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# SOME GRÜSS TYPE RESULTS VIA POMPEIU'S LIKE INEQUALITIES

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Abstract. In this paper, some Grüss type results via Pompeiu's like inequalities are proved.

## 1. Introduction

In 1946, Pompeiu [18] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem* (see also [18, p. 83]).

**Theorem 1** (Pompeiu, 1946 [18])**.** *For every real valued function $f$ differentiable on an interval $[a,b]$ not containing $0$ and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point $\xi$ between $x_1$ and $x_2$ such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1.1}$$

The following inequality is useful to derive some Ostrowski type inequalities, see [9].

**Corollary 1** (Pompeiu's Inequality)**.** *With the assumptions of Theorem 1 and if $\|f - \ell f'\|_\infty = \sup_{t\in(a,b)} |f(t) - tf'(t)| < \infty$ where $\ell(t) = t$, $t \in [a,b]$, then*

$$|tf(x) - xf(t)| \leq \|f - \ell f'\|_\infty |x - t| \tag{1.2}$$

*for any $t, x \in [a,b]$.*

The inequality (1.2) was obtained by the author in [9].

For other Ostrowski type inequalities concerning the $p$-norms $\|f - \ell f'\|_p$ see [1], [2], [17] and [19].

For two Lebesgue integrable functions $f$, $g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt. \tag{1.3}$$

In 1935, Grüss [10] showed that

$$|C(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.4}$$

provided that there exists the real numbers $m, M, n, N$ such that

$$m \leq f(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a,b]. \tag{1.5}$$

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The constant $\frac{1}{4}$ is best possible in (1.3) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [7], states that

$$|C(f,g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \tag{1.6}$$

provided that $f', g'$ exist and are continuous on $[a,b]$ and $\|f'\|_{\infty} = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.6) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_{\infty}[a,b]$ while $\|f'\|_{\infty} = ess\sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1.4) and Čebyšev's one (1.6) is the following inequality obtained by Ostrowski in 1970, [15]:

$$|C(f,g)| \leq \frac{1}{8} (b-a)(M-m) \|g'\|_{\infty}, \tag{1.7}$$

provided that $f$ is *Lebesgue integrable* and satisfies (1.5) while $g$ is absolutely continuous and $g' \in L_{\infty}[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.7).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [12] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.8}$$

provided that $f, g$ are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [3] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a} \left( \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|^p dt \right)^{\frac{1}{p}}, \tag{1.9}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} ess\sup_{t\in[a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right|, \tag{1.10}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty, q = 1$).

Notice that for $q = \infty, p = 1$ in (1.9) we obtain

$$\begin{aligned} |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \\ &\leq \|g\|_{\infty} \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt \end{aligned} \tag{1.11}$$

and if $g$ satisfies (1.5), then

$$
\begin{aligned}
(1.12) \qquad |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\
&\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\
&\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt.
\end{aligned}
$$

The inequality between the first and the last term in (1.12) has been obtained by Cheng and Sun in [8]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [4].

For other recent results on the Grüss inequality, see [11], [13] and [16] and the references therein.

In this paper, some Grüss type results via Pompeiu's like inequalities are proved.

## 2. Some Pompeiu's Type Inequalities

We can generalize the above inequality for the larger class of functions that are absolutely continuous and complex valued as well as for other norms of the difference $f - \ell f'$.

**Theorem 2.** *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ with $b > a > 0$. Then for any $t, x \in [a,b]$ we have*

$$
(2.1) \qquad |tf(x) - xf(t)| \leq \begin{cases} \|f-\ell f'\|_\infty |x-t| & \text{if } f-\ell f' \in L_\infty[a,b], \\[2ex] \left(\frac{1}{2q-1}\right)^{1/q} \|f-\ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \begin{array}{c} \text{if } f-\ell f' \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\[2ex] \|f-\ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases}
$$

*or, equivalently*

$$
(2.2) \qquad \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \leq \begin{cases} \|f-\ell f'\|_\infty \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f-\ell f' \in L_\infty[a,b], \\[2ex] \left(\frac{1}{2q-1}\right)^{1/q} \|f-\ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \begin{array}{c} \text{if } f-\ell f' \in L_p[a,b] \\ p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\[2ex] \|f-\ell f'\|_1 \frac{1}{\min\{t^2,x^2\}}. \end{cases}
$$

*Proof.* If $f$ is absolutely continuous, then $f/\ell$ is absolutely continuous on the interval $[a,b]$ that does not containing 0 and

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \frac{f(x)}{x} - \frac{f(t)}{t}$$

for any $t,x \in [a,b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{s}\right)' ds = \int_t^x \frac{f'(s)s - f(s)}{s^2} ds$$

then we get the following identity

$$tf(x) - xf(t) = xt\int_t^x \frac{f'(s)s - f(s)}{s^2} ds \tag{2.3}$$

for any $t,x \in [a,b]$.

We notice that the equality (2.3) was proved for the smaller class of differentiable function and in a different manner in [17].

Taking the modulus in (2.3) we have

$$\begin{aligned} |tf(x) - xf(t)| &= \left| xt\int_t^x \frac{f'(s)s - f(s)}{s^2} ds \right| \\ &\leq xt\left| \int_t^x \left| \frac{f'(s)s - f(s)}{s^2} \right| ds \right| := I \end{aligned} \tag{2.4}$$

and utilizing Hölder's integral inequality we deduce

$$(2.5) I \leq xt \begin{cases} \sup_{s\in[t,x]([x,t])} |f'(s)s - f(s)| \left|\int_t^x \frac{1}{s^2} ds\right|, \\ \left|\int_t^x |f'(s)s - f(s)|^p ds\right|^{1/p} \left|\int_t^x \frac{1}{s^{2q}} ds\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \left|\int_t^x |f'(s)s - f(s)| ds\right| \sup_{s\in[t,x]([x,t])} \left\{\frac{1}{s^2}\right\}, \end{cases}$$

$$= \begin{cases} \|f - \ell f'\|_\infty |x-t|, \\ \left(\frac{1}{2q-1}\right)^{1/q} \|f - \ell f'\|_p \left|\frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}}\right|^{1/q} & \begin{array}{c} p>1, \\ \frac{1}{p}+\frac{1}{q}=1, \end{array} \\ \|f - \ell f'\|_1 \frac{\max\{t,x\}}{\min\{t,x\}}, \end{cases}$$

and the inequality (2.2) is proved. □

**Remark 1.** *The first inequality in (2.1) also holds in the same form for* $0 > b > a$.

## 3. Some Grüss' Type Inequalities

We have the following result of Grüss type.

**Theorem 3.** *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty [a, b]$, then*

$$\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \right| \leq \frac{1}{12} (b-a)^4 \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty . \tag{3.1}$$

*The constant $\frac{1}{12}$ is best possible.*

*Proof.* From the first inequality in (2.1) we have

$$\begin{aligned} &\left| \int_a^b \int_a^b (t f(x) - x f(t)) (t g(x) - x g(t)) dt dx \right| \\ &\leq \int_a^b \int_a^b |(t f(x) - x f(t)) (t g(x) - x g(t))| dt dx \\ &\leq \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty \int_a^b \int_a^b (x-t)^2 dt dx. \end{aligned} \tag{3.2}$$

Observe that

$$\begin{aligned} &\int_a^b \int_a^b (t f(x) - x f(t)) (t g(x) - x g(t)) dt dx \\ &= \int_a^b \int_a^b \left[ t^2 f(x) g(x) + x^2 f(t) g(t) - t g(t) x f(x) - f(t) t x g(x) \right] dt dx \\ &= 2 \left[ \int_a^b t^2 dt \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \right] \end{aligned}$$

and

$$\int_a^b \int_a^b (x-t)^2 dt dx = \frac{1}{3} \int_a^b \left[ (b-x)^3 + (x-a)^3 \right] dx = \frac{1}{6} (b-a)^4 .$$

Utilising the inequality (3.2), we deduce the desired result (3.1).

Now, assume that the inequality (3.1) holds with a constant $B > 0$ instead of $\frac{1}{12}$, i.e.

$$\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \right| \leq B (b-a)^4 \|f - \ell f'\|_\infty \|g - \ell g'\|_\infty . \tag{3.3}$$

If we take $f(t) = g(t) = 1$, $t \in [a, b]$, then

$$\begin{aligned} &\frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \\ &= \frac{b^3 - a^3}{3} (b-a) - \left( \frac{b^2 - a^2}{2} \right)^2 = \frac{1}{12} (b-a)^4 \end{aligned}$$

and

$$\|f - \ell f'\|_\infty = \|g - \ell g'\|_\infty = 1$$

and by (3.3) we get $B \geq \frac{1}{12}$, which proves the sharpness of the constant. □

The following result for the complementary $(p,q)$-norms, with $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, holds.

**Theorem 4.** *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $b>a>0$. If $f'\in L_p[a,b]$, $g'\in L_q[a,b]$ with $p,q>1, p,q\neq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{b^3-a^3}{3}\int_a^b f(t)g(t)\,dt-\int_a^b tf(t)\,dt\int_a^b tg(t)\,dt\right| \tag{3.4}$$
$$\leq \frac{1}{2(2q-1)^{1/q}(2p-1)^{1/p}}\left\|f-\ell f'\right\|_p\left\|g-\ell g'\right\|_q M_q^{1/q}(a,b)\,M_p^{1/p}(a,b),$$

*where*

$$M_q(a,b):=\int_a^b\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|dtdx.$$

*We have the bounds*

$$M_q(a,b)\leq(b-a)\,N_q^{1/2}(a,b)$$

*and*

$$M_p(a,b)\leq(b-a)\,N_p^{1/2}(a,b)$$

*where, for $r>1$,*

$$N_r(a,b):=\begin{cases}2\left(\frac{b^{2r+1}-a^{2r+1}}{2r+1}\cdot\frac{b^{-2r+3}-a^{-2r+3}}{-2r+3}-\left(\frac{b^2-a^2}{2}\right)^2\right), r\neq\frac{3}{2}\\ \\ \left(b^2-a^2\right)\left(\frac{b^2+a^2}{2}\cdot\ln\frac{b}{a}-\frac{b^2-a^2}{2}\right), r=\frac{3}{2}.\end{cases}$$

*Proof.* From the second inequality in (2.1) we have

$$|tf(x)-xf(t)|\leq\frac{1}{(2q-1)^{1/q}}\left\|f-\ell f'\right\|_p\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}$$

and

$$|tg(x)-xg(t)|\leq\frac{1}{(2p-1)^{1/p}}\left\|g-\ell g'\right\|_q\left|\frac{x^p}{t^{p-1}}-\frac{t^p}{x^{p-1}}\right|^{1/p}$$

for any $t,x\in[a,b]$.

If we multiply these inequalities and integrate, then we get

$$\left|\int_a^b\int_a^b(tf(x)-xf(t))(tg(x)-xg(t))\,dtdx\right| \tag{3.5}$$
$$\leq\int_a^b\int_a^b|(tf(x)-xf(t))(tg(x)-xg(t))|\,dtdx$$
$$\leq\frac{1}{(2q-1)^{1/q}(2p-1)^{1/p}}\left\|f-\ell f'\right\|_p\left\|g-\ell g'\right\|_q$$
$$\times\int_a^b\int_a^b\left|\frac{x^q}{t^{q-1}}-\frac{t^q}{x^{q-1}}\right|^{1/q}\left|\frac{x^p}{t^{p-1}}-\frac{t^p}{x^{p-1}}\right|^{1/p}dtdx.$$

Utilizing Hölder's integral inequality for double integrals we have

$$\int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right|^{1/p} dtdx \tag{3.6}$$
$$\leq \left( \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dtdx \right)^{1/q} \left( \int_a^b \int_a^b \left| \frac{x^p}{t^{p-1}} - \frac{t^p}{x^{p-1}} \right| dtdx \right)^{1/p}$$
$$= M_q^{1/q}(a,b)\, M_p^{1/p}(a,b)$$

for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilising Cauchy-Bunyakowsky-Schwarz integral inequality for double integrals we have

$$M_q(a,b) = \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dtdx$$
$$\leq \left( \int_a^b \int_a^b dtdx \right)^{1/2} \left( \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dtdx \right)^{1/2}$$
$$= (b-a) \left( \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dtdx \right)^{1/2}.$$

Observe that

$$N_q(a,b) := \int_a^b \int_a^b \left( \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right)^2 dtdx$$
$$= \int_a^b \int_a^b \frac{x^{2q}}{t^{2(q-1)}} dtdx - 2 \int_a^b \int_a^b \frac{x^q}{t^{q-1}} \frac{t^q}{x^{q-1}} dtdx + \int_a^b \int_a^b \frac{t^{2q}}{x^{2(q-1)}} dtdx$$
$$= 2 \int_a^b x^{2q} dx \int_a^b t^{-2(q-1)} dt - 2 \left( \int_a^b x dx \right)^2$$
$$= 2 \left( \frac{b^{2q+1} - a^{2q+1}}{2q+1} \cdot \frac{b^{-2q+3} - a^{-2q+3}}{-2q+3} - \left( \frac{b^2 - a^2}{2} \right)^2 \right),$$

provided $q \neq \frac{3}{2}$.

If $q = \frac{3}{2}$, then

$$N_q(a,b) = \left(b^2 - a^2\right) \left[ \frac{b^2 + a^2}{2} \cdot \ln \frac{b}{a} - \frac{b^2 - a^2}{2} \right].$$

Therefore

$$M_q(a,b) \leq (b-a)\, N_q^{1/2}(a,b)$$

and, similarly

$$M_p(a,b) \leq (b-a)\, N_p^{1/2}(a,b).$$

□

**Remark 2.** *The double integral*

$$M_q(a,b) := \int_a^b \int_a^b \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right| dtdx$$

*can be computed exactly by iterating the integrals. However the final form is too complicated to be stated here.*

The Euclidian norms case is as follows:

**Theorem 5.** *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_2 [a, b]$, then*

$$\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) dt - \int_a^b t f(t) dt \int_a^b t g(t) dt \right| \leq \frac{1}{9} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \left[ \left(b^3 + a^3\right) \ln \frac{b}{a} - \frac{2}{3} \left(b^3 - a^3\right) \right]. \tag{3.7}$$

*Proof.* From the second inequality in (2.1) we have

$$|t f(x) - x f(t)| \leq \frac{1}{\sqrt{3}} \|f - \ell f'\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}$$

and

$$|t g(x) - x g(t)| \leq \frac{1}{\sqrt{3}} \|g - \ell g'\|_2 \left| \frac{x^2}{t} - \frac{t^2}{x} \right|^{1/2}$$

for any $t, x \in [a, b]$.

If we multiply these inequalities and integrate, then we get

$$\begin{aligned} &\left| \int_a^b \int_a^b (t f(x) - x f(t)) (t g(x) - x g(t)) dt dx \right| \\ &\leq \int_a^b \int_a^b |(t f(x) - x f(t)) (t g(x) - x g(t))| dt dx \\ &\leq \frac{1}{3} \|f - \ell f'\|_2 \|g - \ell g'\|_2 \int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt dx. \end{aligned} \tag{3.8}$$

Since

$$\begin{aligned} &\int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dt dx \\ &= \int_a^b \left( \int_a^x \left( \frac{x^2}{t} - \frac{t^2}{x} \right) dt + \int_x^b \left( \frac{t^2}{x} - \frac{x^2}{t} \right) dt \right) dx \\ &= \int_a^b \left( x^2 (2 \ln x - \ln a - \ln b) + \frac{b^3 + a^3 - 2x^3}{3x} \right) dx \end{aligned}$$

and

$$\begin{aligned} &\int_a^b x^2 (2 \ln x - \ln a - \ln b) dx \\ &= \int_a^b 2x^2 \ln x dx - \ln (ab) \int_a^b x^2 dx \\ &= \frac{\left(b^3 + a^3\right) \ln \frac{b}{a}}{3} - \frac{2}{9} \left(b^3 - a^3\right) \end{aligned}$$

while

$$\int_a^b \frac{b^3 + a^3 - 2x^3}{3x} dx = \frac{\left(b^3 + a^3\right) \ln \frac{b}{a}}{3} - \frac{2}{9} \left(b^3 - a^3\right),$$

then we conclude that

$$\int_a^b \int_a^b \left| \frac{x^2}{t} - \frac{t^2}{x} \right| dtdx = \frac{2}{3} \left[ \left(b^3 + a^3\right) \ln \frac{b}{a} - \frac{2}{3} \left(b^3 - a^3\right) \right].$$

Making use of the inequality (3.8) we deduce the desired result (3.7). □

**Remark 3.** *It is an open question to the author if $\frac{1}{9}$ is best possible in (3.7).*

**Theorem 6.** *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. Then*

$$\begin{aligned} &\left| \frac{b^3 - a^3}{3} \int_a^b f(t) g(t) \, dt - \int_a^b t f(t) \, dt \int_a^b t g(t) \, dt \right| \\ &\leq \| f - \ell f' \|_1 \| g - \ell g' \|_1 \frac{2b^3 + a^3 - 3ab^2}{6a}. \end{aligned} \tag{3.9}$$

*Proof.* From the third inequality in (2.1) we have

$$\begin{aligned} &\left| \int_a^b \int_a^b (t f(x) - x f(t)) (t g(x) - x g(t)) \, dtdx \right| \\ &\leq \int_a^b \int_a^b |(t f(x) - x f(t)) (t g(x) - x g(t))| \, dtdx \\ &\leq \| f - \ell f' \|_1 \| g - \ell g' \|_1 \int_a^b \int_a^b \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dtdx. \end{aligned} \tag{3.10}$$

Observe that

$$\begin{aligned} &\int_a^b \int_a^b \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dtdx \\ &= \int_a^b \left[ \int_a^x \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt + \int_x^b \left( \frac{\max\{t, x\}}{\min\{t, x\}} \right)^2 dt \right] dx \\ &= \int_a^b \left[ \int_a^x \left( \frac{x}{t} \right)^2 dt + \int_x^b \left( \frac{t}{x} \right)^2 dt \right] dx \\ &= \frac{2b^3 + a^3 - 3ab^2}{6a}, \end{aligned}$$

which together with (3.10) produces the desired inequality (3.9). □

## 4. Some Related Results

The following result holds.

**Theorem 7.** *Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $b > a > 0$. If $f', g' \in L_\infty[a, b]$, then*

$$\begin{aligned} &\left| (b - a) \int_a^b \frac{f(t) g(t)}{t^2} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt \right| \\ &\leq (b - a)^2 \frac{L^2(a, b) - G^2(a, b)}{L^2(a, b) G^2(a, b)} \| f - \ell f' \|_\infty \| g - \ell g' \|_\infty , \end{aligned} \tag{4.1}$$

*where* $G(a,b) := \sqrt{ab}$ *is the geometric mean and*

$$L(a,b) := \frac{b-a}{\ln b - \ln a}$$

*is the Logarithmic mean.*

*The inequality (4.1) is sharp.*

*Proof.* From the first inequality in (2.2) we have

$$\left|\left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right)\right| \leq \|f - \ell f'\|_\infty \|f - \ell f'\|_\infty \left(\frac{1}{t} - \frac{1}{x}\right)^2 \tag{4.2}$$

for any $t, x \in [a,b]$.

Integrating this inequality on $[a,b]^2$ we get

$$\begin{aligned} &\left|\int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right) dtdx\right| \\ &\int_a^b \int_a^b \left|\left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right)\right| dtdx \\ &\leq \|f - \ell f'\|_\infty \|f - \ell f'\|_\infty \int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x}\right)^2 dtdx. \end{aligned} \tag{4.3}$$

We have

$$\begin{aligned} &\int_a^b \int_a^b \left(\frac{f(x)}{x} - \frac{f(t)}{t}\right)\left(\frac{g(x)}{x} - \frac{g(t)}{t}\right) dtdx \\ &= 2\left[(b-a)\int_a^b \frac{f(t)g(t)}{t^2}dt - \int_a^b \frac{f(t)}{t}dt \int_a^b \frac{g(t)}{t}dt\right] \end{aligned}$$

and

$$\int_a^b \int_a^b \left(\frac{1}{t} - \frac{1}{x}\right)^2 dtdx = 2(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)}.$$

Making use of (4.3) we get the desired result (4.1).

If we take $f(t) = g(t) = 1$, then we have

$$\begin{aligned} &(b-a)\int_a^b \frac{f(t)g(t)}{t^2}dt - \int_a^b \frac{f(t)}{t}dt \int_a^b \frac{g(t)}{t}dt \\ &= (b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)} \end{aligned}$$

and

$$\|f - \ell f'\|_\infty = \|g - \ell g'\|_\infty = 1$$

and we obtain in both sides of (4.1) the same quantity

$$(b-a)^2 \frac{L^2(a,b) - G^2(a,b)}{L^2(a,b)G^2(a,b)}.$$

□

The case of Euclidian norms is as follows:

**Theorem 8.** *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous functions on the interval $[a,b]$ with $b>a>0$. If $f',g'\in L_2[a,b]$, then*

$$\left|(b-a)\int_a^b \frac{f(t)g(t)}{t^2}dt-\int_a^b \frac{f(t)}{t}dt\int_a^b \frac{g(t)}{t}dt\right| \le \frac{1}{6}\|f-\ell f'\|_2\|g-\ell g'\|_2\frac{(b-a)^3}{a^2b^2}. \tag{4.4}$$

*Proof.* From the second inequality in (2.2) for $p=q=2$ we have

$$\left|\frac{f(x)}{x}-\frac{f(t)}{t}\right|\le\frac{1}{\sqrt{3}}\|f-\ell f'\|_2\left|\frac{1}{t^3}-\frac{1}{x^3}\right|^{1/2} \tag{4.5}$$

and

$$\left|\frac{g(x)}{x}-\frac{g(t)}{t}\right|\le\frac{1}{\sqrt{3}}\|g-\ell g'\|_2\left|\frac{1}{t^3}-\frac{1}{x^3}\right|^{1/2} \tag{4.6}$$

for any $t,x\in[a,b]$.

On multiplying (4.5) with (4.6) we derive

$$\left|\left(\frac{f(x)}{x}-\frac{f(t)}{t}\right)\left(\frac{g(x)}{x}-\frac{g(t)}{t}\right)\right|\le\frac{1}{3}\|f-\ell f'\|_2\|g-\ell g'\|_2\left|\frac{1}{t^3}-\frac{1}{x^3}\right| \tag{4.7}$$

for any $t,x\in[a,b]$.

Integrating this inequality on $[a,b]^2$ we get

$$\begin{aligned} &\left|\int_a^b\int_a^b\left(\frac{f(x)}{x}-\frac{f(t)}{t}\right)\left(\frac{g(x)}{x}-\frac{g(t)}{t}\right)dtdx\right| \\ &\le\int_a^b\int_a^b\left|\left(\frac{f(x)}{x}-\frac{f(t)}{t}\right)\left(\frac{g(x)}{x}-\frac{g(t)}{t}\right)\right|dtdx \\ &\le\frac{1}{3}\|f-\ell f'\|_2\|g-\ell g'\|_2\int_a^b\int_a^b\left|\frac{1}{t^3}-\frac{1}{x^3}\right|dtdx. \end{aligned} \tag{4.8}$$

We have

$$\begin{aligned} &\int_a^b\int_a^b\left|\frac{1}{t^3}-\frac{1}{x^3}\right|dtdx \\ &=\int_a^b\left[\int_a^x\left(\frac{1}{t^3}-\frac{1}{x^3}\right)dt+\int_x^b\left(\frac{1}{x^3}-\frac{1}{t^3}\right)dt\right]dx \\ &=\int_a^b\left[\int_a^x\left(\frac{1}{t^3}-\frac{1}{x^3}\right)dt+\int_x^b\left(\frac{1}{x^3}-\frac{1}{t^3}\right)dt\right]dx=\frac{(b-a)^3}{a^2b^2}. \end{aligned}$$

From (4.8) we then obtain the desired result (4.4). □

**Remark 4.** *It is an open question to the author if $\frac{1}{6}$ is the best possible constant in (4.4).*

The interested reader may obtain other similar results in terms of the norms $\|f-\ell f'\|_p\|g-\ell g'\|_q$ with $p,q>1, p,q\neq 2$ and $\frac{1}{p}+\frac{1}{q}=1$. However the details are omitted.

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