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SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. In this paper, some general two parameters perturbed Ostrowski type inequalities for functions of bounded variation are established.

1. Introduction

In order to extend the classical *Ostrowski's inequality* for differentiable functions with bounded derivatives to the larger class of functions of bounded variation, the author obtained in 1999 (see [17] or the RGMIA preprint version of [19]) the following result

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{1.1}$$

for any $x \in [a,b]$ and f a function of bounded variation on $[a,b]$. Here $\bigvee_a^b (f)$ denotes the *total variation* of f on $[a,b]$ and the constant $\frac{1}{2}$ is best possible in (1.1). The best inequality one can obtain from (1.1) is the *midpoint inequality*, namely

$$\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \le \frac{1}{2}(b-a) \bigvee_a^b (f), \tag{1.2}$$

for which the constant $\frac{1}{2}$ is also sharp.

For recent related results, see [1]-[4], [6]-[10], [13]-[15], [26]-[30] and [32]-[44].

For a function of bounded variation $v : [a,b] \to \mathbb{C}$ we define the *Cumulative Variation Function* (CVF) $V : [a,b] \to [0,\infty)$ by

$$V(t) := \bigvee_a^t (v),$$

the total variation of v on the interval $[a,t]$ with $t \in [a,b]$.

It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c \in [a,b]$ if and only if the generating function v is continuing in that point. If v is *Lipschitzian* with the constant $L > 0$, i.e.

$$|v(t) - v(s)| \le L\,|t-s| \text{ for any } t,s \in [a,b],$$

then V is also Lipschitzian with the same constant.

The following lemma will be used in the sequel and is of interest in itself as well [11, p. 177]. For a simple proof see also [22].

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Lemma 1. *Let $f,\ u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u). \tag{1.3}$$

The following result may be stated.

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \\
&\le \int_a^x \left(\bigvee_t^x (f)\right) dt + \int_x^b \left(\bigvee_x^t (f)\right) dt \\
&\le (x-a)\bigvee_a^x (f) + (b-x)\bigvee_x^b (f) \\
&\le \begin{cases} \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right] \bigvee_a^b (f), \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right](b-a), \end{cases}
\end{aligned} \tag{1.4}$$

for any $x\in[a,b]$.

The following midpoint inequality holds:

Corollary 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)(b-a)\right| \\
&\le \int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f)\right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f)\right) dt \le \frac{1}{2}(b-a)\bigvee_a^b (f).
\end{aligned} \tag{1.5}$$

The first inequality in (1.5) is sharp and the constant $\frac{1}{2}$ in the second, is best possible.

Motivated by the above results, in this paper we establish some two parameters perturbed Ostrowski type inequalities for functions of bounded variation.

2. Some Identities

We start with the following identity that will play an important role in the following:

Lemma 2. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$\begin{aligned}
&f(x) + \frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x) - (x-a)^2\lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right] + \frac{1}{b-a}\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right],
\end{aligned} \tag{2.1}$$

where the integrals in the right hand side are taken in the Riemann-Stieltjes sense.

Proof. Utilising the integration by parts formula in the Riemann-Stieltjes integral, we have

$$\begin{aligned}(2.2)\qquad &\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\\ &= (t-a)\left[f(t)-\lambda_1(x)t\right]\Big|_a^x - \int_a^x \left[f(t)-\lambda_1(x)t\right]dt\\ &= (x-a)\left[f(x)-\lambda_1(x)x\right] - \int_a^x f(t)\,dt + \frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)\\ &= (x-a)f(x)-\lambda_1(x)x(x-a) - \int_a^x f(t)\,dt + \frac{1}{2}\lambda_1(x)\left(x^2-a^2\right)\\ &= (x-a)f(x) - \int_a^x f(t)\,dt - \frac{1}{2}(x-a)^2\lambda_1(x)\end{aligned}$$

and

$$\begin{aligned}(2.3)\qquad &\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\\ &= (t-b)\left[f(t)-\lambda_2(x)t\right]\Big|_x^b - \int_x^b \left[f(t)-\lambda_2(x)t\right]dt\\ &= (b-x)\left[f(x)-\lambda_2(x)x\right] - \int_x^b f(t)\,dt + \frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)\\ &= (b-x)f(x) - \int_x^b f(t)\,dt - (b-x)\lambda_2(x)x + \frac{1}{2}\lambda_2(x)\left(b^2-x^2\right)\\ &= (b-x)f(x) - \int_x^b f(t)\,dt + \frac{1}{2}(b-x)^2\lambda_2(x).\end{aligned}$$

By adding the equalities (2.2) and (2.3) and dividing by $b-a$ we get the desired representation (2.1). □

Corollary 2. *With the assumption in Lemma 2, we have for any* $\lambda(x)\in\mathbb{C}$ *that*

$$\begin{aligned}(2.4)\quad &f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^x (t-a)\,d\left[f(t)-\lambda(x)t\right]+\frac{1}{b-a}\int_x^b (t-b)\,d\left[f(t)-\lambda(x)t\right].\end{aligned}$$

We have the following midpoint representation:

Corollary 3. *With the assumption in Lemma 2, we have for any* $\lambda_1,\ \lambda_2\in\mathbb{C}$ *that*

$$\begin{aligned}(2.5)\qquad &f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_2-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d\left[f(t)-\lambda_1 t\right]\\ &+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d\left[f(t)-\lambda_2 t\right].\end{aligned}$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\,d[f(t)-\lambda t] + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t)-\lambda t]. \tag{2.6}$$

Remark 1. *If we take $\lambda(x) = 0$ in (2.4) we recapture the Montgomery type identity established in* [19].

3. Inequalities for Functions of Bounded Variation

We denote by $\ell : [a,b] \to [a,b]$ the identity function, namely $\ell(t) = t$ for any $t \in [a,b]$.

We have the following result:

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have the inequality*

$$\begin{aligned}
&\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x) - (x-a)^2\lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\le \frac{1}{b-a}\left[\int_a^x \left(\bigvee_t^x (f-\lambda_1(x)\ell)\right) dt + \int_x^b \left(\bigvee_x^t (f-\lambda_2(x)\ell)\right) dt\right] \\
&\le \frac{1}{b-a}\left[(x-a)\bigvee_a^x (f-\lambda_1(x)\ell) + (b-x)\bigvee_x^b (f-\lambda_2(x)\ell)\right] \\
&\le \begin{cases} \max\left\{\bigvee_a^x (f-\lambda_1(x)\ell), \bigvee_x^b (f-\lambda_2(x)\ell)\right\} \\[2mm] \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left(\bigvee_a^x (f-\lambda_1(x)\ell) + \bigvee_x^b (f-\lambda_2(x)\ell)\right), \end{cases}
\end{aligned} \tag{3.1}$$

where $\bigvee_c^d (g)$ denotes the total variation of g on the interval $[c,d]$.

Proof. Taking the modulus in (2.1) and using the property (1.3) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.2}$$
$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right|$$
$$+\frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\,d\left(\bigvee_a^t\left(f-\lambda_1(x)\ell\right)\right)$$
$$+\frac{1}{b-a}\int_x^b (b-t)\,d\left(\bigvee_a^t\left(f-\lambda_2(x)\ell\right)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\int_a^x (t-a)\,d\left(\bigvee_a^t\left(f-\lambda_1(x)\ell\right)\right)$$
$$=(t-a)\bigvee_a^t\left(f-\lambda_1(x)\ell\right)\Big|_a^x-\int_a^x\left(\bigvee_a^t\left(f-\lambda_1(x)\ell\right)\right)dt$$
$$=(x-a)\bigvee_a^x\left(f-\lambda_1(x)\ell\right)-\int_a^x\left(\bigvee_a^t\left(f-\lambda_1(x)\ell\right)\right)dt$$
$$=\int_a^x\left(\bigvee_t^x\left(f-\lambda_1(x)\ell\right)\right)dt$$

and

$$\int_x^b (b-t)\,d\left(\bigvee_a^t\left(f-\lambda_2(x)\ell\right)\right)$$
$$=(b-t)\bigvee_a^t\left(f-\lambda_2(x)\ell\right)\Big|_x^b+\int_x^b\left(\bigvee_a^t\left(f-\lambda_2(x)\ell\right)\right)dt$$
$$=\int_x^b\left(\bigvee_a^t\left(f-\lambda_2(x)\ell\right)\right)dt-(b-x)\bigvee_a^x\left(f-\lambda_2(x)\ell\right)$$
$$=\int_x^b\left(\bigvee_x^t\left(f-\lambda_2(x)\ell\right)\right)dt.$$

Using (3.2) we deduce the first inequality in (3.1).

We also have

$$\int_a^x\left(\bigvee_t^x\left(f-\lambda_1(x)\ell\right)\right)dt\leq(x-a)\bigvee_a^x\left(f-\lambda_1(x)\ell\right)$$

and

$$\int_x^b \left(\bigvee_x^t (f - \lambda_2(x)\ell) \right) dt \leq (b - x) \bigvee_x^b (f - \lambda_2(x)\ell),$$

which prove the second inequality in (3.1).

The last part is obvious. □

The following result generalizes the inequality (1.4).

Corollary 4. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and $x \in [a,b]$. Then for any $\lambda(x)$ a complex number, we have the inequality*

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) \lambda(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{3.3}$$
$$\leq \frac{1}{b-a} \left[\int_a^x \left(\bigvee_t^x (f - \lambda(x)\ell) \right) dt + \int_x^b \left(\bigvee_x^t (f - \lambda(x)\ell) \right) dt \right]$$
$$\leq \frac{1}{b-a} \left[(x-a) \bigvee_a^x (f - \lambda(x)\ell) + (b-x) \bigvee_x^b (f - \lambda(x)\ell) \right]$$
$$\leq \begin{cases} \frac{1}{2} \bigvee_a^b (f - \lambda(x)\ell) + \frac{1}{2} \left| \bigvee_x^b (f - \lambda(x)\ell) - \bigvee_a^x (f - \lambda(x)\ell) \right| \\[2ex] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f - \lambda(x)\ell). \end{cases}$$

Remark 2. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $\lambda \in \mathbb{C}$ we have the inequalities*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{3.4}$$
$$\leq \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f - \lambda\ell) \right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f - \lambda\ell) \right) dt \right]$$
$$\leq \frac{1}{2} \bigvee_a^b (f - \lambda\ell).$$

This is equivalent to

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{3.5}$$
$$\leq \frac{1}{b-a} \inf_{\lambda \in \mathbb{C}} \left[\int_a^{\frac{a+b}{2}} \left(\bigvee_t^{\frac{a+b}{2}} (f - \lambda\ell) \right) dt + \int_{\frac{a+b}{2}}^b \left(\bigvee_{\frac{a+b}{2}}^t (f - \lambda\ell) \right) dt \right]$$
$$\leq \frac{1}{2} \inf_{\lambda \in \mathbb{C}} \left[\bigvee_a^b (f - \lambda\ell) \right].$$

4. Inequalities for Lipshitzian Functions

We can state the following result:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are complex numbers and there exist the positive numbers $L_1(x)$ and $L_2(x)$ such that $f-\lambda_1(x)\ell$ is Lipschitzian with the constant $L_1(x)$ on the interval $[a,x]$ and $f-\lambda_2(x)\ell$ is Lipschitzian with the constant $L_2(x)$ on the interval $[x,b]$, then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4.1}$$

$$\le\frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^2L_1(x)+\left(\frac{b-x}{b-a}\right)^2L_2(x)\right](b-a)$$

$$\le\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\max\{L_1(x),L_2(x)\}(b-a),\\[2ex] \frac{1}{2}\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p}(b-a),\\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\frac{L_1(x)+L_2(x)}{2}(b-a).\end{cases}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is Riemann integrable and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\le L\int_c^d|g(t)|\,dt. \tag{4.2}$$

Taking the modulus in (2.1) and using the property (4.2) we have

$$\begin{aligned}
&\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right]\right| \\
&+\frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t)-\lambda_2(x)t\right]\right| \\
&\leq \frac{1}{b-a}\left[L_1(x)\int_a^x (t-a)\,dt+L_2(x)\int_x^b (b-t)\,dt\right] \\
&=\frac{L_1(x)(x-a)^2+L_2(x)(b-x)^2}{2(b-a)} \\
&=\frac{1}{2}\left[L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2\right](b-a),
\end{aligned}$$

and the first inequality in (4.1) is proved.

By Hölder's inequality we have

$$L_1(x)\left(\frac{x-a}{b-a}\right)^2+L_2(x)\left(\frac{b-x}{b-a}\right)^2$$

$$\leq\begin{cases}
\left[\left(\frac{x-a}{b-a}\right)^2+\left(\frac{b-x}{b-a}\right)^2\right]\max\left\{L_1(x),L_2(x)\right\} \\
\\
\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{1/q}\left(L_1^p(x)+L_2^p(x)\right)^{1/p}, \\
p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
\\
\max\left\{\left(\frac{x-a}{b-a}\right)^2,\left(\frac{b-x}{b-a}\right)^2\right\}\left[L_1(x)+L_2(x)\right],
\end{cases}$$

which proves, upon simple calculations, the last part of the inequality (4.1). □

Corollary 5. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a complex number and there exist the positive number $L(x)$ such that $f-\lambda(x)\ell$ is Lipschitzian with the constant $L(x)$ on the interval $[a,b]$, then*

$$\begin{aligned}
&\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]L(x)(b-a).
\end{aligned} \tag{4.3}$$

Remark 3. *If λ is a complex number and there exist the positive number L such that $f-\lambda\ell$ is Lipschitzian with the constant L on the interval $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}L(b-a). \tag{4.4}$$

5. Inequalities for Monotonic Functions

Now, the case of monotonic integrators is as follows:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda_1(x)$ and $\lambda_2(x)$ are real numbers such that $f-\lambda_1(x)\ell$ is monotonic nondecreasing on the interval $[a,x]$ and $f-\lambda_2(x)\ell$ is monotonic nondecreasing on the interval $[x,b]$, then*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.1}$$

$$\leq\frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt\right.$$
$$\left.-\frac{1}{2}\left[\lambda_1(x)(x-a)^2+\lambda_2(x)(b-x)^2\right]\right]$$

$$\leq\frac{1}{b-a}\{(x-a)[f(x)-f(a)-\lambda_1(x)(x-a)]$$
$$+(b-x)[f(b)-f(x)-\lambda_2(x)(b-x)]\}$$

$$\leq\begin{cases}\frac{1}{2}[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)] \\ +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda_1(x)(x-a)+\frac{1}{2}\lambda_2(x)(b-x)\right|, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \\ \times[f(b)-f(a)-\lambda_1(x)(x-a)-\lambda_2(x)(b-x)].\end{cases}$$

Proof. It is known that, if $g:[c,d]\to\mathbb{C}$ is continuous and $u:[c,d]\to\mathbb{C}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_c^d g(t)\,du(t)$ exists and

$$\left|\int_c^d g(t)\,du(t)\right|\leq\int_c^d|g(t)|\,du(t). \tag{5.2}$$

Taking the modulus in (2.1) and using the property (5.2) we have

$$
\begin{aligned}
&\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
&\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d\left[f(t) - \lambda_1(x)t\right]\right| \\
&+ \frac{1}{b-a}\left|\int_x^b (t-b)\,d\left[f(t) - \lambda_2(x)t\right]\right| \\
&\leq \frac{1}{b-a}\int_a^x (t-a)\,d\left[f(t) - \lambda_1(x)t\right] \\
&+ \frac{1}{b-a}\int_x^b (b-t)\,d\left[f(t) - \lambda_2(x)t\right].
\end{aligned}
\tag{5.3}
$$

Integrating by parts in the Riemann-Stieltjes integral we have

$$
\begin{aligned}
&\int_a^x (t-a)\,d\left[f(t) - \lambda_1(x)t\right] \\
&= (t-a)\left[f(t) - \lambda_1(x)t\right]\Big|_a^x - \int_a^x \left[f(t) - \lambda_1(x)t\right]dt \\
&= (x-a)\left[f(x) - \lambda_1(x)x\right] - \int_a^x \left[f(t) - \lambda_1(x)t\right]dt \\
&= (x-a)f(x) - \lambda_1(x)x(x-a) - \int_a^x f(t)\,dt + \lambda_1(x)\frac{x^2-a^2}{2} \\
&= (x-a)f(x) - \int_a^x f(t)\,dt - \frac{1}{2}\lambda_1(x)(x-a)^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_x^b (b-t)\,d\left[f(t) - \lambda_2(x)t\right] \\
&= (b-t)\left[f(t) - \lambda_2(x)t\right]\Big|_x^b + \int_x^b \left[f(t) - \lambda_2(x)t\right]dt \\
&= \int_x^b f(t)\,dt - \lambda_2(x)\int_x^b t\,dt - (b-x)\left[f(x) - \lambda_2(x)x\right] \\
&= \int_x^b f(t)\,dt - \lambda_2(x)\frac{b^2-x^2}{2} - (b-x)f(x) + (b-x)\lambda_2(x)x \\
&= \int_x^b f(t)\,dt - (b-x)f(x) - \frac{1}{2}\lambda_2(x)(b-x)^2.
\end{aligned}
$$

If we add these equalities, we get

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] + \int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\, t\right] \\
&= (x-a) f(x) - \int_a^x f(t)\, dt - \frac{1}{2}\lambda_1(x)(x-a)^2 \\
&+ \int_x^b f(t)\, dt - (b-x) f(x) - \frac{1}{2}\lambda_2(x)(b-x)^2 \\
&= (2x-a-b) f(x) + \int_a^b sgn(t-x) f(t)\, dt \\
&- \frac{1}{2}\left[\lambda_1(x)(x-a)^2 + \lambda_2(x)(b-x)^2\right]
\end{aligned}$$

and by (5.3) we get the first inequality in (5.1).

Now, since $f-\lambda_1(x)\,\ell$ is monotonic nondecreasing on the interval $[a,x]$, then

$$\begin{aligned}
&\int_a^x (t-a)\, d\left[f(t)-\lambda_1(x)\, t\right] \\
&\leq (x-a)\left[f(x)-\lambda_1(x)\, x - f(a) + \lambda_1(x)\, a\right] \\
&= (x-a)\left[f(x)-f(a)-\lambda_1(x)(x-a)\right]
\end{aligned}$$

and, since $f-\lambda_2(x)\,\ell$ is monotonic nondecreasing on the interval $[x,b]$, then also

$$\begin{aligned}
&\int_x^b (b-t)\, d\left[f(t)-\lambda_2(x)\, t\right] \\
&\leq (b-x)\left[f(b)-\lambda_2(x)\, b - f(x) + \lambda_2(x)\, x\right] \\
&= (b-x)\left[f(b)-f(x)-\lambda_2(x)(b-x)\right].
\end{aligned}$$

These prove the second inequality in (5.1).

The last part follows by the properties of maximum and the details are omitted. □

Corollary 6. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$ and $x\in(a,b)$. If $\lambda(x)$ is a real number such that $f-\lambda(x)\,\ell$ is monotonic nondecreasing on the* *interval* $[a,b]$, *then*

$$\left|f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.4}$$

$$\leq \frac{1}{b-a}\left[(2x-a-b)f(x)+\int_a^b sgn(t-x)f(t)\,dt - \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\lambda(x)\right]$$

$$\leq \frac{1}{b-a}\{(x-a)[f(x)-f(a)-\lambda(x)(x-a)] + (b-x)[f(b)-f(x)-\lambda(x)(b-x)]\}$$

$$\leq \begin{cases} \frac{f(b)-f(a)}{2}-\frac{1}{2}\lambda(x)(b-a) \\ +\left|f(x)-\frac{f(a)+f(b)}{2}-\frac{1}{2}\lambda(x)(2x-a-b)\right|, \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right] \\ \times[f(b)-f(a)-\lambda(x)(b-a)]. \end{cases}$$

Remark 4. *If λ is a real number such that $f-\lambda\ell$ is monotonic nondecreasing on the interval $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.5}$$

$$\leq \frac{1}{b-a}\left[\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)\,dt-\frac{1}{4}\lambda(b-a)^2\right]$$

$$\leq \frac{1}{2}[f(b)-f(a)-\lambda(b-a)].$$

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