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PERTURBED COMPANIONS OF OSTROWSKI'S INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

S. S. DRAGOMIR[1,2]

Abstract. A perturbed companion of Ostrowski's inequality for functions of bounded variation and applications are given.

1. Introduction

In [16], the author obtained the following companion of Ostrowski's inequality [29]:

Theorem 1. *Assume that the function $f:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.1}$$

$$\leq\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f), \\[2ex] \left[2\left(\frac{x-a}{b-a}\right)^\alpha+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^\alpha\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_a^x(f)\right]^\beta+\left[\bigvee_x^{a+b-x}(f)\right]^\beta+\left[\bigvee_{a+b-x}^b(f)\right]^\beta\right]^{\frac{1}{\beta}}, & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\[2ex] \frac{x+\frac{b-3a}{2}}{b-a}\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$. The constant $\frac{1}{4}$ is best possible in the first branch of the second inequality in (1.1).

The following trapezoid inequality holds.

Corollary 1. *With the assumptions in Theorem 1, one has the trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f). \tag{1.2}$$

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The constant $\frac{1}{2}$ *is best possible in (1.2).*

The inequality (1.2) was first proved in a different manner in [9].

The following midpoint inequality also holds.

Corollary 2. *With the assumptions in Theorem 1, one has the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2}\bigvee_a^b(f). \tag{1.3}$$

The constant $\frac{1}{2}$ *is best possible in (1.3).*

The inequality (1.3) was firstly proved in a different manner in [10].

The best inequality we may get from Theorem 1 on using the bound provided by the first branch in the second inequality in (1.1) is incorporated in the following corollary.

Corollary 3. *With the assumptions in Theorem 1, one has the inequality:*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}\bigvee_a^b(f). \tag{1.4}$$

The constant $\frac{1}{4}$ *is best possible.*

For a monograph devoted to Ostrowski type inequalities, see [22].

For research papers on Ostrowski's inequality see [1]-[21], [23]-[25] and [27].

The main aim of this paper is to provide some bounds for various perturbations of the difference

$$\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt,$$

where f is assumed to be a function of bounded variation on $[a,b]$. Particular instances of interest are also given.

2. Some Identities

The following identity holds.

Lemma 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$. *Then we have the equality*

$$\begin{aligned}
&\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \\
&=\frac{1}{b-a}\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)t\right] \\
&+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d\left[f(t)-\lambda_2(x)t\right] \\
&+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,d\left[f(t)-\lambda_3(x)t\right],
\end{aligned} \tag{2.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2,3$ *complex numbers.*

Proof. Using the integration by parts formula for Riemann-Stieltjes integrals, we have

$$\begin{aligned}&\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)\,t\right]\\&=\int_a^x (t-a)\,df(t)-\lambda_1(x)\int_a^x (t-a)\,dt\\&=(x-a)\,f(x)-\int_a^x f(t)\,dt-\frac{1}{2}\lambda_1(x)\,(x-a)^2,\end{aligned}$$

$$\begin{aligned}&\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d\left[f(t)-\lambda_2(x)\,t\right]\\&=\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)df(t)-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt\\&=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)\\&-\int_x^{a+b-x} f(t)\,dt-\lambda_2(x)\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt\\&=f(a+b-x)\left(\frac{a+b}{2}-x\right)-f(x)\left(x-\frac{a+b}{2}\right)-\int_x^{a+b-x} f(t)\,dt\end{aligned}$$

since, by symmetry

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)dt=0$$

and

$$\begin{aligned}&\int_{a+b-x}^b (t-b)\,d\left[f(t)-\lambda_3(x)\,t\right]\\&=\int_{a+b-x}^b (t-b)\,df(t)-\lambda_3(x)\int_{a+b-x}^b (t-b)\,dt\\&=(x-a)\,f(a+b-x)-\int_{a+b-x}^b f(t)\,dt+\frac{1}{2}\lambda_3(x)\,(x-a)^2.\end{aligned}$$

Summing the above equalities, we deduce

$$\begin{aligned}&\int_a^x (t-a)\,d\left[f(t)-\lambda_1(x)\,t\right]+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d\left[f(t)-\lambda_2(x)\,t\right]\\&+\int_{a+b-x}^b (t-b)\,d\left[f(t)-\lambda_3(x)\,t\right]\\&=(b-a)\,\frac{f(x)+f(a+b-x)}{2}-\int_a^b f(t)\,dt+\frac{1}{2}\left[\lambda_3(x)-\lambda_1(x)\right](x-a)^2,\end{aligned}$$

which is equivalent with the desired identity (2.1). □

The following particular cases are of interest:

Corollary 4. *With the assumption of Lemma 1 we have the equalities*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)d\left[f(t)-\lambda_2 t\right], \tag{2.2}$$

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\,d[f(t)-\lambda_1 t]+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t)-\lambda_3 t], \tag{2.3}$$

and

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\,d[f(t)-\lambda_1 t]$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2 t]$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\,d[f(t)-\lambda_3 t], \tag{2.4}$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

Corollary 5. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$. *Then we have the equality*

$$\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t]$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2(x)t]$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\,d[f(t)-\lambda_1(x)t], \tag{2.5}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2$ *complex numbers.*

Remark 1. *We get from (2.3) the following particular case:*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\,d[f(t)-\lambda_1 t]+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\,d[f(t)-\lambda_1 t], \tag{2.6}$$

for any $\lambda_1 \in \mathbb{C}$*, while from (2.4) we get*

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.7}$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\,d\left[f(t)-\lambda_1 t\right]$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)d\left[f(t)-\lambda_2 t\right]$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(t-b)\,d\left[f(t)-\lambda_1 t\right],$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

3. Inequalities for Functions of Bounded Variation

The following lemma will be used in the sequel and is of interest in itself as well [2, p. 177]. For a simple proof see [18].

Lemma 2. *Let* $f,\ u : [a,b] \to \mathbb{C}$*. If* f *is continuous on* $[a,b]$ *and* u *is of bounded variation on* $[a,b]$*, then the Riemann-Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]}|f(t)|\bigvee_a^b (u). \tag{3.1}$$

We denote by $\ell : [a,b] \to [a,b]$ the *identity function*, namely $\ell(t) = t$ for any $t \in [a,b]$.

We have the following result:

Theorem 2. *Let* $f : [a,b] \to \mathbb{C}$ *be a function of bounded variation on* $[a,b]$*. Then we have the inequalities*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.2}$$
$$\le \frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x\left(f-\lambda_1(x)\ell\right)\right)dt+\int_x^{\frac{a+b}{2}}\left(\bigvee_x^t\left(f-\lambda_2(x)\ell\right)\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_t^{a+b-x}\left(f-\lambda_2(x)\ell\right)\right)dt+\int_{a+b-x}^b\left(\bigvee_{a+b-x}^t\left(f-\lambda_3(x)\ell\right)\right)dt\right]$$
$$\le \frac{1}{b-a}\left[(x-a)\bigvee_a^x\left(f-\lambda_1(x)\ell\right)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}\left(f-\lambda_2(x)\ell\right)\right.$$
$$\left.+(x-a)\bigvee_{a+b-x}^b\left(f-\lambda_3(x)\ell\right)\right]$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left|\frac{x - \frac{3a+b}{4}}{b-a}\right|\right] \\ \times \left[\bigvee_a^x (f - \lambda_1(x)\ell) + \bigvee_x^{a+b-x} (f - \lambda_2(x)\ell) + \bigvee_{a+b-x}^b (f - \lambda_3(x)\ell)\right] \\ \\ \frac{x + \frac{b-3a}{2}}{b-a} \\ \times \max\left\{\bigvee_a^x (f - \lambda_1(x)\ell), \bigvee_x^{a+b-x} (f - \lambda_2(x)\ell), \bigvee_{a+b-x}^b (f - \lambda_3(x)\ell)\right\} \end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ *and* $\lambda_i(x), i = 1, 2, 3$ *complex numbers.*

Proof. Taking the modulus on (2.1) and making use of (3.1), we have

$$\left|\frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{\lambda_3(x) - \lambda_1(x)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.3}$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\,d\left(\bigvee_a^t (f - \lambda_1(x)\ell)\right)$$
$$+ \frac{1}{b-a}\int_x^{a+b-x} \left|t - \frac{a+b}{2}\right| d\left(\bigvee_a^t (f - \lambda_2(x)\ell)\right)$$
$$+ \frac{1}{b-a}\int_{a+b-x}^b (b-t)\,d\left(\bigvee_a^t (f - \lambda_3(x)\ell)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (t-a)\,d\left(\bigvee_a^t (f - \lambda_1(x)\ell)\right) \\
&= (t-a)\bigvee_a^t (f - \lambda_1(x)\ell)\Big|_a^x - \int_a^x \bigvee_a^t (f - \lambda_1(x)\ell)\,dt \\
&= (x-a)\bigvee_a^x (f - \lambda_1(x)\ell) - \int_a^x \bigvee_a^t (f - \lambda_1(x)\ell)\,dt \\
&= \int_a^x \left(\bigvee_t^x (f - \lambda_1(x)\ell)\right) dt.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right| d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&=\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right) d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&+\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right) d\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\\
&=\left(\frac{a+b}{2}-t\right)\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\Bigg|_x^{\frac{a+b}{2}}+\int_x^{\frac{a+b}{2}}\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt\\
&+\left(t-\frac{a+b}{2}\right)\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)\Bigg|_{\frac{a+b}{2}}^{a+b-x}-\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt\\
&=\int_x^{\frac{a+b}{2}}\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt-\left(\frac{a+b}{2}-x\right)\left(\bigvee_a^x (f-\lambda_2(x)\ell)\right)\\
&+\left(\frac{a+b}{2}-x\right)\left(\bigvee_a^{a+b-x} (f-\lambda_2(x)\ell)\right)-\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_a^t (f-\lambda_2(x)\ell)\right)dt\\
&=\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_t^{a+b-x} (f-\lambda_2(x)\ell)\right)dt+\int_x^{\frac{a+b}{2}}\left(\bigvee_x^t (f-\lambda_2(x)\ell)\right)dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{a+b-x}^b (b-t)\, d\left(\bigvee_a^t (f-\lambda_3(x)\ell)\right)\\
&=(b-t)\left(\bigvee_a^t (f-\lambda_3(x)\ell)\right)\Bigg|_{a+b-x}^b+\int_{a+b-x}^b\left(\bigvee_a^t (f-\lambda_3(x)\ell)\right)dt\\
&=\int_{a+b-x}^b\left(\bigvee_a^t (f-\lambda_3(x)\ell)\right)dt-(b-(a+b-x))\left(\bigvee_a^{a+b-x} (f-\lambda_3(x)\ell)\right)\\
&=\int_{a+b-x}^b\left(\bigvee_{a+b-x}^t (f-\lambda_3(x)\ell)\right)dt.
\end{aligned}$$

Making use of (3.3) we deduce the first inequality in (3.2).

Since

$$\int_a^x\left(\bigvee_t^x (f-\lambda_1(x)\ell)\right)dt\le (x-a)\bigvee_a^x (f-\lambda_1(x)\ell),$$

$$\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_{t}^{a+b-x}(f-\lambda_2(x)\ell)\right)dt+\int_{x}^{\frac{a+b}{2}}\left(\bigvee_{x}^{t}(f-\lambda_2(x)\ell)\right)dt$$
$$\leq\left(\frac{a+b}{2}-x\right)\bigvee_{\frac{a+b}{2}}^{a+b-x}(f-\lambda_2(x)\ell)+\left(\frac{a+b}{2}-x\right)\bigvee_{x}^{\frac{a+b}{2}}(f-\lambda_2(x)\ell)$$
$$=\left(\frac{a+b}{2}-x\right)\bigvee_{x}^{a+b-x}(f-\lambda_2(x)\ell)$$

and

$$\int_{a+b-x}^{b}\left(\bigvee_{a+b-x}^{t}(f-\lambda_3(x)\ell)\right)dt\leq(x-a)\bigvee_{a+b-x}^{b}(f-\lambda_3(x)\ell),$$

the second inequality is also proved.

The last inequality is obvious by the maximum properties. □

The following midpoint and trapezoid type inequalities hold:

Corollary 6. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have the inequalities*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.4}$$
$$\leq\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f-\lambda_2\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f-\lambda_2\ell)\right)dt\right]$$
$$\leq\frac{1}{2}\bigvee_a^b(f-\lambda_2\ell),$$

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{3.5}$$
$$\leq\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda_1\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda_3\ell)\right)dt\right]$$
$$\leq\frac{1}{2}\left[\bigvee_a^{\frac{a+b}{2}}(f-\lambda_1\ell)+\bigvee_{\frac{a+b}{2}}^b(f-\lambda_3\ell)\right]$$

and

$$\begin{aligned}
(3.6)\quad & \left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
& \le \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda_1\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^{t}(f-\lambda_2\ell)\right)dt\right. \\
& \left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda_2\ell)\right)dt+\int_{\frac{a+3b}{4}}^{b}\left(\bigvee_{\frac{a+3b}{4}}^{t}(f-\lambda_3\ell)\right)dt\right] \\
& \le \frac{1}{4}\left[\bigvee_a^{\frac{3a+b}{4}}(f-\lambda_1\ell)+\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f-\lambda_2\ell)+\bigvee_{\frac{a+3b}{4}}^{b}(f-\lambda_3\ell)\right]
\end{aligned}$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

Corollary 7. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$. *Then we have the inequalities*

$$\begin{aligned}
(3.7)\quad & \left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
& \le \frac{1}{b-a}\left[\int_a^x\left(\bigvee_t^x(f-\lambda_1(x)\ell)\right)dt+\int_x^{\frac{a+b}{2}}\left(\bigvee_x^t(f-\lambda_2(x)\ell)\right)dt\right. \\
& \left.+\int_{\frac{a+b}{2}}^{a+b-x}\left(\bigvee_t^{a+b-x}(f-\lambda_2(x)\ell)\right)dt+\int_{a+b-x}^{b}\left(\bigvee_{a+b-x}^{t}(f-\lambda_1(x)\ell)\right)dt\right] \\
& \le \frac{1}{b-a}\left[(x-a)\bigvee_a^x(f-\lambda_1(x)\ell)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell)\right. \\
& \left.+(x-a)\bigvee_{a+b-x}^{b}(f-\lambda_1(x)\ell)\right]
\end{aligned}$$

$$\le\begin{cases}
\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right] \\
\times\left[\bigvee_a^x(f-\lambda_1(x)\ell)+\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell)+\bigvee_{a+b-x}^b(f-\lambda_1(x)\ell)\right] \\
\\
\frac{x+\frac{b-3a}{2}}{b-a} \\
\times\max\left\{\bigvee_a^x(f-\lambda_1(x)\ell),\bigvee_x^{a+b-x}(f-\lambda_2(x)\ell),\bigvee_{a+b-x}^b(f-\lambda_1(x)\ell)\right\}
\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2,$ *complex numbers.*

Remark 2. *We have the particular inequalities of interest*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f-\lambda_1\ell)\right)dt+\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f-\lambda_1\ell)\right)dt\right] \\
&\le \frac{1}{2}\bigvee_a^b(f-\lambda_1\ell)
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda_1\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^t(f-\lambda_2\ell)\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda_2\ell)\right)dt+\int_{\frac{a+3b}{4}}^b\left(\bigvee_{\frac{a+3b}{4}}^t(f-\lambda_1\ell)\right)dt\right] \\
&\le \frac{1}{4}\left[\bigvee_a^{\frac{3a+b}{4}}(f-\lambda_1\ell)+\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f-\lambda_2\ell)+\bigvee_{\frac{a+3b}{4}}^b(f-\lambda_1\ell)\right]
\end{aligned} \tag{3.9}$$

for any $\lambda_1,\lambda_2\in\mathbb{C}$.

If we take $\lambda_1=\lambda_2=\lambda$*, then we get*

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{b-a}\left[\int_a^{\frac{3a+b}{4}}\left(\bigvee_t^{\frac{3a+b}{4}}(f-\lambda\ell)\right)dt+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left(\bigvee_{\frac{3a+b}{4}}^t(f-\lambda\ell)\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}\left(\bigvee_t^{\frac{a+3b}{4}}(f-\lambda\ell)\right)dt+\int_{\frac{a+3b}{4}}^b\left(\bigvee_{\frac{a+3b}{4}}^t(f-\lambda\ell)\right)dt\right] \\
&\le \frac{1}{4}\left[\bigvee_a^b(f-\lambda\ell)\right]
\end{aligned} \tag{3.10}$$

for any $\lambda_1,\lambda_2\in\mathbb{C}$.

From (3.8) we deduce the simpler inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{b-a}\int_a^b \left| \bigvee_t^{\frac{a+b}{2}} (f-\lambda\ell) \right| dt \le \frac{1}{2}\bigvee_a^b (f-\lambda\ell) \tag{3.11}$$

while from (3.10) we get

$$\begin{aligned} &\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \left| \bigvee_t^{\frac{3a+b}{4}} (f-\lambda\ell) \right| dt + \int_{\frac{a+b}{2}}^b \left(\left| \bigvee_t^{\frac{a+3b}{4}} (f-\lambda\ell) \right| \right) dt \right] \\ &\le \frac{1}{4}\bigvee_a^b (f-\lambda\ell) \end{aligned} \tag{3.12}$$

for any $\lambda \in \mathbb{C}$*.*

We can state the following result.

Proposition 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$*. If there exists the constants* $\gamma,\Gamma\in\mathbb{C}$ *such that*

$$\bigvee_a^b \left(f - \frac{\gamma+\Gamma}{2}\ell \right) \le \frac{1}{2}|\Gamma-\gamma|,$$

then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4}|\Gamma-\gamma|$$

and

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}|\Gamma-\gamma|.$$

The inequalities follow by (3.11) and (3.12).

Proposition 2. *Assume that* $f:[a,b]\to\mathbb{C}$ *is a function of bounded variation on* $[a,b]$*. If for some* $\lambda\in\mathbb{C}$ *the cumulative variation function* $V_\lambda:[a,b]\to[0,\infty)$*,*

$$V_\lambda(t) := \bigvee_a^t (f-\lambda\ell)$$

is Lipschitzian with the constant $L_\gamma>0$*, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{4}L_\gamma(b-a) \tag{3.13}$$

and

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{8}L_\gamma(b-a). \tag{3.14}$$

Proof. From (3.11) we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\int_a^b\left|\bigvee_t^{\frac{a+b}{2}}(f-\lambda\ell)\right|dt=\frac{1}{b-a}\int_a^b\left|V_\lambda\left(\frac{a+b}{2}\right)-V_\lambda(t)\right|dt \\
&\leq \frac{L_\gamma}{b-a}\int_a^b\left|\frac{a+b}{2}-t\right|dt=\frac{1}{4}L_\gamma(b-a)
\end{aligned}$$

and the inequality (3.13) is proved.

From (3.12) we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
&\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left|\bigvee_t^{\frac{3a+b}{4}}(f-\lambda\ell)\right|dt+\int_{\frac{a+b}{2}}^b\left(\left|\bigvee_t^{\frac{a+3b}{4}}(f-\lambda\ell)\right|\right)dt\right] \\
&=\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\left|V_\lambda\left(\frac{3a+b}{4}\right)-V_\lambda(t)\right|dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^b\left(\left|V_\lambda\left(\frac{a+3b}{4}\right)-V_\lambda(t)\right|\right)dt\right] \\
&\leq \frac{L_\gamma}{b-a}\left[\int_a^{\frac{a+b}{2}}\left|\frac{3a+b}{4}-t\right|dt+\int_{\frac{a+b}{2}}^b\left(\left|\frac{a+3b}{4}-t\right|\right)dt\right] \\
&=\frac{L_\gamma}{b-a}\left[\int_a^{\frac{a+b}{2}}\left|\frac{3a+b}{4}-t\right|dt+\int_{\frac{a+b}{2}}^b\left(\left|\frac{a+3b}{4}-t\right|\right)dt\right] \\
&=\frac{L_\gamma}{b-a}\left[\frac{1}{16}(b-a)^2+\frac{1}{16}(b-a)^2\right]=\frac{L_\gamma}{8}(b-a)
\end{aligned}$$

and the inequality (3.14) is proved. □

4. Inequalities for Lipschitzian Functions

We say that a function $f:[c,d]\to\mathbb{C}$ is *Lipschitzian* with a constant $L>0$ on the interval $[c,d]$ if

$$|f(t)-f(s)|\leq L|t-s|$$

for any $t,s\in[c,d]$.

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$. For $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_i(x),i=1,2,3$ complex numbers, assume that $f-\lambda_1(x)\ell$ is Lipschitzian with the constant $L_1(x)>0$ on $[a,x]$, $f-\lambda_2(x)\ell$ with the constant $L_2(x)>0$ on*

$[x, a+b-x]$ *and* $f-\lambda_3(x)\ell$ *with the constant* $L_3(x)>0$ *on* $[a+b-x, b]$, *then*

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4.1}$$
$$\leq \frac{1}{(b-a)}\left[\frac{1}{2}(x-a)^2 L_1(x)+\left(x-\frac{a+b}{2}\right)^2 L_2(x)+\frac{1}{2}(x-a)^2 L_3(x)\right]$$
$$\leq \left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\max\{L_1(x), L_2(x), L_3(x)\}.$$

Proof. It is known that if $g:[c,d]\to\mathbb{C}$ is Riemann integrable on $[c,d]$ and $u:[c,d]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and we have the inequality

$$\left|\int_a^b f(t)\,du(t)\right|\leq L\int_a^b |f(t)|\,dt. \tag{4.2}$$

Taking the modulus in (2.1) and using the property (4.2) we have

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]+\frac{1}{2}(x-a)^2\frac{\lambda_3(x)-\lambda_1(x)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\leq \frac{1}{b-a}\left|\int_a^x (t-a)\,d[f(t)-\lambda_1(x)t]\right|$$
$$+\frac{1}{b-a}\left|\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)d[f(t)-\lambda_2(x)t]\right|$$
$$+\frac{1}{b-a}\left|\int_{a+b-x}^b (t-b)\,d[f(t)-\lambda_3(x)t]\right|$$
$$\leq \frac{1}{b-a}L_1(x)\int_a^x (t-a)\,dt+\frac{1}{b-a}L_2(x)\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|dt$$
$$+\frac{1}{b-a}L_3(x)\int_{a+b-x}^b (b-t)\,dt$$
$$=\frac{1}{(b-a)}\left[\frac{1}{2}(x-a)^2 L_1(x)+\left(x-\frac{a+b}{2}\right)^2 L_2(x)+\frac{1}{2}(x-a)^2 L_3(x)\right],$$

which proves the first inequality in (4.1).

Since

$$\frac{1}{2}(x-a)^2 L_1(x)+\left(x-\frac{a+b}{2}\right)^2 L_2(x)+\frac{1}{2}(x-a)^2 L_3(x)$$
$$\leq\left[\frac{1}{2}(x-a)^2+\left(x-\frac{a+b}{2}\right)^2+\frac{1}{2}(x-a)^2\right]\max\{L_1(x), L_2(x), L_3(x)\}$$
$$=\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\max\{L_1(x), L_2(x), L_3(x)\},$$

the last part of (4.1) is also proved. □

Corollary 8. *Let $f:[a,b]\to\mathbb{C}$ be a bounded function on $[a,b]$.*

(i) If for$\lambda_2\in\mathbb{C}$ the function $f-\lambda_2\ell$ is Lipschitzian with the constant $L_2>0$ on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}(b-a)L_2. \tag{4.3}$$

(ii) If for $\lambda_1,\lambda_2\in\mathbb{C}$ the function $f-\lambda_1\ell$ is Lipschitzian with the constant $L_1>0$ on $\left[a,\frac{a+b}{2}\right]$ and $f-\lambda_3\ell$ is Lipschitzian with the constant $L_3>0$ on $\left[\frac{a+b}{2},b\right]$, then

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{4}\left(\frac{L_1+L_3}{2}\right)(b-a). \tag{4.4}$$

(iii) If for $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$ the function $f-\lambda_1\ell$ is Lipschitzian with the constant $L_1>0$ on $\left[a,\frac{3a+b}{4}\right]$, $f-\lambda_2\ell$ is Lipschitzian with the constant $L_2>0$ on $\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]$ and $f-\lambda_3\ell$ is Lipschitzian with the constant $L_3>0$ on $\left[\frac{a+3b}{4},b\right]$, then

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{16}\left(\frac{1}{2}L_1+L_2+\frac{1}{2}L_3\right)(b-a). \tag{4.5}$$

Remark 3. *We have the following particular cases of interest.*

If for some $\lambda\in\mathbb{C}$ the function $f-\lambda\ell$ is Lipschitzian with the constant $L_\lambda>0$ on $[a,b]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{4}L_\lambda(b-a) \tag{4.6}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{8}L_\lambda(b-a). \tag{4.7}$$

The following lemma may be stated:

Lemma 3. *Let $u:[a,b]\to\mathbb{R}$ and $l,L\in\mathbb{R}$ with $L>l$. The following statements are equivalent:*

(i) *The function $u-\frac{l+L}{2}\cdot e$, where $e(t)=t$, $t\in[a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzian;*

(ii) *We have the inequalities*

$$l\le\frac{u(t)-u(s)}{t-s}\le L\quad\text{for each }\; t,s\in[a,b]\quad\text{with }\; t\ne s; \tag{4.8}$$

(iii) *We have the inequalities*

$$l(t-s)\le u(t)-u(s)\le L(t-s)\quad\text{for each }\; t,s\in[a,b]\quad\text{with }\; t>s. \tag{4.9}$$

Following [28], we can introduce the definition of (l,L)-Lipschitzian functions:

Definition 1. *The function $u:[a,b]\to\mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 3 is said to be (l,L)-Lipschitzian on $[a,b]$.*

If $L>0$ and $l=-L$, then $(-L,L)$ −Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l,L)-Lipschitzian functions.

Proposition 3. *Let $u:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty<l=\inf_{t\in(a,b)}u'(t)$ and $\sup_{t\in(a,b)}u'(t)=L<\infty$, then u is (l,L)-Lipschitzian on $[a,b]$.*

As consequences of the inequalities (4.6) and (4.7) for real valued functions we can state the following result.

Proposition 4. *Let $l,L\in\mathbb{R}$ with $L>l$ and $f:[a,b]\to\mathbb{R}$ an (l,L)-Lipschitzian function on $[a,b]$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(L-l)(b-a) \tag{4.10}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{16}(L-l)(b-a). \tag{4.11}$$

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975

[3] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[4] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[5] P. Cerone, S. S. Dragomir, and C. E. M. Pearce, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[6] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[7] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[8] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[9] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60**(1) (1999), 495-508.

[10] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.,* **22** (2000), 13-18.

[11] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[12] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[13] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[14] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[15] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[16] S. S. Dragomir, A companion of Ostrowski's inequality for functions of bounded variation and applications, Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suppl. Art. 28. [Online http://rgmia.org/papers/v5e/COIFBVApp.pdf].

[17] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[18] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[19] S. S. Dragomir, Some perturbed Ostrowski type inequalities for functions of bounded variation, Preprint *RGMIA Res. Rep. Coll.* **16** (2013), Art. 93. [http://rgmia.org/papers/v16/v16a93.pdf].

[20] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[21] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[22] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrechy/Boston/London, 2002.

[23] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[24] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[25] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[26] A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.,* **115** (2002), 260-288.

[27] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.

[28] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.

[29] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa