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SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS (III)

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Abstract. In this paper, some new perturbed Ostrowski type inequalities for absolutely continuous functions are established.

1. Introduction

In order to obtain various perturbed Ostrowski type inequalities, in the earlier paper [26] we established the following equality:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt, \tag{1.1}$$

where the integrals in the right hand side are taken in the Lebesgue sense.

The following equality in terms of one parameter holds:

Corollary 1. *With the assumption in Lemma 1, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt. \tag{1.2}$$

Remark 1. *If we take $\lambda(x)=0$ in (1.2), then we get Montgomery's identity for absolutely continuous functions, namely*

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^x (t-a)f'(t)\,dt+\frac{1}{b-a}\int_x^b (t-b)f'(t)\,dt, \tag{1.3}$$

for $x\in[a,b]$.

We have the following midpoint representation:

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Corollary 2. *With the assumption in Lemma 1, we have for any λ_1, $\lambda_2 \in \mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_2-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_2\right]dt. \tag{1.4}$$

In particular, if $\lambda_1=\lambda_2=\lambda$, then we have the equality

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda\right]d. \tag{1.5}$$

The identity (1.1) has many particular cases of interest.

If $x\in(a,b)$ is a point of differentiability for the absolutely continuous function $f:[a,b]\to\mathbb{C}$, then we have the equality:

$$f(x)+\left(\frac{a+b}{2}-x\right)f'(x)-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt. \tag{1.6}$$

In particular we have

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \\ &+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \end{aligned} \tag{1.7}$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x\in(a,b)$, if we take in (1.1)

$$\lambda_1(x)=\frac{f(x)-f(a)}{x-a} \text{ and } \lambda_2(x)=\frac{f(b)-f(x)}{b-x},$$

then we get, after some elementary calculations,

$$\begin{aligned} &\frac{1}{2}\left[f(x)+\frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\frac{f(x)-f(a)}{x-a}\right]dt \\ &+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\frac{f(b)-f(x)}{b-x}\right]dt. \end{aligned} \tag{1.8}$$

In particular, we have

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(b)+f(a)}{2}\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.9}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}}\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]dt.$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (1.1) for $\lambda_1(x)=f'_+(a)$ and $\lambda_2(x)=f'_-(b)$

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.10}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'_+(a)\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'_-(b)\right]dt,$$

for all $x\in[a,b]$.

In particular, we have

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.11}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-f'_+(a)\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'_-(b)\right]dt.$$

If we take in (1.1) $\lambda_2(x)=\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$, provided this derivative exists and is finite, then we get

$$f(x)+\left(\frac{a+b}{2}-x\right)f'\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.12}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt,$$

for all $x\in[a,b]$.

In [26] we obtained the following perturbed Ostrowski type inequality:

Theorem 1. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$
\begin{aligned}
(1.13)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \\
& \left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right| \\
& \leq \frac{1}{4}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right] \\
& \leq \frac{1}{4}(b-a) \\
& \times \begin{cases}
\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\
\\
\left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\
p > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\
\\
\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f')
\end{cases}
\end{aligned}
$$

for any $x \in [a,b]$.

Another perturbed Ostrowski type inequality obtained in [27] is as follows:

Theorem 2. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$
\begin{aligned}
(1.14)\qquad & \left| f(x) + \left(\frac{a+b}{2} - x\right) f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\
& \leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_t^x (f')\,dt + \int_x^b (b-t)\bigvee_x^t (f')\,dt\right] \\
& \leq \frac{1}{2}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f')\,dt + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right]
\end{aligned}
$$

$$\leq \frac{1}{2}(b-a)$$
$$\times \begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f')+\frac{1}{2}\left|\bigvee_a^x (f')-\bigvee_x^b (f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q+\left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f'), \end{cases}$$

for any $x \in [a,b]$.

For other Ostrowski type inequalities see [1]-[19] and [23]-[46].

Motivated by the above results, we establish in this paper other perturbed Ostrowski type inequalities for complex valued differentiable functions.

2. Inequalities for Derivatives of Bounded Variation

Assume that the function $f: I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, from (1.10) we have the equality

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt, \tag{2.1}$$

for any $x \in [a,b]$.

In particular, for $x=\frac{a+b}{2}$, we have

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-f'(a)\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\left[f'(b)-f'(t)\right]dt. \tag{2.2}$$

Theorem 3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any $x \in [a,b]$*

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \tag{2.3}$$
$$\le \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\, dt + \int_x^b (b-t) \bigvee_t^b (f')\, dt \right]$$
$$\le \frac{1}{b-a} \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}} (x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f') \right)^p dt \right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f') \right) dt \end{cases}$$
$$+ \frac{1}{b-a} \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_t^b (f') \right)^p dt \right)^{1/p}, \\ (b-x) \int_x^b \left(\bigvee_t^b (f') \right) dt. \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in (2.1) we have

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \tag{2.4}$$
$$\le \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(a) \right| dt + \frac{1}{b-a} \int_x^b (b-t) \left| f'(b) - f'(t) \right| dt,$$

for any $x \in [a,b]$.

Since the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then

$$\left| f'(t) - f'(a) \right| \le \bigvee_a^t (f') \text{ for any } t \in [a,x]$$

and

$$\left| f'(b) - f'(t) \right| \le \bigvee_t^b (f') \text{ for any } t \in [x,b].$$

Therefore

$$\int_a^x (t-a) \left| f'(t) - f'(a) \right| dt \le \int_a^x (t-a) \bigvee_a^t (f')\, dt$$

and

$$\int_x^b (b-t)\left|f'(b)-f'(t)\right| dt \le \int_x^b (b-t) \bigvee_t^b (f')\, dt$$

for any $x \in [a,b]$.

Adding these two inequalities and dividing by $b-a$ we get the first inequality in (2.3).

Using Hölder's integral inequality we have

$$\int_a^x (t-a) \bigvee_a^t (f')\, dt \le \begin{cases} \bigvee_a^x (f') \int_a^x (t-a)\, dt, \\ \left(\int_a^x (t-a)^q\, dt\right)^{1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f')\right) dt, \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}} (x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f')\right) dt \end{cases}$$

and

$$\int_x^b (b-t) \bigvee_t^b (f')\, dt \le \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_x^b (f')\right)^p dt\right)^{1/p}, \\ (b-x) \int_x^b \left(\bigvee_x^b (f')\right) dt. \end{cases}$$

□

Remark 2. *From the first branch in (2.3) we have the sequence of inequalities*

$$\begin{aligned}
(2.5)\quad & \left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\
& \leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\,dt + \int_x^b (b-t) \bigvee_t^b (f')\,dt \right] \\
& \leq \frac{1}{2}(b-a) \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b (f') \right] \\
& \leq \frac{1}{2}(b-a) \\
& \times \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b (f') + \frac{1}{2} \left| \bigvee_a^x (f') - \bigvee_x^b (f') \right| \right], \\ \\ \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x (f') \right]^q + \left[\bigvee_x^b (f') \right]^q \right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f'), \end{cases}
\end{aligned}$$

for any $x \in [a, b]$.

From the second branch in (2.3) we have

$$\begin{aligned}
(2.6)\quad & \left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\
& \leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\,dt + \int_x^b (b-t) \bigvee_t^b (f')\,dt \right] \\
& \leq \frac{1}{(q+1)^{1/q}} \left\{ \left(\frac{x-a}{b-a} \right)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f') \right)^p dt \right)^{1/p} \right. \\
& \left. + \left(\frac{b-x}{b-a} \right)^{1+1/q} \left(\int_x^b \left(\bigvee_t^b (f') \right)^p dt \right)^{1/p} \right\} (b-a)^{1/q}
\end{aligned}$$

$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[\int_a^x\left(\bigvee_a^t(f')\right)^p dt+\int_x^b\left(\bigvee_t^b(f')\right)^p dt\right]^{1/p}(b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[(x-a)\left(\bigvee_a^x(f')\right)^p+(b-x)\left(\bigvee_x^b(f')\right)^p\right]^{1/p}(b-a)^{1/q}$$

for any $x\in[a,b]$ *and* $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

From the third branch in (2.3) we have

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.7}$$
$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')\,dt+\int_x^b (b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq\left(\frac{x-a}{b-a}\right)\int_a^x\left(\bigvee_a^t(f')\right)dt+\left(\frac{b-x}{b-a}\right)\int_x^b\left(\bigvee_t^b(f')\right)dt$$
$$\leq\begin{cases}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\int_a^x\left(\bigvee_a^t(f')\right)dt+\int_x^b\left(\bigvee_t^b(f')\right)dt\right]\\[2ex] \left[\left(\frac{x-a}{b-a}\right)^q+\left(\frac{b-x}{b-a}\right)^q\right]^{1/q}\\ \times\left[\left[\int_a^x\left(\bigvee_a^t(f')\right)dt\right]^p+\left[\int_x^b\left(\bigvee_t^b(f')\right)dt\right]^p\right]^{1/p}\\[2ex] \max\left\{\int_a^x\left(\bigvee_a^t(f')\right)dt,\int_x^b\left(\bigvee_t^b(f')\right)dt\right\}\end{cases}$$

for any $x\in[a,b]$ *and* $p>1,\ \frac{1}{p}+\frac{1}{q}=1.$

Remark 3. *We observe that, if we take* $x=\frac{a+b}{2}$ *in (2.5) then we get the perturbed midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.8}$$
$$\leq\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(t-a)\bigvee_a^t(f')\,dt+\int_{\frac{a+b}{2}}^b(b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq\frac{1}{8}(b-a)\bigvee_a^b(f').$$

3. Inequalities for Lipschitzian Derivatives

We start with the following result.

Theorem 4. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in (a,b)$. If $\alpha_i > -1$ and $L_{\alpha_i} > 0$ with $i = 1,2$ are such that*

$$|f'(t) - f'(a)| \le L_{\alpha_1} (t-a)^{\alpha_1} \text{ for any } t \in [a,x) \tag{3.1}$$

and

$$|f'(b) - f'(t)| \le L_{\alpha_2} (b-t)^{\alpha_2} \text{ for any } t \in (x,b], \tag{3.2}$$

then we have

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{\alpha_1 + 2} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2} (b-x)^{\alpha_2+2} \right]. \tag{3.3}$$

Proof. Using the conditions (3.1) and (3.2) we have

$$\begin{aligned} &\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \\ &\le \frac{1}{b-a} \int_a^x (t-a) |f'(t) - f'(a)|\,dt + \frac{1}{b-a} \int_x^b (b-t) |f'(b) - f'(t)|\,dt \\ &\le \frac{1}{b-a} L_{\alpha_1} \int_a^x (t-a)^{\alpha_1+1} dt + \frac{1}{b-a} L_{\alpha_2} \int_x^b (b-t)^{\alpha_2+1} dt \\ &= \frac{1}{b-a} L_{\alpha_1} \frac{(x-a)^{\alpha_1+2}}{\alpha_1+2} + \frac{1}{b-a} L_{\alpha_2} \frac{(b-x)^{\alpha_2+2}}{\alpha_2+2} \\ &= \frac{1}{b-a} \left[\frac{L_{\alpha_1}}{\alpha_1+2} (x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2} (b-x)^{\alpha_2+2} \right] \end{aligned}$$

and the inequality (3.3) is obtained. □

Corollary 3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$|f'(t) - f'(s)| \le H |t-s|^r$$

for any $t,s \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given, then

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \le \frac{H}{r+2} \left[\left(\frac{x-a}{b-a} \right)^{r+2} + \left(\frac{b-x}{b-a} \right)^{r+2} \right] (b-a)^{r+1}, \tag{3.4}$$

for any $x \in [a,b]$.

In particular, if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{3}L\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right](b-a)^2, \tag{3.5}$$

for any $x \in [a,b]$.

Remark 4. *With the assumptions of Corollary 3 we have the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{H}{2^{r+1}(r+2)}(b-a)^{r+1}. \tag{3.6}$$

If if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)\left[f'(b)-f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}L(b-a)^2. \tag{3.7}$$

4. Inequalities for Differentiable Functions with the Property (S)

Let $f : I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Then f' is monotonic nondecreasing and by the equality (2.1) we have

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \ge 0 \tag{4.1}$$

or, equivalently

$$\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right] \ge \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) \tag{4.2}$$

for any $x \in [a,b]$.

We observe that the inequalities (4.1) and (4.2) remain valid for the larger class of differentiable functions f that that satisfy the *property* (S) on the interval $[a,b]$, namely

$$f'(a) \le f'(t) \le f'(b) \tag{S}$$

for any $t \in [a,b]$.

Theorem 5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$.*

(i) Let $x \in [a,b]$. If f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$f'(x)\left(\frac{a+b}{2}-x\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{4.3}$$

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x\in[a,b]$

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]. \tag{4.4}$$

Proof. (i) Since f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
&f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\\
&=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt\\
&\leq\frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(x)\right]dt\\
&=\frac{f'(x)-f'(a)}{b-a}\int_a^x (t-a)\,dt+\frac{f'(b)-f'(x)}{b-a}\int_x^b (b-t)\,dt\\
&=\frac{f'(x)-f'(a)}{b-a}\cdot\frac{(x-a)^2}{2}+\frac{f'(b)-f'(x)}{b-a}\cdot\frac{(b-x)^2}{2}\\
&=\frac{1}{2(b-a)}\left[(f'(x)-f'(a))(x-a)^2+(f'(b)-f'(x))(b-x)^2\right]\\
&=\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-f'(x)\left(\frac{a+b}{2}-x\right),
\end{aligned}$$

which proves the inequality (4.3).

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x\in[a,b]$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt+\frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt\\
&\leq\frac{x-a}{b-a}\int_a^x \left[f'(t)-f'(a)\right]dt+\frac{b-x}{b-a}\int_x^b \left[f'(b)-f'(t)\right]dt\\
&=\frac{1}{b-a}(x-a)\left[f(x)-f(a)-f'(a)(x-a)\right]\\
&+\frac{1}{b-a}(b-x)\left[f'(b)(b-x)-f(b)+f(x)\right]\\
&=\frac{1}{b-a}\left[f(x)(x-a)-f(a)(x-a)-f'(a)(x-a)^2\right]\\
&+\frac{1}{b-a}\left[f'(b)(b-x)^2-f(b)(b-x)+f(x)(b-x)\right]\\
&=\frac{1}{b-a}\left\{f'(b)(b-x)^2-f'(a)(x-a)^2-f(a)(x-a)-f(b)(b-x)\right.\\
&\left.+f(x)(b-a)\right\}\\
&=\frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a}+f(x)-\frac{f(a)(x-a)+f(b)(b-x)}{b-a},
\end{aligned}$$

which proves the inequality (4.4). □

Remark 5. *The inequality (4.3) was obtained for the case of convex functions in* [20] *while (4.4) was established for convex functions in* [21] *with different proofs.*

Further, we use the Čebyšev inequality for synchronous functions (functions with same monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \geq \frac{1}{d-c}\int_c^d g(t)\,dt \cdot \frac{1}{d-c}\int_c^d h(t)\,dt. \tag{4.5}$$

Theorem 6. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in [a,b]$. If f is convex on the interval $[a,x]$ and $[x,b]$, then*

$$\frac{1}{2}\left[f(x) + \frac{f(a)(x-a) + f(b)(b-x)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{4.6}$$

Proof. We have

$$\begin{aligned} &f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \end{aligned} \tag{4.7}$$

for any $x \in [a,b]$.

Since f' is monotonic nondecreasing on $[a,x]$, then by Čebyšev inequality (3.4) we have

$$\begin{aligned} \int_a^x (t-a)\left[f'(t) - f'(a)\right]dt &\geq \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \int_a^x \left[f'(t) - f'(a)\right]dt \\ &= \frac{1}{2}(x-a)\left[f(x) - f(a) - f'(a)(x-a)\right] \\ &= \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \end{aligned}$$

and, by the same inequality,

$$\begin{aligned} \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt &\geq \frac{1}{b-x}\int_x^b (b-t)\,dt \cdot \int_x^b \left[f'(b) - f'(t)\right]dt \\ &= \frac{1}{2}(b-x)\left[f'(b)(b-x) - f(b) + f(x)\right] \\ &= \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right]. \end{aligned}$$

If we add these two inequalities, then we get

$$\begin{aligned} &\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \\ \geq\ & \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \\ &+ \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right] \\ =\ & \frac{1}{2}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] + \frac{1}{2}f(x)(b-a) \\ &- \frac{1}{2}\left[f(a)(x-a) + f(b)(b-x)\right]. \end{aligned}$$

Dividing by $b-a$ and utilizing the equality (4.7) we deduce the inequality (4.6). □

Remark 6. *If the function is convex on the whole interval $[a,b]$, then, the inequality (4.6) is true for any $x \in [a,b]$.*

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