;t) \le \begin{cases} \max\{A_r(t), B_r(t)\}\bigvee_a^b(u) \\ \\ [A_r^\alpha(t)+B_r^\alpha(t)]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^t(u)\right)^\beta + \left(\bigvee_t^b(u)\right)^\beta\right]^{\frac{1}{\beta}} & \alpha>1, \frac{1}{\alpha}+\frac{1}{\beta}=1 \\ \\ [A_r(t)+B_r(t)]\max\left\{\bigvee_a^t(u), \bigvee_t^b(u)\right\} \end{cases} \tag{2.4}$$

for any $t \in [a,b]$*.*

Proof. We have the equality

$$T_{\mathcal{C}}(f,u;a,b;t) = \int_t^b \left[f\left(e^{ib}\right) - f\left(e^{is}\right)\right]du(s) + \int_a^t \left[f\left(e^{is}\right) - f\left(e^{ia}\right)\right]du(s) \tag{2.5}$$

for any $t \in [a,b]$.

It is known that if $p:[c,d]\to\mathbb{C}$ is a continuous function and $v:[c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)\,dv(t)\right| \le \max_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.6}$$

Taking the modulus in the equality (2.5) and utilising the property (2.6) we deduce

$$\begin{aligned} &|T_{\mathcal{C}}(f,u;a,b;t)| \\ &\le \left|\int_t^b \left[f\left(e^{ib}\right)-f\left(e^{is}\right)\right]du(s)\right| + \left|\int_a^t \left[f\left(e^{is}\right)-f\left(e^{ia}\right)\right]du(s)\right| \\ &\le \max_{s\in[t,b]}\left|f\left(e^{ib}\right)-f\left(e^{is}\right)\right|\bigvee_t^b(u) + \max_{s\in[a,t]}\left|f\left(e^{is}\right)-f\left(e^{ia}\right)\right|\bigvee_a^t(u) \\ &\le H\left[\max_{s\in[t,b]}\left|e^{ib}-e^{is}\right|^r\bigvee_t^b(u) + \max_{s\in[a,t]}\left|e^{is}-e^{ia}\right|^r\bigvee_a^t(u)\right] \end{aligned} \tag{2.7}$$

for any $t \in [a,b]$.

Since

$$\left|e^{is}-e^{it}\right|^{2}=\left|e^{is}\right|^{2}-2\operatorname{Re}\left(e^{i(s-t)}\right)+\left|e^{it}\right|^{2}$$
$$=2-2\cos(s-t)=4\sin^{2}\left(\frac{s-t}{2}\right)$$

for any $t,s\in\mathbb{R}$, then

$$\left|e^{is}-e^{it}\right|^{r}=2^{r}\left|\sin\left(\frac{s-t}{2}\right)\right|^{r} \tag{2.8}$$

for any $t,s\in\mathbb{R}$.

For $[a,b]\subseteq[0,2\pi]$ we have

$$\left|e^{ib}-e^{is}\right|^{r}=2^{r}\sin^{r}\left(\frac{b-s}{2}\right)$$

and

$$\left|e^{is}-e^{ia}\right|^{r}=2^{r}\sin^{r}\left(\frac{s-a}{2}\right)$$

for any $s\in[a,b]$.

Utilising the inequality (2.7) we deduce the desired result (2.2).

By making use of the Hölder inequality

$$mp+nq\leq\begin{cases}\max\{m,n\}(p+q)\\ \\ \left(m^{1/\alpha}+n^{1/\alpha}\right)^{\alpha}\left(p^{1/\beta}+q^{1/\beta}\right), & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\end{cases}$$

that holds for $m,p,n,q\geq0$, we deduce the inequality (2.4). □

Define the functional

$$\begin{aligned}M_{\mathcal{C}}(f,u;a,b)&:=T_{\mathcal{C}}\left(f,u;a,b;\frac{a+b}{2}\right)\\&=f\left(e^{ib}\right)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]+f\left(e^{ia}\right)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]\\&\quad-\int_{a}^{b}f\left(e^{is}\right)du(s).\end{aligned} \tag{2.9}$$

For $t=\frac{a+b}{2}$ we have

$$A_{r}\left(\frac{a+b}{2}\right):=\max_{s\in\left[a,\frac{a+b}{2}\right]}\left\{\sin^{r}\left(\frac{s-a}{2}\right)\right\}=\sin^{r}\left(\frac{b-a}{4}\right),$$

and

$$B_{r}\left(\frac{a+b}{2}\right):=\max_{s\in\left[\frac{a+b}{2},b\right]}\left\{\sin^{r}\left(\frac{b-s}{2}\right)\right\}=\sin^{r}\left(\frac{b-a}{4}\right).$$

We can then state the following particular case of interest for applications as shown below.

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\left|M_{\mathcal{C}}(f,u;a,b)\right|\leq2^{r}H\sin^{r}\left(\frac{b-a}{4}\right)\bigvee_{a}^{b}(u)\leq\frac{1}{2^{r}}H(b-a)^{r}\bigvee_{a}^{b}(u). \tag{2.10}$$

In particular, if f is Lipschitzian with the constant $K>0$, then

$$|M_{\mathcal{C}}(f,u;a,b)| \leq 2K\sin\left(\frac{b-a}{4}\right)\bigvee_a^b(u) \leq \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{2.11}$$

The constant 2 in the first inequality (2.11) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. We must only prove the sharpness of the constant 2 in the first inequality (2.11).

Assume that there is an $E>0$ such that

$$\begin{aligned} &\left|f\left(e^{ib}\right)\left[u(b)-u\left(\frac{a+b}{2}\right)\right]+f\left(e^{ia}\right)\left[u\left(\frac{a+b}{2}\right)-u(a)\right]\right. \\ &\left.-\int_a^b f\left(e^{is}\right)du(s)\right| \\ &\leq EK\sin\left(\frac{b-a}{4}\right)\bigvee_a^b(u) \end{aligned} \tag{2.12}$$

for an interval $[a,b]\subseteq[0,2\pi]$, a K-Lipschitzian function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and a function of bounded variation $u:[a,b]\to\mathbb{C}$.

If we take $[a,b]=[0,2\pi]$, $f(z)=z$ then $K=1$ and the inequality (2.12) becomes

$$\left|u(2\pi)-u(0)-\int_0^{2\pi}e^{is}du(s)\right| \leq E\bigvee_0^{2\pi}(u) \tag{2.13}$$

for any function of bounded variation $u:[0,2\pi]\to\mathbb{C}$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_0^{2\pi}e^{is}du(s)=e^{is}u(s)\Big|_0^{2\pi}-i\int_0^{2\pi}e^{is}u(s)\,ds=u(2\pi)-u(0)-i\int_0^{2\pi}e^{is}u(s)\,ds$$

and the inequality (2.13) becomes

$$\left|\int_0^{2\pi}e^{is}u(s)\,ds\right| \leq E\bigvee_0^{2\pi}(u) \tag{2.14}$$

for any function of bounded variation $u:[0,2\pi]\to\mathbb{C}$.

Now, if we take the function

$$u(s):=\begin{cases} -1 & \text{if } s\in[0,\pi] \\ \\ 1 & \text{if } s\in[\pi,2\pi], \end{cases}$$

then u is of bounded variation, $\bigvee_0^{2\pi}(u)=2$ and

$$\begin{aligned} \int_0^{2\pi}e^{is}u(s)\,ds &= -\int_0^{\pi}e^{is}ds+\int_{\pi}^{2\pi}e^{is}ds \\ &= -\frac{1}{i}e^{i\pi}+\frac{1}{i}e^{0}+\frac{1}{i}e^{2\pi}-\frac{1}{i}e^{i\pi}=\frac{4}{i} \end{aligned}$$

and the inequality (2.14) becomes $4\leq 2E$ showing that $E\geq 2$. $\square$

Remark 1. *If the lenght of the interval $[a,b]$ is less than π, i.e. $0<b-a\leq\pi$, then*

$$A_r(t):=\sin^r\left(\frac{t-a}{2}\right) \text{ and } B_r(t):=\sin^r\left(\frac{b-t}{2}\right)$$

and by (2.2) and by (2.4) we have

$$\begin{aligned}
&|T_{\mathcal{C}}(f,u;a,b;t)| \qquad (2.15)\\
&\leq 2^r H\left[\sin^r\left(\frac{t-a}{2}\right)\bigvee_a^t(u)+\sin^r\left(\frac{b-t}{2}\right)\bigvee_t^b(u)\right]\\
&\leq\begin{cases}
\max\left\{\sin^r\left(\frac{t-a}{2}\right),\sin^r\left(\frac{b-t}{2}\right)\right\}\bigvee_a^b(u) & \\
\left[\sin^{\alpha r}\left(\frac{t-a}{2}\right)+\sin^{\alpha r}\left(\frac{b-t}{2}\right)\right]^{\frac{1}{\alpha}} & \\
\times\left[\left(\bigvee_a^t(u)\right)^{\beta}+\left(\bigvee_t^b(u)\right)^{\beta}\right]^{\frac{1}{\beta}} & \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1\\
\left[\sin^r\left(\frac{t-a}{2}\right)+\sin^r\left(\frac{b-t}{2}\right)\right]\max\left\{\bigvee_a^t(u),\bigvee_t^b(u)\right\} &
\end{cases}
\end{aligned}$$

for any $t\in[a,b]$.

Remark 2. *If $a=0$ and $b=2\pi$, then*

$$T_{\mathcal{C}}(f,u;0,2\pi;\pi)=f(1)[u(2\pi)-u(0)]-\int_0^{2\pi}f\left(e^{is}\right)du(s),$$

and

$$A_r(\pi):=\max_{s\in[0,\pi]}\left\{\sin^r\left(\frac{s}{2}\right)\right\}=1$$

while

$$B_r(\pi):=\max_{s\in[\pi,2\pi]}\left\{\sin^r\left(\frac{2\pi-s}{2}\right)\right\}=1.$$

Therefore from (2.2) we have

$$\left|f(1)[u(2\pi)-u(0)]-\int_0^{2\pi}f\left(e^{is}\right)du(s)\right|\leq 2^r H\bigvee_a^b(u). \tag{2.16}$$

Theorem 2. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1). If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$, then*

$$|T_{\mathcal{C}}(f,u;a,b;t)|\leq 2^r LHC_r(a,b;t) \tag{2.17}$$

for any $t\in[a,b]$, where

$$\begin{aligned}
C_r(a,b;t) \quad : \quad &= \int_t^b\sin^r\left(\frac{b-s}{2}\right)ds+\int_a^t\sin^r\left(\frac{s-a}{2}\right)ds \qquad (2.18)\\
&\leq \frac{(b-t)^{r+1}+(t-a)^{r+1}}{(r+1)2^r}
\end{aligned}$$

for any $t \in [a,b]$.

In particular, if $f : \mathcal{C}(0,1) \to \mathbb{C}$ *is Lipschitzian with the constant* $K > 0$, *then we have*

$$|T_{\mathcal{C}}(f,u;a,b;t)| \leq 8LK\left[\sin^2\left(\frac{b-t}{4}\right) + \sin^2\left(\frac{t-a}{4}\right)\right] \tag{2.19}$$

for any $t \in [a,b]$.

Proof. It is well known that if $p : [a,b] \to \mathbb{C}$ is a Riemann integrable function and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right| \leq M\int_a^b |p(t)|\,dt. \tag{2.20}$$

Utilising this property and the equality (2.5) we have

$$\begin{aligned} &|T_{\mathcal{C}}(f,u;a,b;t)| \\ &\leq \left|\int_t^b \left[f\left(e^{ib}\right) - f\left(e^{is}\right)\right] du(s)\right| + \left|\int_a^t \left[f\left(e^{is}\right) - f\left(e^{ia}\right)\right] du(s)\right| \\ &\leq L\left[\int_t^b \left|f\left(e^{ib}\right) - f\left(e^{is}\right)\right| ds + \int_a^t \left|f\left(e^{is}\right) - f\left(e^{ia}\right)\right| ds\right] \\ &\leq LH\left[\int_t^b \left|e^{ib} - e^{is}\right|^r ds + \int_a^t \left|e^{is} - e^{ia}\right|^r ds\right] \\ &= 2^r LH\left[\int_t^b \sin^r\left(\frac{b-s}{2}\right) ds + \int_a^t \sin^r\left(\frac{s-a}{2}\right) ds\right] \end{aligned} \tag{2.21}$$

for any $t \in [a,b]$.

On making use of the elementary inequality $\sin x \leq x, x \in [0,\pi]$ we have

$$\begin{aligned} &\int_t^b \sin^r\left(\frac{b-s}{2}\right) ds + \int_a^t \sin^r\left(\frac{s-a}{2}\right) ds \\ &\leq \int_t^b \left(\frac{b-s}{2}\right)^r ds + \int_a^t \left(\frac{s-a}{2}\right)^r ds \\ &= \frac{(b-t)^{r+1} + (t-a)^{r+1}}{(r+1)\,2^r} \end{aligned}$$

for any $t \in [a,b]$. This proves the inequality (2.18).

For $r = 1$

$$\begin{aligned} C_1(a,b;t) &:= \int_t^b \sin\left(\frac{b-s}{2}\right) ds + \int_a^t \sin\left(\frac{s-a}{2}\right) ds \\ &= 2 - 2\cos\left(\frac{b-t}{2}\right) + 2 - \cos\left(\frac{t-a}{2}\right) \\ &= 4\left[\sin^2\left(\frac{b-t}{4}\right) + \sin^2\left(\frac{t-a}{4}\right)\right] \end{aligned}$$

for any $t \in [a,b]$.

Using (2.17) for $r = 1$ we deduce (2.19). □

Remark 3. *For $a=0$ and $b=2\pi$ we have*

$$\begin{aligned}\sin^2\left(\frac{b-t}{4}\right)+\sin^2\left(\frac{t-a}{4}\right)&=\sin^2\left(\frac{\pi}{2}-\frac{t}{4}\right)+\sin^2\left(\frac{t}{4}\right)\\&=\cos^2\left(\frac{t}{4}\right)+\sin^2\left(\frac{t}{4}\right)=1\end{aligned}$$

and by (2.19) we deduce that

$$\left|f(1)\left[u(2\pi)-u(0)\right]-\int_0^{2\pi}f\left(e^{is}\right)du(s)\right|\le 8LK \tag{2.22}$$

for any $t\in[a,b]$.

The case of midpoint rule $t=\frac{a+b}{2}$ is as follows:

Corollary 2. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ is Lipschitzian with the constant $K>0$ and $u:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[a,b]$. Then we have*

$$\left|M_{\mathcal{C}}(f,u;a,b)\right|\le 16LK\sin^2\left(\frac{b-a}{8}\right). \tag{2.23}$$

The case of monotonic nondecreasing integrators that is important for applications for unitary operators is as follows.

Theorem 3. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1). If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$\left|T_{\mathcal{C}}(f,u;a,b;t)\right|\le 2^r H D_r(u;a,b;t) \tag{2.24}$$

for any $t\in[a,b]$, where

$$\begin{aligned}D_r(u;a,b;t)&:=\int_t^b\sin^r\left(\frac{b-s}{2}\right)du(s)+\int_a^t\sin^r\left(\frac{s-a}{2}\right)du(s)\\&\le\frac{1}{2^r}\left[\int_t^b(b-s)^r\,du(s)+\int_a^t(s-a)^r\,du(s)\right]\end{aligned} \tag{2.25}$$

for any $t\in[a,b]$.

Proof. It is well known that if $p:[a,b]\to\mathbb{C}$ is a continuous function and $v:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and the following inequality holds

$$\left|\int_a^b p(t)\,dv(t)\right|\le\int_a^b|p(t)|\,dv(t). \tag{2.26}$$

Utilising the property (2.26), we have from (2.5) that

$$
\begin{aligned}
(2.27)\qquad & \left|T_{\mathcal{C}}(f,u;a,b;t)\right| \\
&\leq \left|\int_t^b \left[f\left(e^{ib}\right)-f\left(e^{is}\right)\right]du(s)\right|+\left|\int_a^t \left[f\left(e^{is}\right)-f\left(e^{ia}\right)\right]du(s)\right| \\
&\leq \left[\int_t^b \left|f\left(e^{ib}\right)-f\left(e^{is}\right)\right|du(s)+\int_a^t \left|f\left(e^{is}\right)-f\left(e^{ia}\right)\right|du(s)\right] \\
&\leq H\left[\int_t^b \left|e^{ib}-e^{is}\right|^r du(s)+\int_a^t \left|e^{is}-e^{ia}\right|^r du(s)\right] \\
&= 2^r H\left[\int_t^b \sin^r\left(\frac{b-s}{2}\right)du(s)+\int_a^t \sin^r\left(\frac{s-a}{2}\right)du(s)\right]
\end{aligned}
$$

for any $t\in[a,b]$, which proves (2.24).

Moreover, by the elementary inequality $\sin x\leq x, x\in[0,\pi]$ and the monotonicity of u we also have

$$
\begin{aligned}
&\int_t^b \sin^r\left(\frac{b-s}{2}\right)du(s)+\int_a^t \sin^r\left(\frac{s-a}{2}\right)du(s) \\
&\leq \int_t^b \left(\frac{b-s}{2}\right)^r du(s)+\int_a^t \left(\frac{s-a}{2}\right)^r du(s)
\end{aligned}
$$

which proves (2.25). □

Corollary 3. *Assume that f is as in Theorem 3. If the function $u:[0,2\pi]\to\mathbb{R}$ is monotonic nondecreasing on $[0,2\pi]$, then*

$$
\begin{aligned}
(2.28)\qquad & \left|f(1)\left[u(2\pi)-u(0)\right]-\int_0^{2\pi} f\left(e^{is}\right)du(s)\right| \\
&\leq 2^r H\int_0^{2\pi}\sin^r\left(\frac{s}{2}\right)du(s)=2^{r/2}H\int_0^{2\pi}(1-\cos s)^{r/2}du(s).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
D_r(f,u;0,2\pi;t) &:= \int_t^{2\pi}\sin^r\left(\frac{2\pi-s}{2}\right)du(s)+\int_0^t \sin^r\left(\frac{s}{2}\right)du(s) \\
&= \int_t^{2\pi}\sin^r\left(\pi-\frac{s}{2}\right)du(s)+\int_0^t \sin^r\left(\frac{s}{2}\right)du(s) \\
&= \int_t^{2\pi}\sin^r\left(\frac{s}{2}\right)du(s)+\int_0^t \sin^r\left(\frac{s}{2}\right)du(s) \\
&= \int_0^{2\pi}\sin^r\left(\frac{s}{2}\right)du(s)
\end{aligned}
$$

for any $t\in[0,2\pi]$.

Since for $s\in[0,2\pi]$ we have

$$
\sin\left(\frac{s}{2}\right)=\left(\frac{1-\cos s}{2}\right)^{1/2}
$$

then the last part of (2.28) is obtained. □

3. Applications for Functions of Unitary Operators

We have the following vector inequality for functions of unitary operators.

Theorem 4. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the Hölder's type condition (2.1). If the operator $U:H\to H$ on the Hilbert space H is unitary and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ is its spectral family, then*

$$|f(1)\langle x,y\rangle-\langle f(U)x,y\rangle|\le 2^r H\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le 2^r H\|x\|\|y\| \tag{3.1}$$

for any $x,y\in H$.

Proof. For given $x,y\in H$, define the function $u(\lambda):=\langle E_\lambda x,y\rangle,\lambda\in[0,2\pi]$. We will show that u is of bounded variation and

$$\bigvee_0^{2\pi}(u)=:\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)\le\|x\|\|y\|. \tag{3.2}$$

It is well known that, if P is a nonnegative selfadjoint operator on H, i.e., $\langle Px,x\rangle\ge 0$ for any $x\in H$, then the following inequality is a *generalization of the Schwarz inequality* in H

$$|\langle Px,y\rangle|^2\le\langle Px,x\rangle\langle Py,y\rangle, \tag{3.3}$$

for any $x,y\in H$.

Now, if $d:0=t_0<t_1<...<t_{n-1}<t_n=2\pi$ is an arbitrary partition of the interval $[0,2\pi]$, then we have by Schwarz's inequality for nonnegative operators (3.3) that

$$\begin{aligned}&\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \\ &=\sup_d\left\{\sum_{i=0}^{n-1}\left|\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,y\right\rangle\right|\right\} \\ &\le\sup_d\left\{\sum_{i=0}^{n-1}\left[\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle^{1/2}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle^{1/2}\right]\right\}:=I.\end{aligned} \tag{3.4}$$

By the Cauchy-Buniakovski-Schwarz inequality for sequences of real numbers we also have that

$$\begin{aligned}I&\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\ &\le\sup_d\left\{\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)x,x\right\rangle\right]^{1/2}\left[\sum_{i=0}^{n-1}\left\langle\left(E_{t_{i+1}}-E_{t_i}\right)y,y\right\rangle\right]^{1/2}\right\} \\ &=\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,x\right\rangle\right)\right]^{1/2}\left[\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}y,y\right\rangle\right)\right]^{1/2}=\|x\|\|y\|\end{aligned} \tag{3.5}$$

for any $x,y\in H$.

Utilising the inequality (2.16) we can write that

$$\left| f(1)\langle E_{2\pi}x,y\rangle - \int_0^{2\pi} f\left(e^{is}\right) d\langle E_s x,y\rangle \right| \leq 2^r H \bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \tag{3.6}$$

for any $x, y \in H$.

On making use of the representation theorem (1.7) and the inequality (3.2) we deduce the desired result (3.1). $\square$

Theorem 5. *With the assumptions of Theorem 4 we have*

$$\left| f(1)\|x\|^2 - \langle f(U)x,x\rangle \right| \leq 2^{r/2} H \left\langle [1_H - \operatorname{Re}(U)]^{r/2} x, x \right\rangle, \tag{3.7}$$

for any $x \in H$, where

$$\operatorname{Re}(U) := \frac{U+U^*}{2}.$$

Proof. Utilising the inequality (2.28), we have

$$\begin{aligned} &\left| f(1)[\langle E_{2\pi}x,x\rangle - \langle E_0 x,x\rangle] - \int_0^{2\pi} f\left(e^{is}\right) d\langle E_s x,x\rangle \right| \\ &\leq 2^{r/2} H \int_0^{2\pi} (1-\cos s)^{r/2} d\langle E_s x,x\rangle \end{aligned} \tag{3.8}$$

for any $x \in H$.

Since

$$\begin{aligned} \int_0^{2\pi} (1-\cos s)^{r/2} d\langle E_s x,x\rangle &= \int_0^{2\pi} \left(1-\operatorname{Re}\left(e^{is}\right)\right)^{r/2} d\langle E_s x,x\rangle \\ &= \left\langle (1_H - \operatorname{Re}(U))^{r/2} x, x\right\rangle \end{aligned}$$

for any $x \in H$, then by (3.8) we get the desired result (3.7). $\square$

Example 1. *In order to provide some simple examples for the inequalities above we choose two complex functions as follows.*

a) *Consider the power function $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = z^m$ where m is a nonzero integer. Then, obviously, for any z, w belonging to the unit circle $\mathcal{C}(0,1)$ we have the inequality*

$$|f(z) - f(w)| \leq |m|\,|z-w|$$

which shows that f is Lipschitzian with the constant $L = |m|$ on the circle $\mathcal{C}(0,1)$.

Then from (3.7), we get for any unitary operator U that

$$\left| \|x\|^2 - \langle U^m x,x\rangle \right| \leq 2^{1/2}|m| \left\langle (1_H - \operatorname{Re}(U))^{1/2} x, x\right\rangle, \tag{3.9}$$

for any $x \in H$.

For $m = 1$ we also get from (3.1) that

$$|\langle x,y\rangle - \langle Ux,y\rangle| \leq 2\bigvee_0^{2\pi}\left(\left\langle E_{(\cdot)}x,y\right\rangle\right) \leq 2\|x\|\,\|y\| \tag{3.10}$$

for any $x, y \in H$.

b) *For* $a \neq \pm 1, 0$ *consider the function* $f : \mathcal{C}(0,1) \to \mathbb{C}$, $f_a(z) = \frac{1}{1-az}$. *Observe that*

$$|f_a(z) - f_a(w)| = \frac{|a| |z-w|}{|1-az| |1-aw|} \tag{3.11}$$

for any $z, w \in \mathcal{C}(0,1)$.

If $z = e^{it}$ *with* $t \in [0, 2\pi]$, *then we have*

$$\begin{aligned} |1-az|^2 &= 1 - 2a\operatorname{Re}(\bar{z}) + a^2 |z|^2 = 1 - 2a\cos t + a^2 \\ &\geq 1 - 2|a| + a^2 = (1-|a|)^2 \end{aligned}$$

therefore

$$\frac{1}{|1-az|} \leq \frac{1}{|1-|a||} \text{ and } \frac{1}{|1-aw|} \leq \frac{1}{|1-|a||} \tag{3.12}$$

for any $z, w \in \mathcal{C}(0,1)$.

Utilising (3.11) and (3.12) we deduce

$$|f_a(z) - f_a(w)| \leq \frac{|a|}{(1-|a|)^2} |z-w| \tag{3.13}$$

for any $z, w \in \mathcal{C}(0,1)$, *showing that the function* f_a *is Lipschitzian with the constant* $L_a = \frac{|a|}{(1-|a|)^2}$ *on the circle* $\mathcal{C}(0,1)$.

Now, if we employ the inequality (3.1), then we can state the inequality

$$\begin{aligned} \left| (1-a)^{-1} \langle x, y\rangle - \left\langle (1_H - aU)^{-1} x, y \right\rangle \right| &\leq \frac{2|a|}{(1-|a|)^2} \bigvee_0^{2\pi} \left(\langle E_{(\cdot)} x, y \rangle \right) \\ &\leq \frac{2|a|}{(1-|a|)^2} \|x\| \|y\| \end{aligned} \tag{3.14}$$

for any unitary operator U *and for any* $x, y \in H$.

On making use of the inequality (3.7) we also have

$$\begin{aligned} &\left| (1-a)^{-1} \|x\|^2 - \left\langle (1_H - aU)^{-1} x, x \right\rangle \right| \\ &\leq \frac{2^{1/2} |a|}{(1-|a|)^2} \left\langle [1_H - \operatorname{Re}(U)]^{1/2} x, x \right\rangle, \end{aligned} \tag{3.15}$$

for any $x \in H$.

4. A Quadrature Rule

We consider the following partition of the interval $[a, b]$

$$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and the intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$. Define $h_k := x_{k+1} - x_k$, $0 \leq k \leq n-1$ and $\nu(\Delta_n) = \max\{h_k : 0 \leq k \leq n-1\}$ the norm of the partition Δ_n.

For the continuous function $f:\mathcal{C}(0,1)\to\mathbb{C}$ and the function $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ of bounded variation on $[a,b]$, define the *generalised trapezoid quadrature rule*

$$\begin{aligned}(4.1)\qquad & T_n(f,u,\Delta_n,\xi)\\ &:=\sum_{k=0}^{n-1}\left\{f\left(e^{ix_{k+1}}\right)\left[u(x_{k+1})-u(\xi_k)\right]+f\left(e^{ix_k}\right)\left[u(\xi_k)-u(x_k)\right]\right\}\end{aligned}$$

and the remainder $R_n(f,u,\Delta_n,\xi)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right)du(t)$ by $T_n(f,u,\Delta_n,\xi)$. Then we have

$$\int_a^b f\left(e^{it}\right)du(t)=T_n(f,u,\Delta_n,\xi)+R_n(f,u,\Delta_n,\xi). \tag{4.2}$$

The following result provides *a priory* bounds for $R_n(f,u,\Delta_n,\xi)$ in several instances of f and u as above.

Proposition 1. *Assume that $f:\mathcal{C}(0,1)\to\mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(z)-f(w)|\le H|z-w|^r$$

for any $w,z\in\mathcal{C}(0,1)$, where $H>0$ and $r\in(0,1]$ are given.

If $[a,b]\subseteq[0,2\pi]$ and the function $u:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then for any partition $\Delta_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ with the norm $\nu(\Delta_n)\le\pi$ we have the error bound

$$\begin{aligned}(4.3)\qquad & |R_n(f,u,\Delta_n,\xi)|\\ &\le 2^rH\sum_{k=0}^{n-1}\sin^r\left(\frac{1}{2}\left[\frac{x_{k+1}-x_k}{2}+\left|\xi_k-\frac{x_{k+1}+x_k}{2}\right|\right]\right)\bigvee_{x_k}^{x_{k+1}}(u)\\ &\le 2^rH\sum_{k=0}^{n-1}\sin^r\left(\frac{x_{k+1}-x_k}{2}\right)\bigvee_{x_k}^{x_{k+1}}(u)\le 2^r\sum_{k=0}^{n-1}\sin^r\left(\frac{x_{k+1}-x_k}{2}\right)\bigvee_{x_k}^{x_{k+1}}(u)\\ &\le 2^rH\sin^r\left(\frac{\nu(\Delta_n)}{2}\right)\bigvee_a^b(u)\le\nu^r(\Delta_n)H\bigvee_a^b(u)\end{aligned}$$

for any intermediate points $\xi_k\in[x_k,x_{k+1}]$ where $0\le k\le n-1$.

Proof. Since $\nu(\Delta_n) \leq \pi$, then on writing inequality (2.15) on each interval $[x_k, x_{k+1}]$ and for any intermediate points $\xi_k \in [x_k, x_{k+1}]$ where $0 \leq k \leq n-1$, we have

$$
\begin{aligned}
(4.4)\qquad & \left| \int_{x_k}^{x_{k+1}} f\left(e^{it}\right) du(t) \right. \\
& \left. - f\left(e^{ix_{k+1}}\right)\left[u(x_{k+1}) - u(\xi_k)\right] - f\left(e^{ix_k}\right)\left[u(\xi_k) - u(x_k)\right] \right| \\
& \leq 2^r H \left[\sin^r\left(\frac{\xi_k - x_k}{2}\right) \bigvee_{x_k}^{\xi_k}(u) + \sin^r\left(\frac{x_{k+1} - \xi_k}{2}\right) \bigvee_{\xi_k}^{x_{k+1}}(u) \right] \\
& \leq 2^r H \max\left\{ \sin^r\left(\frac{\xi_k - x_k}{2}\right), \sin^r\left(\frac{x_{k+1} - \xi_k}{2}\right) \right\} \bigvee_{x_k}^{x_{k+1}}(u) \\
& = 2^r H \sin^r\left(\max\left\{ \left(\frac{\xi_k - x_k}{2}\right), \left(\frac{x_{k+1} - \xi_k}{2}\right) \right\} \right) \bigvee_{x_k}^{x_{k+1}}(u) \\
& \leq 2^r H \sin^r\left(\frac{1}{2}\left[\frac{x_{k+1} - x_k}{2} + \left| \xi_k - \frac{x_{k+1} + x_k}{2} \right| \right] \right) \bigvee_{x_k}^{x_{k+1}}(u) \\
& \leq 2^r H \sin^r\left(\frac{x_{k+1} - x_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \leq 2^r H \sin^r\left(\frac{x_{k+1} - x_k}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \\
& \leq 2^r H \sin^r\left(\frac{\nu(\Delta_n)}{2}\right) \bigvee_{x_k}^{x_{k+1}}(u) \leq \nu^r(\Delta_n) H \bigvee_{x_k}^{x_{k+1}}(u)
\end{aligned}
$$

Summing over k from 0 to $n-1$ in (4.4) and utilizing the generalized triangle inequality, we deduce (4.3). □

For the continuous function $f : \mathcal{C}(0,1) \to \mathbb{C}$ and the function $u : [a,b] \subseteq [0, 2\pi] \to \mathbb{C}$ of bounded variation on $[a,b]$, define the *trapezoid midpoint quadrature rule*

$$
\begin{aligned}
(4.5)\qquad M_n(f, u, \Delta_n) := & \sum_{k=0}^{n-1} f\left(e^{ix_{k+1}}\right)\left[u(x_{k+1}) - u\left(\frac{x_k + x_{k+1}}{2}\right) \right] \\
& + \sum_{k=0}^{n-1} f\left(e^{ix_k}\right)\left[u\left(\frac{x_k + x_{k+1}}{2}\right) - u(x_k) \right]
\end{aligned}
$$

and the remainder $T_n(f, u, \Delta_n)$ in approximating the Riemann-Stieltjes integral $\int_a^b f\left(e^{it}\right) du(t)$ by $M_n(f, u, \Delta_n)$. Then we have

$$
(4.6)\qquad \int_a^b f\left(e^{it}\right) du(t) = M_n(f, u, \Delta_n) + T_n(f, u, \Delta_n).
$$

Proposition 2. *Assume that f and Δ_n are as in Proposition 1, then we have the error bound*

$$
\begin{aligned}
(4.7)\qquad |T_n(f, u, \Delta_n)| & \leq 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{x_{k+1} - x_k}{4}\right) \bigvee_{x_k}^{x_{k+1}}(u) \\
& \leq 2^r H \sin^r\left(\frac{\nu(\Delta_n)}{4}\right) \bigvee_a^b(u) \leq \frac{1}{2^r} \nu^r(\Delta_n) H \bigvee_a^b(u).
\end{aligned}
$$

We consider the following partition of the interval $[0, 2\pi]$

$$\Gamma_n : 0 = \lambda_0 < \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi$$

and the intermediate points $\xi_k \in [\lambda_k, \lambda_{k+1}]$ where $0 \le k \le n-1$. Define $h_k := \lambda_{k+1} - \lambda_k$, $0 \le k \le n-1$ and $\nu(\Gamma_n) = \max\{h_k : 0 \le k \le n-1\}$ the norm of the partition Γ_n.

If U is a unitary operator on the Hilbert space H and $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, the spectral family of U, then we can introduce the following sums

$$\begin{aligned}(4.8)\qquad & T_n(f, \Gamma_n, \xi; x, y) \\ &:= \sum_{k=0}^{n-1} \left\{ f\left(e^{i\lambda_{k+1}}\right) \left\langle \left(E_{\lambda_{k+1}} - E_{\xi_k}\right) x, y\right\rangle + f\left(e^{i\lambda_k}\right) \left\langle \left(E_{\xi_k} - E_{\lambda_k}\right) x, y\right\rangle \right\}\end{aligned}$$

for $x, y \in H$.

For a function $f : \mathcal{C}(0,1) \to \mathbb{C}$ that satisfies the Hölder's type condition (2.1), we can approximate the function f of unitary operator U as follows

$$(4.9)\qquad \langle f(U)x, y\rangle = T_n(f, \Gamma_n, \xi; x, y) + R_n(f, \Gamma_n, \xi; x, y)$$

for $x, y \in H$, where the reminder satisfies the bounds

$$\begin{aligned}(4.10)\qquad & |R_n(f, \Gamma_n, \xi; x, y)| \\ &\le 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{1}{2}\left[\frac{\lambda_{k+1}-\lambda_k}{2} + \left|\xi_k - \frac{\lambda_{k+1}+\lambda_k}{2}\right|\right]\right) \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\le 2^r H \sum_{k=0}^{n-1} \sin^r\left(\frac{\lambda_{k+1}-\lambda_k}{2}\right) \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\le 2^r \sum_{k=0}^{n-1} \sin^r\left(\frac{\lambda_{k+1}-\lambda_k}{2}\right) \bigvee_{\lambda_k}^{\lambda_{k+1}} \left(\langle E_{(\cdot)}x, y\rangle\right) \\ &\le 2^r H \sin^r\left(\frac{\nu(\Gamma_n)}{2}\right) \bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right) \le \nu^r(\Gamma_n) H \bigvee_0^{2\pi} \left(\langle E_{(\cdot)}x, y\rangle\right)\end{aligned}$$

for any $x, y \in H$.

The interested reader may apply the above results for various Lipschitzian functions $f : \mathcal{C}(0,1) \to \mathbb{C}$. However, the details are not presented here.

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