OSTROWSKI VIA A TWO FUNCTIONS POMPEIU'S INEQUALITY

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ABSTRACT. In this paper, some generalizations of Pompeiu's inequality for two complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type results. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

1. INTRODUCTION

In 1946, Pompeiu [6] derived a variant of *Lagrange's mean value theorem*, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$(1.1) \qquad \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

The following inequality is useful to derive some Ostrowski type inequalities.

Corollary 1 (Pompeiu's Inequality). *With the assumptions of Theorem 1 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - t f'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$(1.2) \qquad |t f(x) - x f(t)| \leq \|f - \ell f'\|_\infty |x - t|$$

for any $t, x \in [a, b]$.

The inequality (1.2) was stated by the author in [3].

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 2 (Ostrowski, 1938 [4]). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $|f'(t)| \leq M < \infty$ for all $t \in (a, b)$. Then for any $x \in [a, b]$, we have the inequality*

$$(1.3) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a).$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 3 (Dragomir, 2005 [3])**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *with* $[a, b]$ *not containing* 0. *Then for any* $x \in [a, b]$, *we have the inequality*

$$(1.4) \qquad \left| \frac{a + b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b - a} \int_a^b f(t)\, dt \right|$$

$$\leq \frac{b - a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] \| f - \ell f' \|_\infty,$$

where $\ell(t) = t$, $t \in [a, b]$.

The constant $\frac{1}{4}$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

In [7], E. C. Popa using a mean value theorem obtained a generalization of (1.4) as follows:

Theorem 4 (Popa, 2007 [7])**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *Assume that* $\alpha \notin [a, b]$. *Then for any* $x \in [a, b]$, *we have the inequality*

$$(1.5) \qquad \left| \left(\frac{a + b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b - a} \int_a^b f(t)\, dt \right|$$

$$\leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] (b - a) \| f - \ell_\alpha f' \|_\infty,$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \leq p \leq \infty$ which for $p = \infty$ gives Dragomir's result.

Theorem 5 (Pečarić & Ungar, 2006 [5])**.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b) *with* $0 < a < b$. *Then for* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *we have the inequality*

$$(1.6) \qquad \left| \frac{a + b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b - a} \int_a^b f(t)\, dt \right| \leq PU(x, p) \| f - \ell f' \|_p,$$

for $x \in [a, b]$, *where*

$$PU(x, p) \quad : \quad = (b - a)^{\frac{1}{p} - 1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1 - 2q)(2 - q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1 - 2q)(1 + q)} \right)^{1/q} \right.$$

$$\left. + \left(\frac{b^{2-q} - x^{2-q}}{(1 - 2q)(2 - q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1 - 2q)(1 + q)} \right)^{1/q} \right].$$

In the cases $(p, q) = (1, \infty), (\infty, 1)$ *and* $(2, 2)$ *the quantity* $PU(x, p)$ *has to be taken as the limit as* $p \to 1, \infty$ *and* 2, *respectively.*

For other inequalities in terms of the p-norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$ and $\alpha \notin [a, b]$ see [1] and [2].

In this paper, some new Pompeiu's type inequalities for two complex-valued absolutely continuous functions are provided. They are applied to obtain some Ostrowski type inequalities. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

2. A General Pompeiu's Inequality

We start with the following generalization of (1.2).

Theorem 6. *Let* $f, g : [a, b] \to \mathbb{C}$ *be absolutely continuous functions on the interval* $[a, b]$ *with* $g(t) \neq 0$ *for all* $t \in [a, b]$. *Then for any* $t, x \in [a, b]$ *we have*

(2.1) $\left| \dfrac{f(x)}{g(x)} - \dfrac{f(t)}{g(t)} \right|$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_\infty[a, b], \\[2em] \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} \text{if } f'g - fg' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2em] \|f'g - fg'\|_1 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \end{cases}$$

or, equivalently

(2.2) $|g(t) f(x) - f(t) g(x)|$

$$\leq \begin{cases} \|f'g - fg'\|_\infty |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_\infty[a, b], \\[2em] \|f'g - fg'\|_p |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{array}{c} \text{if } f'g - fg' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2em] \|f'g - fg'\|_1 |g(t) g(x)| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\}. \end{cases}$$

Proof. If f and g are absolutely continuous and $g(t) \neq 0$ for all $t \in [a, b]$, then f/g is absolutely continuous on the interval $[a, b]$ and

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds,$$

then we get the following identity

(2.3) $\dfrac{f(x)}{g(x)} - \dfrac{f(t)}{g(t)} = \displaystyle\int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds$

for any $t, x \in [a, b]$.

Taking the modulus in (2.3) we have

$$(2.4) \qquad \left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| = \left| \int_t^x \frac{f'(s) g(s) - f(s) g'(s)}{g^2(s)} ds \right|$$

$$\leq \left| \int_t^x \frac{|f'(s) g(s) - f(s) g'(s)|}{|g(s)|^2} ds \right| := I$$

and utilizing Hölder's integral inequality we deduce

$$I \leq \begin{cases} \sup_{s \in [t,x]([x,t])} |f'(s) g(s) - f(s) g'(s)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\[2mm] \left| \int_t^x |f'(s) g(s) - f(s) g'(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \quad \begin{array}{l} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2mm] \left| \int_t^x |f'(s) g(s) - f(s) g'(s)| ds \right| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\}, \end{cases}$$

$$= \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\[2mm] \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \quad \begin{array}{l} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2mm] \|f'g - fg'\|_1 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \end{cases}$$

and the inequality (2.1) is proved. □

The following particular case extends Pompeiu's inequality to other p-norms than $p = \infty$ obtained in (1.2).

Corollary 2. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $t, x \in [a, b]$ we have*

$$(2.5) \quad \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right|$$

$$\leq \begin{cases} \|f - \ell f'\|_\infty \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f - \ell f' \in L_\infty [a, b], \\[3mm] \frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \begin{array}{l} \text{if } f - \ell f' \in L_p [a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[3mm] \|f - \ell f'\|_1 \frac{1}{\min\{t^2, x^2\}} \end{cases}$$

or, equivalently

$$(2.6) \quad |tf(x) - xf(t)|$$

$$\leq \begin{cases} \|f - \ell f'\|_\infty \, |x - t| & \text{if } f - \ell f' \in L_\infty\,[a,b]\,, \\[2mm] \frac{1}{(2q-1)^{1/q}} \, \|f - \ell f'\|_p \, \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \text{if } f - \ell f' \in L_p\,[a,b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\[2mm] \|f - \ell f'\|_1 \, \frac{\max\{t,x\}}{\min\{t,x\}}, & \end{cases}$$

where $\ell(t) = t, t \in [a,b]\,.$

The proof follows by (2.1) for $g(t) = \ell(t) = t,\ t \in [a,b]\,.$
The general case for power functions is as follows.

Corollary 3. *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *with* $b > a > 0.$ *If* $r \in \mathbb{R},\ r \neq 0,$ *then for any* $t, x \in [a,b]$ *we have*

$$(2.7) \quad \left|\frac{f(x)}{x^r} - \frac{f(t)}{t^r}\right|$$

$$\leq \begin{cases} \frac{1}{|r|} \, \|f'\ell - rf\|_\infty \, \left|\frac{1}{x^r} - \frac{1}{t^r}\right|, & \text{if } f'\ell - rf \in L_\infty\,[a,b]\,, \\[3mm] \|f'\ell - rf\|_p \\ \quad \times \begin{cases} \frac{1}{|1 - q(r+1)|^{1/q}} \left|\frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}}\right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\[2mm] |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \quad \text{if } f'\ell - rf \in L_p\,[a,b]\,, \\[3mm] \|f'\ell - rf\|_1 \, \frac{1}{\min\{x^{r+1}, t^{r+1}\}}, & \end{cases}$$

or, equivalently

$$(2.8) \quad |t^r f(x) - x^r f(t)|$$

$$\leq \begin{cases} \frac{1}{|r|} \, \|f'\ell - rf\|_\infty \, |t^r - x^r|, & \text{if } f'\ell - rf \in L_\infty\,[a,b]\,, \\[3mm] \|f'\ell - rf\|_p \\ \quad \times \begin{cases} \frac{t^r x^r}{|1 - q(r+1)|^{1/q}} \left|\frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}}\right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\[2mm] t^r x^r \, |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \quad \text{if } f'\ell - rf \in L_p\,[a,b]\,, \\[3mm] \|f'\ell - rf\|_1 \, \frac{t^r x^r}{\min\{x^{r+1}, t^{r+1}\}}, & \end{cases}$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1.$

The proof follows by (2.1) for $g(t) = t^r$, $t \in [a, b]$. The details for calculations are omitted.

We have the following result for exponential.

Corollary 4. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then for any $t, x \in [a, b]$ we have*

(2.9) $\left| \dfrac{f(x)}{\exp(i\alpha x)} - \dfrac{f(t)}{\exp(i\alpha t)} \right|$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty |x - t| & \text{if } f' - i\alpha f \in L_\infty[a, b], \\[2mm] \|f' - i\alpha f\|_p |x - t|^{1/q} & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2mm] \|f' - i\alpha f\|_1 \end{cases}$$

or, equivalently

(2.10) $|\exp(i\alpha t) f(x) - f(t) \exp(i\alpha x)|$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty |x - t| & \text{if } f' - i\alpha f \in L_\infty[a, b], \\[2mm] \|f' - i\alpha f\|_p |x - t|^{1/q} & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[2mm] \|f' - i\alpha f\|_1 . \end{cases}$$

3. An Inequality Generalizing Ostrowski's

The following result holds:

Theorem 7. *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$. If $0 < m \leq |g(t)| \leq M < \infty$ for any $t \in [a, b]$, then*

(3.1) $\left| f(x) \displaystyle\int_a^b g(t)\, dt - g(x) \int_a^b f(t)\, dt \right|$

$$\leq \left(\frac{M}{m} \right)^2 \begin{cases} \|f'g - fg'\|_\infty (b - a)^2 \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b - a} \right)^2 \right] & \text{if } f'g - fg' \in L_\infty[a, b], \\[3mm] \|f'g - fg'\|_p \left[\frac{(b - x)^{1 + 1/q} + (x - a)^{1 + 1/q}}{1 + 1/q} \right] & \begin{array}{c} \text{if } f'g - fg' \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[3mm] \|f'g - fg'\|_1 (b - a) \end{cases}$$

for any $x \in [a, b]$.

Proof. Utilizing (2.2) we have

$$(3.2) \quad \left| f\left(x\right) \int_a^b g\left(t\right) dt - g\left(x\right) \int_a^b f\left(t\right) dt \right|$$

$$\leq \int_a^b \left| g\left(t\right) f\left(x\right) - f\left(t\right) g\left(x\right) \right| dt$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty |g\left(x\right)| \int_a^b \left(|g\left(t\right)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\\\ \|f'g - fg'\|_p |g\left(x\right)| \int_a^b \left(|g\left(t\right)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\\\ \|f'g - fg'\|_1 |g\left(x\right)| \int_a^b \left(|g\left(t\right)| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases}$$

for any $x \in [a,b]$, which is of interest in itself.

Since $0 < m \leq |g\left(t\right)| \leq M < \infty$ for any $t \in [a,b]$, then

$$|g\left(x\right)| \int_a^b \left(|g\left(t\right)| \left| \int_t^x \frac{1}{|g\left(s\right)|^2} ds \right| \right) dt \leq \left(\frac{M}{m}\right)^2 \int_a^b |x-t| dt$$

$$= \left(\frac{M}{m}\right)^2 \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right],$$

$$|g\left(x\right)| \int_a^b \left(|g\left(t\right)| \left| \int_t^x \frac{1}{|g\left(s\right)|^{2q}} ds \right|^{1/q} \right) dt$$

$$\leq \left(\frac{M}{m}\right)^2 \int_a^b |x-t|^{1/q} dt = \left(\frac{M}{m}\right)^2 \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}$$

and

$$|g\left(x\right)| \int_a^b \left(|g\left(t\right)| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{|g\left(s\right)|^2} \right\} \right) dt \leq \left(\frac{M}{m}\right)^2 \int_a^b dt = \left(\frac{M}{m}\right)^2 (b-a)$$

for any $x \in [a,b]$ and by (3.2) we get the desired result (3.1). $\qquad\square$

Remark 1. *If we take $g\left(t\right) = 1, t \in [a,b]$ in the first inequality (3.1) we recapture Ostrowski's inequality.*

Corollary 5. *With the assumptions in Theorem 7 we have the midpoint inequalities*

$$(3.3) \quad \left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right) \int_a^b f(t)\,dt \right|$$

$$\leq \left(\frac{M}{m}\right)^2 \begin{cases} \frac{1}{4}(b-a)^2 \, \|f'g - fg'\|_\infty & \text{if } f'g - fg' \in L_\infty[a,b], \\[4mm] \frac{1}{2^{1/q}(1+1/q)}(b-a)^{1+1/q} \, \|f'g - fg'\|_p & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases}$$

The following result also holds:

Theorem 8. *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$, $g(x) \neq 0$ for $x \in [a,b]$ and $g^{-2} \in L_\infty[a,b]$. Then*

$$(3.4) \quad \left| \frac{f(x)}{g(x)} \int_a^b g(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$\leq \|g^{-2}\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \, |x-t|\,dt, & \text{if } f'g - fg' \in L_\infty[a,b], \\[4mm] \|f'g - fg'\|_p \int_a^b |g(t)| \, |x-t|^{1/q}\,dt & \begin{array}{c} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[4mm] \|f'g - fg'\|_1 \int_a^b |g(t)|\,dt \end{cases}$$

for any $x \in [a,b]$.

Proof. Utilizing (2.2) we have

$$(3.5) \quad \left| \frac{f(x)}{g(x)} \int_a^b g(t)\,dt - \int_a^b f(t)\,dt \right|$$

$$\leq \begin{cases} \|f'g - fg'\|_\infty \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2}\,ds \right| \right) dt, \\[4mm] \|f'g - fg'\|_p \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}}\,ds \right|^{1/q} \right) dt, \\[4mm] \|f'g - fg'\|_1 \int_a^b \left(|g(t)| \sup_{s\in[t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases}$$

for any $x \in [a,b]$.

Since

$$\left| \int_t^x \frac{1}{|g(s)|^2}\,ds \right| \leq \|g^{-2}\|_\infty \, |x-t|,$$

$$\left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \leq \left\| g^{-2} \right\|_\infty |x - t|^{1/q}$$

and

$$\sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \leq \left\| g^{-2} \right\|_\infty$$

for any $x, t \in [a, b]$, then on making use of (3.5) we get the desired result (3.4). $\square$

We have the midpoint inequalities:

Corollary 6. *With the assumptions of Theorem 8 we have*

$$(3.6) \quad \left| \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \int_a^b g(t)\, dt - \int_a^b f(t)\, dt \right|$$

$$\leq \left\| g^{-2} \right\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right| dt, & \text{if } f'g - fg' \in L_\infty[a,b], \\[4mm] \|f'g - fg'\|_p \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right|^{1/q} dt & \begin{array}{l} \text{if } f'g - fg' \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases}$$

We have the following exponential version of Ostrowski's inequality as well:

Theorem 9. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then for any $x \in [a,b]$ we have*

$$(3.7) \quad \left| \frac{\exp(i\alpha(b-x)) - \exp(-i\alpha(x-a))}{i\alpha} f(x) - \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a} \right)^2 \right], & \text{if } f' - i\alpha f \in L_\infty[a,b], \\[4mm] \|f' - i\alpha f\|_p \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1 + 1/q}, & \begin{array}{l} \text{if } f' - i\alpha f \\ \in L_p[a,b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[4mm] \|f' - i\alpha f\|_1. & \end{cases}$$

Proof. If we write the inequality (3.2) for $g(t) = \exp(i\alpha t)$, $t \in [a, b]$, then we get

$$\left| f(x) \int_a^b \exp(i\alpha t)\, dt - \exp(i\alpha x) \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \|f' - i\alpha f\|_\infty \int_a^b |x - t|\, dt, \text{ if } f' - i\alpha f \in L_\infty[a, b] \\[2mm] \|f' - i\alpha f\|_p \, |g(x)| \int_a^b |x - t|^{1/q}\, dt, & \begin{array}{c} \text{if } f' - i\alpha f \\ \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array} \\[2mm] \|f' - i\alpha f\|_1, \end{cases}$$

which, after simple calculation, is equivalent with (3.7).

The details are omitted. $\square$

Corollary 7. *With the assumptions of Theorem 9 we have the midpoint inequalities*

$$(3.8) \quad \left| \frac{\exp\left(i\alpha\left(\frac{b-a}{2}\right)\right) - \exp\left(-i\alpha\left(\frac{b-a}{2}\right)\right)}{i\alpha} f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \frac{1}{4}\|f' - i\alpha f\|_\infty (b-a)^2, & \text{if } f' - i\alpha f \in L_\infty[a, b], \\[4mm] \frac{1}{2^{1/q}(1 + 1/q)}(b-a)^{1+1/q}\|f' - i\alpha f\|_p, & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{array} \end{cases}$$

or, equivalently

$$(3.9) \quad \left| \frac{2\sin\left(\alpha\left(\frac{b-a}{2}\right)\right)}{\alpha} f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\, dt \right|$$

$$\leq \begin{cases} \frac{1}{4}\|f' - i\alpha f\|_\infty (b-a)^2, & \text{if } f' - i\alpha f \in L_\infty[a, b], \\[4mm] \frac{1}{2^{1/q}(1 + 1/q)}(b-a)^{1+1/q}\|f' - i\alpha f\|_p, & \begin{array}{c} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{array} \end{cases}$$

4. AN APPLICATION FOR CBS-INEQUALITY

The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$(4.1) \quad \left| \int_a^b f(t)\, g(t)\, dt \right|^2 \leq \int_a^b |f(t)|^2\, dt \int_a^b |g(t)|^2\, dt,$$

provided that $f, g \in L_2[a, b]$.

We have the following result concerning some reverses of the CBS-inequality:

Theorem 10. *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $g(t) \neq 0$ for all $t \in [a, b]$. Then*

$$(4.2) \quad 0 \leq \int_a^b |g(t)|^2 \, dt \int_a^b |f(t)|^2 \, dt - \left| \int_a^b f(t) g(t) \, dt \right|^2$$

$$\leq \frac{1}{2} \times \begin{cases} \|f'\overline{g} - f\overline{g}'\|_\infty^2 \left(\int_a^b |g(t)|^2 \, dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^2} dt \right)^2, & \text{if } \begin{array}{l} f'\overline{g} - f\overline{g}' \in L_\infty[a, b], \\ \frac{1}{|g|^2} \in L[a, b] \end{array} \\[3em] \|f'\overline{g} - f\overline{g}'\|_p^2 \left(\int_a^b |g(t)|^2 \, dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}, & \text{if } \begin{array}{l} f'\overline{g} - f\overline{g}' \in L_p[a, b], \\ \frac{1}{|g|^{2q}} \in L[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{array} \\[3em] \|f'\overline{g} - f\overline{g}'\|_1^2 \left(\int_a^b |g(t)|^2 \, dt \right)^2 ess \sup_{t \in [a, b]} \left\{ \frac{1}{|g(t)|^4} \right\}, & \text{if } \frac{1}{|g|} \in L_\infty[a, b]. \end{cases}$$

Proof. Utilising the inequality (2.2) we have

$$(4.3) \quad \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|$$

$$\leq \begin{cases} \|f'\overline{g} - f\overline{g}'\|_\infty |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'\overline{g} - f\overline{g}' \\ & \in L_\infty[a, b], \\[2em] \|f'\overline{g} - f\overline{g}'\|_p |g(t) g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \text{if } f'\overline{g} - f\overline{g}' \\ & \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\[2em] \|f'\overline{g} - f\overline{g}'\|_1 |g(t) g(x)| \sup_{s \in [t, x]([x, t])} \left\{ \frac{1}{|g(s)|^2} \right\}. \end{cases}$$

for any $t, x \in [a, b]$.

Taking the square in (4.3) and integrating over $(t, x) \in [a, b]^2$ we have

$$(4.4) \quad \int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 \, dt dx$$

$$\leq \begin{cases} \|f'\overline{g} - f\overline{g}'\|_\infty^2 \int_a^b \int_a^b |g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 \, dt dx, \\[2em] \|f'\overline{g} - f\overline{g}'\|_p^2 \int_a^b \int_a^b |g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \, dt dx, \\[2em] \|f'\overline{g} - f\overline{g}'\|_1^2 \int_a^b \int_a^b |g(t) g(x)|^2 \sup_{s \in [t, x]([x, t])} \left\{ \frac{1}{|g(s)|^4} \right\} \, dt dx. \end{cases}$$

Observe that

$$\int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 dt dx$$

$$= \int_a^b \int_a^b \left(|g(t)|^2 |f(x)|^2 - 2 \operatorname{Re} \left[\overline{g(t)} f(x) \overline{f(t) \overline{g(x)}} \right] + |g(x)|^2 |f(t)|^2 \right) dt dx$$

$$= \int_a^b |g(t)|^2 dt \int_a^b |f(x)|^2 dx - 2 \operatorname{Re} \left[\int_a^b \overline{f(t) g(t)} dt \int_a^b f(x) g(x) dx \right]$$

$$+ \int_a^b |g(x)|^2 dx \int_a^b |f(t)|^2 dt$$

$$= 2 \left[\int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \right],$$

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 \right] dt dx$$

$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^2} dt \right)^2,$$

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dt dx$$

$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q},$$

and

$$\int_a^b \int_a^b \left[|g(t) g(x)|^2 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dt dx$$

$$\leq \left(\int_a^b |g(t)|^2 dt \right)^2 ess \sup_{t \in [a,b]} \left\{ \frac{1}{|g(t)|^4} \right\},$$

then by (4.4) we get the desired result (4.2). □

References

[1] A. M. Acu and F. D. Sofonea, On an inequality of Ostrowski type. *J. Sci. Arts* **2011**, no. 3(16), 281–287.

[2] A. M. Acu, A. Baboş and F. D. Sofonea, The mean value theorems and inequalities of Ostrowski type. *Sci. Stud. Res. Ser. Math. Inform.* **21** (2011), no. 1, 5–16.

[3] S. S. Dragomir, An inequality of Ostrowski type via Pompeiu's mean value theorem. *J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 83, 9 pp.

[4] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[5] J. Pečarić and Š. Ungar, On an inequality of Ostrowski type, *J. Ineq. Pure & Appl. Math.* **7** (2006). No. 4. Art. 151.

[6] D. Pompeiu, Sur une proposition analogue au théorème des accroissements finis, *Mathematica* (Cluj, Romania), **22**(1946), 143-146.

[7] E. C. Popa, An inequality of Ostrowski type via a mean value theorem, *General Mathematics* Vol. **15**, No. 1, 2007, 93-100.

[8] P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

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