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MULTIPLICATIVE OSTROWSKI AND TRAPEZOID INEQUALITIES

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Abstract. We introduce the multiplicative Ostrowski and trapezoid inequalities, i.e. providing bounds for the comparison of a function f and its integral mean in the following sense:

$$f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \quad \text{and} \quad f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right].$$

We consider the case for absolutely continuous and logarithmic convex functions. We apply these inequalities to provide approximations for the integral of f and the first moment of f around zero, i.e.

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b x f(x)\,dx$$

for an absolutely continuous function f on $[a,b]$.

1. Introduction

Comparison between functions and integral means is incorporated in Ostrowski type inequalities. The first result in this direction is due to Ostrowski [27].

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M, \quad x\in[a,b]. \tag{1}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

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More inequalities of Ostrowski type have been generalised for functions which are not necessarily differentiable, namely, absolutely continuous, Hölder continuous and convex functions. We refer to Section 2 for the details of these inequalities.

Inequalities providing upper bounds for the quantity

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|,\qquad x\in[a,b] \tag{2}$$

are known in the literature as *generalized trapezoid inequalities*. It has been shown in Dragomir [7] (cf. Cerone, Dragomir and Pearce [5]) that

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f) \tag{3}$$

for any $x\in[a,b]$, provided that f is of bounded variation on $[a,b]$. In particular, we have the *trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2}\bigvee_a^b(f). \tag{4}$$

The constant $\frac{1}{2}$ is the best possible. The trapezoid inequalities have also been developed for other types of functions, such as absolutely continuous and convex functions. We refer to Section 2 for the details of these inequalities.

Motivated by the above results, we intend to develop the Ostrowski and trapezoid inequalities. In particular, we are interested in the multiplicative Ostrowski and trapezoid inequalities, i.e. providing bounds for the comparison of a function f and its integral mean in the following sense:

$$f(x)\exp\left[-\frac{1}{b-a}\int_a^b\log f(t)\,dt\right]\quad\text{and}\quad f(b)^{\frac{b-x}{b-a}}f(a)^{\frac{x-a}{b-a}}\exp\left[-\frac{1}{b-a}\int_a^b\log f(t)\,dt\right].$$

We summarise the results concerning absolutely continuous functions and logarithmic convex functions in Section 3. In Section 4, we apply these inequalities to provide approximations for the integral of f and the first moment of f around zero, i.e.

$$\int_a^b f(x)\,dx\quad\text{and}\quad\int_a^b xf(x)\,dx$$

for an absolutely continuous function f on $[a,b]$.

2. Results concerning the Ostrowski and trapezoid inequalities

This section serves as a reference point for the developments of the Ostrowski and trapezoid inequalities. Readers who are familiar with these developments may skip this section.

2.1. **Ostrowski inequality.** The following results for absolutely continuous functions hold (cf. Dragomir and Wang [20] – [22]).

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x\in[a,b]$, we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}}\left[\left(\frac{x-a}{b-a}\right)^{\alpha+1}+\left(\frac{b-x}{b-a}\right)^{\alpha+1}\right]^{\frac{1}{\alpha}}(b-a)^{\frac{1}{\alpha}}\|f'\|_\beta, & \text{if } f'\in L_\beta[a,b]\\ & \frac{1}{\alpha}+\frac{1}{\beta}=1,\ \alpha>1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ $(r\in[1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|g(t)| \quad and \quad \|g\|_{[a,b],r}:=\left(\int_a^b|g(t)|^r\,dt\right)^{\frac{1}{r}},\ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from Fink's result [23]. If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (cf. Dragomir et al. [18] and the references therein for earlier contributions):

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be of $r-H-$Hölder type, i.e.,*

$$|f(x)-f(y)|\leq H|x-y|^r,\ \textit{for all}\ \ x,y\in[a,b], \tag{5}$$

where $r\in(0,1]$ and $H>0$ are fixed. Then, for all $x\in[a,b]$, we have the inequality:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{H}{r+1}\left[\left(\frac{b-x}{b-a}\right)^{r+1}+\left(\frac{x-a}{b-a}\right)^{r+1}\right](b-a)^r. \tag{6}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitz continuous functions (with constant $L > 0$) (cf. Dragomir [8])

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)\,L, \tag{7}$$

where $x \in [a, b]$. Here the constant $\frac{1}{4}$ is also best possible.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (cf. Dragomir [11]).

Theorem 2.3. *Assume that $f : [a, b] \to \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b (f)$ its total variation. Then,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f) \tag{8}$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we further assume that f is monotonically increasing, then the inequality (8) may be improved in the following manner [9] (cf. Dragomir and Rassias [19]).

Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:*

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &\le \frac{1}{b-a}\left\{ [2x-(a+b)]\,f(x) + \int_a^b \operatorname{sgn}(t-x)\,f(t)\,dt \right\} \\ &\le \frac{1}{b-a}\left\{ (x-a)\,[f(x)-f(a)] + (b-x)\,[f(b)-f(x)] \right\} \\ &\le \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b)-f(a)]. \end{aligned} \tag{9}$$

All the inequalities in (9) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [13]:

Theorem 2.5. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ one has the inequality*

$$\begin{aligned} \frac{1}{2}\left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] &\le \int_a^b f(t)\,dt - (b-a)\,f(x) \\ &\le \frac{1}{2}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right]. \end{aligned} \tag{10}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $x = a$ *or* $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, we refer to Anastassiou [1]; Cerone and Dragomir [2], [4]; Cerone, Dragomir and Roumeliotis [6]; Dragomir [8], [9] and [16]. Inequalities for the Riemann-Stieltjes integral may be found in Dragomir [10], [12]; while the generalization for isotonic functionals was provided in Dragomir [15]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph by Dragomir [17].

2.2. **Trapezoid inequality.** If f is absolutely continuous on $[a,b]$, then (see [3, p. 93])

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{11}$$
$$\leq\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}, & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_{[a,b],1},\end{cases}$$

for any $x\in[a,b]$. Here, $\|\cdot\|_{[a,b],p}$ are the usual Lebesgue norms.

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\begin{cases}\frac{1}{4}(b-a)\|f'\|_\infty, & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_p, & \text{if } f'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{2}\|f'\|_1.\end{cases} \tag{12}$$

The constants $\frac{1}{4}$, $\frac{1}{2(q+1)^{1/q}}$ and $\frac{1}{2}$ are the best possible.

Finally, for convex functions $f:[a,b]\to\mathbb{R}$, we have [14]

$$\begin{aligned}\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right] &\leq (b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\\ &\leq\frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_-(a)\right]\end{aligned} \tag{13}$$

for any $x \in (a, b)$, provided that $f'_{-}(b)$ and $f'_{+}(a)$ are finite. As above, the second inequality also holds for $x = a$ and $x = b$ and the constant $\frac{1}{2}$ is the best possible on both sides of (13). In particular, we have

$$\frac{1}{8}(b-a)^2\left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}(b-a)\left[f'_{-}(b) - f'_{-}(a)\right]. \tag{14}$$

The constant $\frac{1}{8}$ is best possible in both inequalities. For other recent results on the trapezoid inequality, we refer to Dragomir [13], Kechriniotis and Assimakis [24], Liu [25], Mercer [26] and Ujevíc [28].

3. Results

We present our main results in this section.

3.1. Multiplicative Ostrowski inequalities.

We start with the first of our main theorems.

Theorem 3.1. *Let $f : [a, b] \to (0, \infty)$ be an absolutely continuous function and $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma f(t) \leq f'(t) \leq \Gamma f(t), \quad \text{for almost all } t \in [a, b].$$

Then, we have

$$\exp\left[\frac{\gamma(x-a)^2 - \Gamma(b-x)^2}{2(b-a)}\right] \leq f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \leq \exp\left[\frac{\Gamma(x-a)^2 - \gamma(b-x)^2}{2(b-a)}\right], \tag{15}$$

for any $x \in [a, b]$. In particular, we have

$$\exp\left[-\frac{1}{8}(\Gamma-\gamma)(b-a)\right] \leq f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \leq \exp\left[\frac{1}{8}(\Gamma-\gamma)(b-a)\right]. \tag{16}$$

The constant $\frac{1}{8}$ is best possible in (16).

Proof. We use the Montgomery identity

$$g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\left[\int_a^x (t-a)g'(t)\,dt + \int_x^b (t-b)g'(t)\,dt\right] \tag{17}$$

where $g : [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$. If we write (17) for the functions $g(t) = \log f(t)$, then we get

$$\log f(x) = \frac{1}{b-a}\int_a^b \log f(t)\,dt + \frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right]. \tag{18}$$

Taking the exponential of (18), and multiplying the result by $\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$, we have

$$f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] = \exp\left\{\frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right]\right\} \tag{19}$$

which can be considered as the **multiplicative Montgomery identity**. Now, since

$$\gamma \le \frac{f'(t)}{f(t)} \le \Gamma, \quad \text{for almost all } t \in [a,b],$$

it implies that

$$\gamma\int_a^x (t-a)\,dt \le \int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt \le \Gamma\int_a^x (t-a)\,dt,$$

which is equivalent to

$$\frac{1}{2}\gamma(x-a)^2 \le \int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt \le \frac{1}{2}\Gamma(x-a)^2. \tag{20}$$

Also,

$$\Gamma\int_x^b (t-b)\,dt \le \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt \le \gamma\int_x^b (t-b)\,dt,$$

which is equivalent to

$$-\frac{1}{2}\Gamma(b-x)^2 \le \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt \le -\frac{1}{2}\gamma(b-x)^2. \tag{21}$$

Adding inequalities (20) and (21) and dividing the resulted inequalities by $b-a>0$ gives us

$$\begin{aligned}\frac{1}{2(b-a)}\left[\gamma(x-a)^2-\Gamma(b-x)^2\right] &\le \frac{1}{b-a}\left[\int_a^x (t-a)\frac{f'(t)}{f(t)}\,dt + \int_x^b (t-b)\frac{f'(t)}{f(t)}\,dt\right]\\ &\le \frac{1}{2(b-a)}\left[\Gamma(x-a)^2-\gamma(b-x)^2\right],\end{aligned} \tag{22}$$

for $x\in[a,b]$. Utilising (19) and (22) we get (15); with (16) as its special case, i.e. when $x=\frac{a+b}{2}$. The proof for the best constant is given in Remark 3.2 (via the sharpness of $\frac{1}{4}$ in (23)). □

Remark 3.2. If $|f'(t)|\le Mf(t)$ for almost every $t\in[a,b]$, then by (15), for $\gamma=-M$ and $\Gamma=M$, we get

$$\begin{aligned}\exp\left[-\frac{M}{b-a}\left(\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right)\right] &\le f(x)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\\ &\le \exp\left[\frac{M}{b-a}\left(\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right)\right].\end{aligned}$$

In particular, we have

$$\exp\left[-\frac{1}{4}M(b-a)\right] \leq f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \leq \exp\left[\frac{1}{4}M(b-a)\right], \tag{23}$$

with $\frac{1}{4}$ as a best constant. To verify this, suppose that (23) holds for constants A, B instead of $-\frac{1}{4}$ and $\frac{1}{4}$, respectively, i.e.

$$\exp\left[AM(b-a)\right] \leq f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \tag{24}$$

$$\exp\left[BM(b-a)\right] \geq f\left(\frac{a+b}{2}\right)\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]. \tag{25}$$

Suppose in (24), $f(x) = \exp(|x - \frac{a+b}{2}|)$, thus $M = 1$, and we have

$$\exp\left[A(b-a)\right] \leq \exp\left[-\frac{1}{4}(b-a)\right].$$

Since the exponential function is strictly increasing, we now have $A(b-a) \leq -\frac{1}{4}(b-a)$; which asserts that $A \leq -\frac{1}{4}$ since $a < b$. Now suppose in (25) that $f(x) = \exp(-|x - \frac{a+b}{2}|)$, again, $M = 1$ and we have

$$\exp\left[B(b-a)\right] \geq \exp\left[\frac{1}{4}(b-a)\right].$$

By similar arguments, we conclude that $B \geq \frac{1}{4}$.

3.2. **Multiplicative trapezoid inequalities.** Similarly to the results in Subsection 3.1, we have the results for multiplicative trapezoid inequalities in the following.

Theorem 3.3. *Let $f : [a,b] \to (0,\infty)$ be an absolutely continuous function and $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma f(t) \leq f'(t) \leq \Gamma f(t), \quad \text{for almost all } t \in [a,b].$$

Then, we have

$$\begin{aligned}\exp\left[\frac{\gamma(b-x)^2 - \Gamma(x-a)^2}{2(b-a)}\right] &\leq f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\ &\leq \exp\left[\frac{\Gamma(b-x)^2 - \gamma(x-a)^2}{2(b-a)}\right],\end{aligned} \tag{26}$$

for any $x \in [a,b]$. In particular, we have

$$\exp\left[-\frac{1}{8}(\Gamma-\gamma)(b-a)\right] \leq \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \leq \exp\left[\frac{1}{8}(\Gamma-\gamma)(b-a)\right]. \tag{27}$$

The constant $\frac{1}{8}$ *is best possible in* (27).

Proof. We use the generalised trapezoid identity

$$\frac{(b-x)g(b)+(x-a)g(a)}{b-a}-\frac{1}{b-a}\int_a^b g(t)\,dt=\frac{1}{b-a}\int_a^b (t-x)g'(t)\,dt, \tag{28}$$

that holds for any $x\in[a,b]$ and g an absolutely continuous function. If we write (28) for the function $g(t)=\log f(t)$, then we get

$$\begin{aligned}
&\frac{(b-x)\log f(b)+(x-a)\log f(a)}{b-a}-\frac{1}{b-a}\int_a^b \log f(t)\,dt\\
=\;&\frac{1}{b-a}\int_a^b (t-x)\frac{f'(t)}{f(t)}\,dt\\
=\;&\frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt+\int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right],
\end{aligned}\tag{29}$$

for $x\in[a,b]$. The identity (29) is equivalent to

$$\begin{aligned}
&f(b)^{\frac{b-x}{b-a}}f(a)^{\frac{x-a}{b-a}}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\\
=\;&\exp\left\{\frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt+\int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right]\right\},
\end{aligned}\tag{30}$$

for $x\in[a,b]$, which is the multiplicative generalised trapezoid identity. Similarly to the expositions in the proof of Theorem 3.1 and using the assumption that

$$\gamma\le\frac{f'(t)}{f(t)}\le\Gamma,\quad \text{for almost all } t\in[a,b],$$

we have

$$\begin{aligned}
\frac{1}{2(b-a)}\left[\gamma(b-x)^2-\Gamma(x-a)^2\right]\;&\le\frac{1}{b-a}\left[\int_a^x (t-x)\frac{f'(t)}{f(t)}\,dt+\int_x^b (t-x)\frac{f'(t)}{f(t)}\,dt\right]\\
&\le\frac{1}{2(b-a)}\left[\Gamma(b-x)^2-\gamma(x-a)^2\right].
\end{aligned}\tag{31}$$

Taking the exponential of (31) and utilising (30) we get the desired result (26); with (27) as a special case when $x=\frac{a+b}{2}$. The proof for the best constant is given in Remark 3.4 (by inequality (32)). □

Remark 3.4. If $|f'(t)|\le Mf(t)$ for almost all $t\in[a,b]$, then by (26) we get (for $\gamma=-M$, $\Gamma=M$):

$$\begin{aligned}
\exp\left\{-\frac{M}{(b-a)}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right\}\;&\le f(b)^{\frac{b-x}{b-a}}f(a)^{\frac{x-a}{b-a}}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\\
&\le\exp\left\{\frac{M}{(b-a)}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right\}
\end{aligned}$$

for any $x \in [a,b]$. In particular, we have

$$\exp\left[-\frac{1}{4}M(b-a)\right] \le \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \le \exp\left[\frac{1}{4}M(b-a)\right], \tag{32}$$

with $\frac{1}{4}$ as the best constant. To verify this, suppose that (32) holds for constants C, D instead of $-\frac{1}{4}$ and $\frac{1}{4}$, respectively, i.e.

$$\exp\left[CM(b-a)\right] \le \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \tag{33}$$

$$\exp\left[DM(b-a)\right] \ge \sqrt{f(a)f(b)}\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]. \tag{34}$$

Suppose in (33), $f(x) = \exp(-|x - \frac{a+b}{2}|)$, thus $M = 1$, and we have

$$\exp\left[C(b-a)\right] \le \exp\left[-\frac{1}{4}(b-a)\right].$$

Since the exponential function is strictly increasing, we now have $C(b-a) \le -\frac{1}{4}(b-a)$; which asserts that $C \le -\frac{1}{4}$ since $a < b$. Now suppose in (34) that $f(x) = \exp(|x - \frac{a+b}{2}|)$, again, $M = 1$ and we have

$$\exp\left[D(b-a)\right] \ge \exp\left[\frac{1}{4}(b-a)\right].$$

By similar arguments, we conclude that $D \ge \frac{1}{4}$.

4. Applications

In this section, we apply the results from Section 3 to provide approximations for the integral of f and the first moment of f around zero. We start with the following subsection regarding the inequalities for logarithmic convex functions, as tools to help us in providing the above mentioned approximations.

4.1. **Inequalities for logarithmic convex functions.** If $f : [a,b] \to (0,\infty)$ is logarithmic convex, that is, $\log f$ is convex, then $\log f$ is differentiable almost everywhere and

$$\frac{f'_+(a)}{f(a)} \le \log f(t) = \frac{f'(t)}{f(t)} \le \frac{f'_-(b)}{f(b)}, \quad t \in (a,b).$$

Also, by Hermite-Hadamard's inequality we have the bounds

$$\log f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \log f(t)\,dt \le \frac{\log f(b) + \log f(a)}{2} = \log\sqrt{f(b)f(a)}. \tag{35}$$

From (15), we have

$$\begin{aligned} & \exp\left[\frac{\frac{f'_+(a)}{f(a)}(x-a)^2 - \frac{f'_-(b)}{f(b)}(b-x)^2}{2(b-a)}\right] \exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right) \\ \le\ & f(x) \\ \le\ & \exp\left[\frac{\frac{f'_-(b)}{f(b)}(x-a)^2 - \frac{f'_+(a)}{f(a)}(b-x)^2}{2(b-a)}\right] \exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right), \end{aligned} \tag{36}$$

for all $x \in [a,b]$. Utilising (35), we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \exp\left[\frac{\frac{f'_+(a)}{f(a)}(x-a)^2 - \frac{f'_-(b)}{f(b)}(b-x)^2}{2(b-a)}\right] \\ \le\ & f(x) \\ \le\ & \sqrt{f(a)f(b)} \exp\left[\frac{\frac{f'_-(b)}{f(b)}(x-a)^2 - \frac{f'_+(a)}{f(a)}(b-x)^2}{2(b-a)}\right], \end{aligned} \tag{37}$$

for all $x \in [a,b]$. From (26), we have

$$\begin{aligned} & \exp\left[\frac{\frac{f'_+(a)}{f(a)}(b-x)^2 - \frac{f'_-(b)}{f(b)}(x-a)^2}{2(b-a)}\right] \exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right) \\ \le\ & f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \\ \le\ & \exp\left[\frac{\frac{f'_-(b)}{f(b)}(b-x)^2 - \frac{f'_+(a)}{f(a)}(x-a)^2}{2(b-a)}\right] \exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right), \end{aligned} \tag{38}$$

for all $x \in [a,b]$. Utilising (35), we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \exp\left[\frac{\frac{f'_+(a)}{f(a)}(b-x)^2 - \frac{f'_-(b)}{f(b)}(x-a)^2}{2(b-a)}\right] \\ \le\ & f(b)^{\frac{b-x}{b-a}} f(a)^{\frac{x-a}{b-a}} \\ \le\ & \sqrt{f(a)f(b)} \exp\left[\frac{\frac{f'_-(b)}{f(b)}(b-x)^2 - \frac{f'_+(a)}{f(a)}(x-a)^2}{2(b-a)}\right], \end{aligned} \tag{39}$$

for all $x \in [a,b]$.

4.2. **Integral approximations.** Recall the error function:

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt; \\ \operatorname{erfi}(z) &= -i\,\operatorname{erf}(iz). \end{aligned}$$

Proposition 4.1. *Let $f:[a,b]\to(0,\infty)$ be an absolutely continuous function and $\gamma,\Gamma\in\mathbb{R}$ such that*

$$\gamma f(t)\le f'(t)\le\Gamma f(t),\quad \textit{for almost all } t\in[a,b].$$

Then we have the following estimates for the integral of f on $[a,b]$:

$$\begin{aligned}&\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\exp\left[\frac{1}{b-a}\int_a^b\log f(t)\,dt+\frac{\gamma\Gamma}{2\alpha}\right]\\ \le&\int_a^b f(x)\,dx\\ \le&\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\exp\left[\frac{1}{b-a}\int_a^b\log f(t)\,dt-\frac{\gamma\Gamma}{2\alpha}\right];\end{aligned}$$

where $\alpha=(\Gamma-\gamma)/(b-a)$. Furthermore, if f is log convex, then we have

$$\begin{aligned}&\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]f\left(\frac{a+b}{2}\right)\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\\ \le&\int_a^b f(x)\,dx\\ \le&\sqrt{\frac{\pi}{2\alpha}}\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\sqrt{f(a)f(b)}\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right).\end{aligned}$$

Proof. First, we note some useful identities to help us in our calculations:

$$\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}=-\frac{\Gamma-\gamma}{2(b-a)}\left(x-\frac{b\Gamma-a\gamma}{\Gamma-\gamma}\right)^2+\frac{(b-a)\gamma\Gamma}{2(\Gamma-\gamma)};\tag{40}$$

$$\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}=\frac{\Gamma-\gamma}{2(b-a)}\left(x+\frac{b\gamma-a\Gamma}{\Gamma-\gamma}\right)^2-\frac{(b-a)\gamma\Gamma}{2(\Gamma-\gamma)}.\tag{41}$$

To simplify our calculations, we let

$$\alpha=\frac{\Gamma-\gamma}{b-a},\quad\beta_1=\frac{b\Gamma-a\gamma}{\Gamma-\gamma},\quad\beta_2=\frac{a\Gamma-b\gamma}{\Gamma-\gamma}$$

so now (40) and (41) become

$$\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}=-\frac{\alpha}{2}(x-\beta_1)^2+\frac{\gamma\Gamma}{2\alpha};\tag{42}$$

$$\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}=\frac{\alpha}{2}(x-\beta_2)^2-\frac{\gamma\Gamma}{2\alpha}.\tag{43}$$

We integrate (15) with respect to x over $[a,b]$. We observe the integral

$$\begin{aligned}\int_a^b \exp\left[\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}\right]dx &= \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b \exp\left[-\frac{\alpha}{2}(x-\beta_1)^2\right]dx \\ &= \sqrt{\frac{2}{\alpha}}\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}}\exp(-u^2)\,du \\ &= \sqrt{\frac{2}{\alpha}}\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\frac{\sqrt{\pi}}{2}\left[-\operatorname{erf}\left(-\frac{\Gamma}{\sqrt{2\alpha}}\right)+\operatorname{erf}\left(-\frac{\gamma}{\sqrt{2\alpha}}\right)\right] \\ &= \sqrt{\frac{\pi}{2\alpha}}\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].\end{aligned}$$

Performing similar calculations, we get that:

$$\begin{aligned}\int_a^b \exp\left[\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}\right]dx &= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b \exp\left[\frac{\alpha}{2}(x-\beta_2)^2\right]dx \\ &= -\sqrt{\frac{2}{\alpha}}\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\int_{i\frac{\gamma}{\sqrt{2\alpha}}}^{i\frac{\Gamma}{\sqrt{2\alpha}}} i\exp(-u^2)\,du \\ &= \sqrt{\frac{\pi}{2\alpha}}\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].\end{aligned}$$

Thus (15) becomes:

$$\begin{aligned}&\sqrt{\frac{\pi}{2\alpha}}\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right] \\ &\le \int_a^b f(x)\,dx\exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\ &\le \sqrt{\frac{\pi}{2\alpha}}\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].\end{aligned}$$

Multiplying the above by $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ gives us the desired result. The last set of inequalities follows from (37), coupled with the fact that both functions erf and erfi is monotonically increasing. □

Proposition 4.2. *Let $0<a<b$ and $f:[a,b]\to(0,\infty)$ be an absolutely continuous function and $\gamma,\Gamma\in\mathbb{R}$ such that*

$$\gamma f(t)\le f'(t)\le \Gamma f(t),\quad \textit{for almost all } t\in[a,b].$$

Then we have the following estimates for $\int_a^b xf(x)\,dx$*:*

$$\begin{aligned}
&\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\left\{\frac{1}{\alpha}\left(\exp\left(-\frac{\Gamma(b-a)}{2}\right)-\exp\left(\frac{\gamma(b-a)}{2}\right)\right)\right.\\
&\left.+\sqrt{\frac{\pi}{2\alpha}}\beta_1\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\right\}\\
\leq\ & \int_a^b xf(x)\,dx\\
\leq\ & \exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]\left\{\frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right)-\exp\left(-\frac{\gamma(b-a)}{2}\right)\right)\right.\\
&\left.+\sqrt{\frac{\pi}{2\alpha}}\beta_2\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]\right\},
\end{aligned}$$

where

$$\alpha=\frac{\Gamma-\gamma}{b-a},\quad \beta_1=\frac{b\Gamma-a\gamma}{\Gamma-\gamma},\quad \beta_2=\frac{a\Gamma-b\gamma}{\Gamma-\gamma}.$$

Proof. We multiply (15) with $x\geq 0$ and integrate the resulting inequality with respect to x over $[a,b]$. We observe the integral

$$\begin{aligned}
&\int_a^b x\exp\left[\frac{\gamma(x-a)^2-\Gamma(b-x)^2}{2(b-a)}\right]dx\\
=\ & \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b x\exp\left[-\frac{\alpha}{2}(x-\beta_1)^2\right]dx\\
=\ & \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{2}{\alpha}\int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}} u\exp(-u^2)\,du+\sqrt{\frac{2}{\alpha}}\beta_1\int_{-\frac{\Gamma}{\sqrt{2\alpha}}}^{-\frac{\gamma}{\sqrt{2\alpha}}}\exp(-u^2)\,du\right]\\
=\ & \exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{1}{\alpha}\left(-\exp\left(-\frac{\gamma^2}{2\alpha}\right)+\exp\left(-\frac{\Gamma^2}{2\alpha}\right)\right)+\sqrt{\frac{\pi}{2\alpha}}\beta_1\left(\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right)\right]\\
=\ & \frac{1}{\alpha}\left(-\exp\left(\frac{\gamma(b-a)}{2}\right)+\exp\left(-\frac{\Gamma(b-a)}{2}\right)\right)+\sqrt{\frac{\pi}{2\alpha}}\beta_1\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]
\end{aligned}$$

Performing similar calculations, we get that

$$\begin{aligned}
&\int_a^b x\exp\left[\frac{\Gamma(x-a)^2-\gamma(b-x)^2}{2(b-a)}\right]dx\\
=\ & \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\int_a^b x\exp\left[\frac{\alpha}{2}(x-\beta_2)^2\right]dx\\
=\ & \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[-\frac{2}{\alpha}\int_{\frac{\gamma}{\sqrt{2\alpha}}i}^{\frac{\Gamma}{\sqrt{2\alpha}}i} u\exp(-u^2)\,du+\sqrt{\frac{2}{\alpha}}\beta_2\int_{\frac{\gamma}{\sqrt{2\alpha}}i}^{\frac{\Gamma}{\sqrt{2\alpha}}i}(-i)\exp(-u^2)\,du\right]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\frac{1}{\alpha}\left(\exp\left(\frac{\Gamma^2}{2\alpha}\right)-\exp\left(\frac{\gamma^2}{2\alpha}\right)\right)+\sqrt{\frac{\pi}{2\alpha}}\beta_2\left(\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right)\right] \\
&= \frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right)-\exp\left(-\frac{\gamma(b-a)}{2}\right)\right)+\sqrt{\frac{\pi}{2\alpha}}\beta_2\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right]
\end{aligned}
$$

Thus (15) becomes:

$$
\begin{aligned}
&\frac{1}{\alpha}\left(\exp\left(-\frac{\Gamma(b-a)}{2}\right)-\exp\left(\frac{\gamma(b-a)}{2}\right)\right)\sqrt{\frac{\pi}{2\alpha}}\beta_1\exp\left(\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erf}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erf}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right] \\
\leq\ & \int_a^b xf(x)\,dx\ \exp\left[-\frac{1}{b-a}\int_a^b \log f(t)\,dt\right] \\
\leq\ & \frac{1}{\alpha}\left(\exp\left(\frac{\Gamma(b-a)}{2}\right)-\exp\left(-\frac{\gamma(b-a)}{2}\right)\right)+\sqrt{\frac{\pi}{2\alpha}}\beta_2\exp\left(-\frac{\gamma\Gamma}{2\alpha}\right)\left[\operatorname{erfi}\left(\frac{\Gamma}{\sqrt{2\alpha}}\right)-\operatorname{erfi}\left(\frac{\gamma}{\sqrt{2\alpha}}\right)\right].
\end{aligned}
$$

Multiplying the above by $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ gives us the desired result. □

Remark 4.3. The inequalities in Proposition 4.2 can be simplified in the similar manner to that of Proposition 4.1 by assuming that f is logarithmic convex and using the estimates for $\exp\left[\frac{1}{b-a}\int_a^b \log f(t)\,dt\right]$ in (37).

5. Competing Intersets

The authors declare that they have no competing interests.

6. Authors' contributions

PC, SSD and EK contributed equally in all stages of writing the paper. All authors read and approved the final manuscript.

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