

**SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR
ABSOLUTELY CONTINUOUS FUNCTIONS (II)**

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ABSTRACT. In this paper, further perturbed Ostrowski type inequalities for absolutely continuous functions are established.

1. INTRODUCTION

In order to obtain various perturbed Ostrowski type inequalities, in the earlier paper [24] we established the following equality:

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous on $[a, b]$ and $x \in [a, b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$(1.1) \quad f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 \lambda_2(x) - (x-a)^2 \lambda_1(x) \right] - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a) [f'(t) - \lambda_1(x)] dt + \frac{1}{b-a} \int_x^b (t-b) [f'(t) - \lambda_2(x)] dt,$$

where the integrals in the right hand side are taken in the Lebesgue sense.

The following equality in terms of one parameter holds:

Corollary 1. *With the assumption in Lemma 1, we have for any $\lambda(x) \in \mathbb{C}$ that*

$$(1.2) \quad f(x) + \left(\frac{a+b}{2} - x \right) \lambda(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a) [f'(t) - \lambda(x)] dt + \frac{1}{b-a} \int_x^b (t-b) [f'(t) - \lambda(x)] dt.$$

Remark 1. *If we take $\lambda(x) = 0$ in (1.2), then we get Montgomery's identity for absolutely continuous functions, namely*

$$(1.3) \quad f(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a) f'(t) dt + \frac{1}{b-a} \int_x^b (t-b) f'(t) dt,$$

for $x \in [a, b]$.

We have the following midpoint representation:

1991 *Mathematics Subject Classification.* 25D15, 25D10.

Key words and phrases. Ostrowski's inequality, Integral inequalities, Perturbed Ostrowski Type Inequalities.

Corollary 2. *With the assumption in Lemma 1, we have for any $\lambda_1, \lambda_2 \in \mathbb{C}$ that*

$$(1.4) \quad f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)(\lambda_2 - \lambda_1) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda_1] dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda_2] dt.$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then we have the equality

$$(1.5) \quad f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a)[f'(t) - \lambda] dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b)[f'(t) - \lambda] dt.$$

The identity (1.1) has many particular cases of interest.

If $x \in (a, b)$ is a point of differentiability for the absolutely continuous function $f : [a, b] \rightarrow \mathbb{C}$, then we have the equality:

$$(1.6) \quad f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a)[f'(t) - f'(x)] dt + \frac{1}{b-a} \int_x^b (t-b)[f'(t) - f'(x)] dt.$$

In particular we have

$$(1.7) \quad f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) \left[f'(t) - f'\left(\frac{a+b}{2}\right) \right] dt \\ + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b) \left[f'(t) - f'\left(\frac{a+b}{2}\right) \right] dt$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x \in (a, b)$, if we take in (1.1)

$$\lambda_1(x) = \frac{f(x) - f(a)}{x-a} \quad \text{and} \quad \lambda_2(x) = \frac{f(b) - f(x)}{b-x},$$

then we get, after some elementary calculations,

$$(1.8) \quad \frac{1}{2} \left[f(x) + \frac{(b-x)f(b) + (x-a)f(a)}{b-a} \right] - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a) \left[f'(t) - \frac{f(x) - f(a)}{x-a} \right] dt \\ + \frac{1}{b-a} \int_x^b (t-b) \left[f'(t) - \frac{f(b) - f(x)}{b-x} \right] dt.$$

In particular, we have

$$\begin{aligned}
(1.9) \quad & \frac{1}{2} \left[f \left(\frac{a+b}{2} \right) + \frac{f(b) + f(a)}{2} \right] - \frac{1}{b-a} \int_a^b f(t) dt \\
&= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) \left[f'(t) - \frac{f \left(\frac{a+b}{2} \right) - f(a)}{\frac{b-a}{2}} \right] dt \\
&+ \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b) \left[f'(t) - \frac{f(b) - f \left(\frac{a+b}{2} \right)}{\frac{b-a}{2}} \right] dt.
\end{aligned}$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (1.1) for $\lambda_1(x) = f'_+(a)$ and $\lambda_2(x) = f'_-(b)$

$$\begin{aligned}
(1.10) \quad & f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right] - \frac{1}{b-a} \int_a^b f(t) dt \\
&= \frac{1}{b-a} \int_a^x (t-a) [f'(t) - f'_+(a)] dt \\
&+ \frac{1}{b-a} \int_x^b (t-b) [f'(t) - f'_-(b)] dt,
\end{aligned}$$

for all $x \in [a, b]$.

In particular, we have

$$\begin{aligned}
(1.11) \quad & f \left(\frac{a+b}{2} \right) + \frac{1}{8} (b-a) [f'_-(b) - f'_+(a)] - \frac{1}{b-a} \int_a^b f(t) dt \\
&= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) [f'(t) - f'_+(a)] dt \\
&+ \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (t-b) [f'(t) - f'_-(b)] dt.
\end{aligned}$$

If we take in (1.1) $\lambda_2(x) = \lambda_2(x) = f' \left(\frac{a+b}{2} \right)$, provided this derivative exists and is finite, then we get

$$\begin{aligned}
(1.12) \quad & f(x) + \left(\frac{a+b}{2} - x \right) f' \left(\frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(t) dt \\
&= \frac{1}{b-a} \int_a^x (t-a) \left[f'(t) - f' \left(\frac{a+b}{2} \right) \right] dt \\
&+ \frac{1}{b-a} \int_x^b (t-b) \left[f'(t) - f' \left(\frac{a+b}{2} \right) \right] dt,
\end{aligned}$$

for all $x \in [a, b]$.

In [24] we obtained the following perturbed Ostrowski type inequalities:

Theorem 1. Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on \mathring{I} and $[a, b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \rightarrow \mathbb{C}$ is of bounded variation on $[a, b]$, then

$$(1.13) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{2} \left(\frac{a+b}{2} - x \right) f'(x) + \frac{1}{4(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] \right| \leq \frac{1}{4} \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x(f') + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b(f') \right] (b-a) \leq \frac{1}{4} (b-a) \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b(f') + \frac{1}{2} \left| \bigvee_a^x(f') - \bigvee_x^b(f') \right| \right], \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x(f') \right]^q + \left[\bigvee_x^b(f') \right]^q \right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x-\frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f') \end{cases}$$

for any $x \in [a, b]$.

We say that $v : [a, b] \rightarrow \mathbb{C}$ is *Lipschitzian* with the constant $L > 0$, if it satisfies the condition

$$|v(t) - v(s)| \leq L|t - s| \text{ for any } t, s \in [a, b].$$

Theorem 2. Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on \mathring{I} and $[a, b] \subset \mathring{I}$. Let $x \in (a, b)$. If the derivative $f' : \mathring{I} \rightarrow \mathbb{C}$ is Lipschitzian with the constant $K_1(x)$ on $[a, x]$ and constant $K_2(x)$ on $[x, b]$, then

$$(1.14) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{2} \left(\frac{a+b}{2} - x \right) f'(x) + \frac{1}{4(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] \right| \leq \frac{1}{8} \left[\left(\frac{x-a}{b-a} \right)^3 K_1(x) + \left(\frac{b-x}{b-a} \right)^3 K_2(x) \right] (b-a)^2$$

$$\leq \frac{1}{8} (b-a)^2 \times \begin{cases} \left[\left(\frac{x-a}{b-a} \right)^3 + \left(\frac{b-x}{b-a} \right)^3 \right] \max \{K_1(x), K_2(x)\}, \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} [K_1^q(x) + K_2^q(x)]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^3 [K_1(x) + K_2(x)]. \end{cases}$$

For other Ostrowski type inequalities see [1]-[19] and [21]-[43].

Motivated by the above results, we establish in this paper other perturbed Ostrowski type inequalities for complex valued differentiable functions.

2. INEQUALITIES FOR DERIVATIVES OF BOUNDED VARIATION

Assume that the function $f : I \rightarrow \mathbb{C}$ is differentiable on the interior of I , denoted $\overset{\circ}{I}$, and $[a, b] \subset \overset{\circ}{I}$. Then, from (1.6) we have the equality

$$(2.1) \quad f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \\ = \frac{1}{b-a} \int_a^x (t-a) [f'(t) - f'(x)] dt + \frac{1}{b-a} \int_x^b (t-b) [f'(t) - f'(x)] dt$$

for any $x \in [a, b]$.

We have the following result:

Theorem 3. *Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on $\overset{\circ}{I}$ and $[a, b] \subset \overset{\circ}{I}$. If the derivative $f' : \overset{\circ}{I} \rightarrow \mathbb{C}$ is of bounded variation on $[a, b]$, then*

$$(2.2) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_t^x(f') dt + \int_x^b (b-t) \bigvee_x^t(f') dt \right] \\ \leq \frac{1}{2} (b-a) \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x(f') dt + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b(f') \right]$$

$$\leq \frac{1}{2}(b-a) \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b(f') + \frac{1}{2} \left| \bigvee_a^x(f') - \bigvee_x^b(f') \right| \right], \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x(f') \right]^q + \left[\bigvee_x^b(f') \right]^q \right]^{1/q}, \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f'), \end{cases}$$

for any $x \in [a, b]$.

Proof. Taking the modulus in (2.1) we have

$$(2.3) \quad \begin{aligned} & \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left| \int_a^x (t-a) [f'(t) - f'(x)] dt \right| \\ & \quad + \frac{1}{b-a} \left| \int_x^b (t-b) [f'(t) - f'(x)] dt \right| \\ & \leq \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(x)| dt \\ & \quad + \frac{1}{b-a} \int_x^b (b-t) |f'(t) - f'(x)| dt. \end{aligned}$$

Since the derivative $f' : \hat{I} \rightarrow \mathbb{C}$ is of bounded variation on $[a, x]$ and $[x, b]$, then

$$|f'(t) - f'(x)| \leq \bigvee_t^x(f') \quad \text{for } t \in [a, x]$$

and

$$|f'(t) - f'(x)| \leq \bigvee_x^t(f') \quad \text{for } t \in [x, b].$$

Therefore

$$\begin{aligned} \int_a^x (t-a) |f'(t) - f'(x)| dt & \leq \int_a^x (t-a) \bigvee_t^x(f') dt \\ & \leq \frac{1}{2} (x-a)^2 \bigvee_a^x(f') \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t) |f'(t) - f'(x)| dt &\leq \int_x^b (b-t) \bigvee_x^t (f') dt \\ &\leq \frac{1}{2} (b-x)^2 \bigvee_x^b (f'), \end{aligned}$$

which, by (2.3) produce the first two inequalities in (2.2).

The last part follows by Hölder's inequality

$$mn + pq \leq (m^\alpha + p^\alpha)^{1/\alpha} (n^\beta + q^\beta)^{1/\beta},$$

where $m, n, p, q \geq 0$ and $\alpha > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. □

Corollary 3. *With the assumptions of Theorem 3, we have*

$$\begin{aligned} (2.4) \quad &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{b-a} \left[\int_a^{\frac{a+b}{2}} (t-a) \bigvee_t^{\frac{a+b}{2}} (f') dt + \int_{\frac{a+b}{2}}^b (b-t) \bigvee_{\frac{a+b}{2}}^t (f') dt \right] \\ &\leq \frac{1}{8} (b-a) \bigvee_a^b (f') dt. \end{aligned}$$

Remark 2. *If $p \in (a, b)$ is a median point in bounded variation for the derivative, i.e. $\bigvee_a^p (f') = \bigvee_p^b (f')$, then under the assumptions of Theorem 3 we have*

$$\begin{aligned} (2.5) \quad &\left| f(p) + \left(\frac{a+b}{2} - p\right) f'(p) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{b-a} \left[\int_a^p (t-a) \bigvee_t^p (f') dt + \int_p^b (b-t) \bigvee_p^t (f') dt \right] \\ &\leq \frac{1}{4} (b-a) \left[\frac{1}{4} + \left(\frac{p - \frac{a+b}{2}}{b-a}\right)^2 \right] \bigvee_a^b (f'). \end{aligned}$$

3. INEQUALITIES FOR LIPSCHITZIAN DERIVATIVES

We start with the following result.

Theorem 4. *Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on $\overset{\circ}{I}$ and $[a, b] \subset \overset{\circ}{I}$. Let $x \in (a, b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1, 2$ are such that*

$$(3.1) \quad |f'(t) - f'(x)| \leq L_{\alpha_1} (x-t)^{\alpha_1} \quad \text{for any } t \in [a, x]$$

and

$$(3.2) \quad |f'(t) - f'(x)| \leq L_{\alpha_2} (t-x)^{\alpha_2} \quad \text{for any } t \in (x, b],$$

then we have

$$(3.3) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{(\alpha_1+1)(\alpha_1+2)} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)} (b-x)^{\alpha_2+2} \right].$$

Proof. Taking the modulus in (2.1) we have

$$(3.4) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(x)| dt \\ + \frac{1}{b-a} \int_x^b (b-t) |f'(t) - f'(x)| dt.$$

Using the properties (3.1) and (3.2) we have

$$\int_a^x (t-a) |f'(t) - f'(x)| dt \leq L_{\alpha_1} \int_a^x (t-a) (x-t)^{\alpha_1} dt \\ = L_{\alpha_1} (x-a)^{\alpha_1+2} \int_0^1 u(1-u)^{\alpha_1} du \\ = L_{\alpha_1} (x-a)^{\alpha_1+2} \int_0^1 u^{\alpha_1} (1-u) du \\ = \frac{1}{(\alpha_1+1)(\alpha_1+2)} L_{\alpha_1} (x-a)^{\alpha_1+2}$$

and

$$\int_x^b (b-t) |f'(t) - f'(x)| dt \leq L_{\alpha_2} \int_x^b (b-t) (t-x)^{\alpha_2} dt \\ = \frac{1}{(\alpha_2+1)(\alpha_2+2)} L_{\alpha_2} (b-x)^{\alpha_2+2}.$$

Utilising (3.4) we get the desired result (3.3). \square

Corollary 4. Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on \mathring{I} and $[a, b] \subset \mathring{I}$. If the derivative is f' of r -H-Hölder type on $[a, b]$, i.e. we have the condition

$$|f'(t) - f'(s)| \leq H |t - s|^r$$

for any $t, s \in [a, b]$, where $r \in (0, 1]$ and $H > 0$ are given, then

$$(3.5) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{H}{(r+1)(r+2)} \left[\left(\frac{x-a}{b-a} \right)^{r+2} + \left(\frac{b-x}{b-a} \right)^{r+2} \right] (b-a)^{r+1},$$

for any $x \in [a, b]$.

In particular, if f' is Lipschitzian with the constant $L > 0$, then

$$(3.6) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ \leq \frac{1}{6} L \left[\left(\frac{x-a}{b-a} \right)^3 + \left(\frac{b-x}{b-a} \right)^3 \right] (b-a)^2,$$

for any $x \in [a, b]$.

4. INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS

The case of convex functions is as follows:

Theorem 5. Let $f : I \rightarrow \mathbb{C}$ be a differentiable convex function on \hat{I} and $[a, b] \subset \hat{I}$. Then for any $x \in [a, b]$ we have

$$(4.1) \quad 0 \leq \frac{1}{b-a} \int_a^b f(t) dt - f(x) - \left(\frac{a+b}{2} - x \right) f'(x) \leq \begin{cases} I_1(x) \\ I_2(x) \\ I_3(x) \end{cases}$$

where

$$I_1(x) := \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - f(x) - 2f'(x) \left(\frac{a+b}{2} - x \right), \\ I_2(x) := \frac{1}{2} \frac{f'(b)(b-x)^2 - f'(a)(x-a)^2}{b-a} - f'(x) \left(\frac{a+b}{2} - x \right)$$

and

$$I_3(x) := \frac{1}{2} \left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} - f(x) \right] - f'(x) \left(\frac{a+b}{2} - x \right)$$

Proof. We have the equality

$$(4.2) \quad \frac{1}{b-a} \int_a^b f(t) dt - f(x) - \left(\frac{a+b}{2} - x \right) f'(x) \\ = \frac{1}{b-a} \int_a^x (t-a) [f'(x) - f'(t)] dt + \frac{1}{b-a} \int_x^b (b-t) [f'(t) - f'(x)] dt$$

for any $x \in [a, b]$.

Since f is a differentiable convex function on \hat{I} , then f' is monotonic nondecreasing on \hat{I} and then

$$\int_a^x (t-a) [f'(x) - f'(t)] dt \geq 0$$

and

$$\int_x^b (b-t) [f'(t) - f'(x)] dt \geq 0,$$

which proves the first inequality in (4.1).

We have

$$\int_a^x (t-a) [f'(x) - f'(t)] dt \leq (x-a) \int_a^x [f'(x) - f'(t)] dt \\ = (x-a) [f'(x)(x-a) - f(x) + f(a)]$$

and

$$\begin{aligned} \int_x^b (b-t)[f'(t) - f'(x)] dt &\leq (b-x) \int_x^b [f'(t) - f'(x)] dt \\ &= (b-x)[f(b) - f(x) - f'(x)(b-x)]. \end{aligned}$$

Adding these inequalities we get

$$\begin{aligned} &\int_a^x (t-a)[f'(x) - f'(t)] dt + \int_x^b (b-t)[f'(t) - f'(x)] dt \\ &\leq (x-a)[f'(x)(x-a) - f(x) + f(a)] \\ &\quad + (b-x)[f(b) - f(x) - f'(x)(b-x)] \\ &= (b-x)f(b) + (x-a)f(a) - (b-a)f(x) \\ &\quad + f'(x)[2x - (a+b)](b-a) \end{aligned}$$

and by (4.2) we get the second inequality for $I_1(x)$.

We also have

$$\begin{aligned} \int_a^x (t-a)[f'(x) - f'(t)] dt &\leq \int_a^x (t-a)[f'(x) - f'(a)] dt \\ &= \frac{1}{2}[f'(x) - f'(a)](x-a)^2 \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t)[f'(t) - f'(x)] dt &\leq \int_x^b (b-t)[f'(b) - f'(x)] dt \\ &= \frac{1}{2}[f'(b) - f'(x)](b-x)^2. \end{aligned}$$

Adding these inequalities we get

$$\begin{aligned} &\int_a^x (t-a)[f'(x) - f'(t)] dt + \int_x^b (b-t)[f'(t) - f'(x)] dt \\ &\leq \frac{1}{2}[f'(x) - f'(a)](x-a)^2 + \frac{1}{2}[f'(b) - f'(x)](b-x)^2 \\ &= \frac{1}{2}[f'(b)(b-x)^2 - f'(a)(x-a)^2 + f'(x)(b-a)[2x - (a+b)]] \end{aligned}$$

and by (4.2) we get the second inequality for $I_2(x)$.

Further, we use the Čebyšev inequality for asynchronous functions (functions of opposite monotonicity), namely

$$\frac{1}{d-c} \int_c^d g(t)h(t) dt \leq \frac{1}{d-c} \int_c^d g(t) dt \cdot \frac{1}{d-c} \int_c^d h(t) dt.$$

Therefore

$$\begin{aligned}
& \frac{1}{x-a} \int_a^x (t-a) [f'(x) - f'(t)] dt \\
& \leq \frac{1}{x-a} \int_a^x (t-a) dt \cdot \frac{1}{x-a} \int_a^x [f'(x) - f'(t)] dt \\
& = \frac{(x-a)^2}{2(x-a)} \cdot \frac{f'(x)(x-a) - f(x) + f(a)}{x-a} \\
& = \frac{1}{2} [f'(x)(x-a) - f(x) + f(a)]
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{b-x} \int_x^b (b-t) [f'(t) - f'(x)] dt \\
& \leq \frac{1}{b-x} \int_x^b (b-t) dt \cdot \frac{1}{b-x} \int_x^b [f'(t) - f'(x)] dt \\
& = \frac{(b-x)^2}{2(b-x)} \cdot \frac{f(b) - f(x) - f'(x)(b-x)}{b-x} \\
& = \frac{1}{2} [f(b) - f(x) - f'(x)(b-x)].
\end{aligned}$$

Adding these inequalities, we have

$$\begin{aligned}
& \frac{1}{b-a} \int_a^x (t-a) [f'(x) - f'(t)] dt + \frac{1}{b-a} \int_x^b (b-t) [f'(t) - f'(x)] dt \\
& \leq \frac{1}{2} \frac{[f'(x)(x-a) - f(x) + f(a)](x-a)}{b-a} \\
& + \frac{1}{2} \frac{[f(b) - f(x) - f'(x)(b-x)](b-x)}{b-a} \\
& = \frac{1}{2(b-a)} [[f'(x)(x-a) - f(x) + f(a)](x-a)] \\
& + \frac{1}{2(b-a)} [[f(b) - f(x) - f'(x)(b-x)](b-x)] \\
& = \frac{1}{2} \left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} - f(x) \right] + f'(x) \left(x - \frac{a+b}{2} \right)
\end{aligned}$$

which proves the inequality for $I_3(x)$. \square

Remark 3. From the first inequality in (4.1) we have

$$(4.3) \quad \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{(b-x)f(b) + (x-a)f(a)}{b-a} - f'(x) \left(\frac{a+b}{2} - x \right)$$

for any $x \in [a, b]$.

From the second inequality in (4.1) we have

$$(4.4) \quad \frac{1}{b-a} \int_a^b f(t) dt - f(x) \leq \frac{1}{2} \cdot \frac{f'(b)(b-x)^2 - f'(a)(x-a)^2}{b-a}$$

for any $x \in [a, b]$.

From the third inequality in (4.1) we have

$$(4.5) \quad \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{1}{2} \left[\frac{f(b)(b-x) + f(a)(x-a)}{b-a} + f(x) \right]$$

for any $x \in [a, b]$.

5. INEQUALITIES FOR ABSOLUTELY CONTINUOUS DERIVATIVES

We use the *Lebesgue p-norms* defined as follows:

$$\|g\|_{[c,d],p} := \left(\int_c^d |g(s)|^p dt \right)^{1/p}, \quad g \in L_p[c, d], \quad p \geq 1$$

and

$$\|g\|_{[c,d],\infty} := \operatorname{ess\,sup}_{s \in [c,d]} |g(s)|, \quad g \in L_\infty[c, d].$$

The case of absolutely continuous derivatives is as follows:

Theorem 6. *Let $f : I \rightarrow \mathbb{C}$ be a differentiable function on \dot{I} and $[a, b] \subset \dot{I}$. If the derivative f' is absolutely continuous on $[a, b]$, then for any $x \in [a, b]$*

$$(5.1) \quad \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right|$$

$$\leq \frac{1}{b-a} \times \begin{cases} \frac{1}{6} (x-a)^3 \|f''\|_{[a,x],\infty}, \\ \frac{q}{(q+1)(q+2)} (x-a)^{1/q+2} \|f''\|_{[a,x],p}, \\ \frac{1}{2} (x-a)^2 \|f''\|_{[a,x],1}, \end{cases}$$

$$+ \frac{1}{b-a} \times \begin{cases} \frac{1}{6} (b-x)^3 \|f''\|_{[x,b],\infty}, \\ \frac{q}{(q+1)(q+2)} (b-x)^{1/q+2} \|f''\|_{[x,b],p}, \\ \frac{1}{2} (b-x)^2 \|f''\|_{[x,b],1}, \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in (2.1) we have

$$(5.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt + \left(\frac{a+b}{2} - x \right) f'(x) \right|$$

$$\leq \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(x)| dt + \frac{1}{b-a} \int_x^b (b-t) |f'(t) - f'(x)| dt$$

$$= \frac{1}{b-a} \int_a^x (t-a) \left| \int_x^t f''(s) ds \right| + \frac{1}{b-a} \int_x^b (b-t) \left| \int_x^t f''(s) ds \right|$$

$$\leq \frac{1}{b-a} \int_a^x (t-a) \int_t^x |f''(s)| ds + \frac{1}{b-a} \int_x^b (b-t) \int_x^t |f''(s)| ds.$$

Using Hölder's integral inequality we have for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \int_a^x (t-a) \int_t^x |f''(s)| ds &\leq \begin{cases} \int_a^x (t-a)(x-t) \|f''\|_{[t,x],\infty} dt \\ \int_a^x (t-a)(x-t)^{1/q} \|f''\|_{[t,x],p} dt \\ \int_a^x (t-a) \|f''\|_{[t,x],1} dt \end{cases} \\ &\leq \begin{cases} \|f''\|_{[a,x],\infty} \int_a^x (t-a)(x-t) dt \\ \|f''\|_{[a,x],p} \int_a^x (t-a)(x-t)^{1/q} dt \\ \|f''\|_{[a,x],1} \int_a^x (t-a) dt \end{cases} \\ &= \begin{cases} \frac{1}{6} (x-a)^3 \|f''\|_{[a,x],\infty} \\ \frac{q}{(q+1)(q+2)} (x-a)^{1/q+2} \|f''\|_{[a,x],p} \\ \frac{1}{2} (x-a)^2 \|f''\|_{[a,x],1} \end{cases} \end{aligned}$$

and, similarly

$$\int_x^b (b-t) \int_x^t |f''(s)| ds \leq \begin{cases} \frac{1}{6} (b-x)^3 \|f''\|_{[x,b],\infty} \\ \frac{q}{(q+1)(q+2)} (b-x)^{1/q+2} \|f''\|_{[x,b],p} \\ \frac{1}{2} (b-x)^2 \|f''\|_{[x,b],1} \end{cases}$$

Utilizing the inequality (5.2) we get the desired result (5.1). \square

Remark 4. *Since*

$$\begin{aligned} &\frac{1}{6} (x-a)^3 \|f''\|_{[a,x],\infty} + \frac{1}{6} (b-x)^3 \|f''\|_{[x,b],\infty} \\ &\leq \frac{1}{6} \left[(x-a)^3 + (b-x)^3 \right] \max \left\{ \|f''\|_{[a,x],\infty}, \|f''\|_{[x,b],\infty} \right\} \\ &= \frac{1}{6} (b-a) \left[(x-a)^2 - (x-a)(b-x) + (b-x)^2 \right] \|f''\|_{[a,b],\infty}, \end{aligned}$$

then by (5.1) we get

$$\begin{aligned} (5.3) \quad &\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{1}{6} \left[\left(\frac{x-a}{b-a} \right)^2 - \left(\frac{x-a}{b-a} \right) \left(\frac{b-x}{b-a} \right) + \left(\frac{b-x}{b-a} \right)^2 \right] \\ &\quad \times (b-a)^2 \|f''\|_{[a,b],\infty}, \end{aligned}$$

for any $x \in [a, b]$.

Since

$$\begin{aligned} & (x-a)^{1/q+2} \|f''\|_{[a,x],p} + (b-x)^{1/q+2} \|f''\|_{[x,b],p} \\ & \leq \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{1/q} \left[\|f''\|_{[a,x],p}^p + \|f''\|_{[x,b],p}^p \right]^{1/p} \\ & = \left[(x-a)^{2q+1} + (b-x)^{2q+1} \right]^{1/q} \|f''\|_{[a,b],p}, \end{aligned}$$

then by (5.1) we get

$$\begin{aligned} (5.4) \quad & \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{q}{(q+1)(q+2)} \left[\left(\frac{x-a}{b-a} \right)^{2q+1} + \left(\frac{b-x}{b-a} \right)^{2q+1} \right]^{1/q} \\ & \quad \times (b-a)^{1+1/q} \|f''\|_{[a,b],p}, \end{aligned}$$

for any $x \in [a, b]$.

Since

$$\begin{aligned} & (x-a)^2 \|f''\|_{[a,x],1} + (b-x)^2 \|f''\|_{[x,b],1} \\ & \leq \max \left\{ (x-a)^2, (b-x)^2 \right\} \left[\|f''\|_{[a,x],1} + \|f''\|_{[x,b],1} \right] \\ & = \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^2 \|f''\|_{[a,b],1}, \end{aligned}$$

then by (5.1) we get

$$\begin{aligned} & \left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right]^2 (b-a) \|f''\|_{[a,b],1} \end{aligned}$$

for any $x \in [a, b]$.

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