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SYMMETRIZED CONVEXITY AND HERMITE-HADAMARD TYPE INEQUALITIES

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Abstract. In this paper we extend the Hermite-Hadamard inequality to the class of symmetrized convex functions. The corresponding version for h-convex functions is also investigated. Some examples of interest are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad a,b\in\mathbb{R},\ a\neq b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [42]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [42]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [10]-[19], [21]-[24], [31]-[34] and [45].

In this paper we show that the Hermite-Hadamard inequality can be extended to a larger class of functions containing the convex functions. The corresponding version for h-convex functions is also investigated. Some examples of interest are provided as well.

2. Symmetrized Convexity

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, which is defined by

$$\breve{f}(t) := \frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b].$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t) := \frac{1}{2}\left[f(t)-f(a+b-t)\right], t\in[a,b].$$

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It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1, t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2) &= \frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\ &= \frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\ &\leq \frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\ &= \frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\ &= \alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

for any $t\in\mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t)=3(a+b)$, $t\in\mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2}>0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2}<0$. Therefore if $a<0<b$ with $\frac{a+b}{2}>0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity.

Definition 1. *We say that the function $f:[a,b]\to\mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if $\breve{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b]\subsetneq SCon[a,b]. \tag{2.1}$$

Also, if $[c,d]\subset[a,b]$ and $f\in SCon[a,b]$, then this does not imply in general that $f\in SCon[c,d]$.

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{2.2}$$

Proof. Since $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then by writing the Hermite-Hadamard inequality for the function $\breve{f}$ we have

$$\breve{f}\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b \breve{f}(t)\,dt\leq\frac{\breve{f}(a)+\breve{f}(b)}{2}. \tag{2.3}$$

However

$$\breve{f}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right),\ \frac{\breve{f}(a)+\breve{f}(b)}{2}=\frac{f(a)+f(b)}{2},$$

and

$$\int_a^b \breve{f}(t)\,dt = \frac{1}{2}\int_a^b [f(t) + f(a+b-t)]\,dt = \int_a^b f(t)\,dt.$$

Then by (2.3) we get (2.2). □

The following result holds:

Theorem 2. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}[f(x) + f(a+b-x)] \le \frac{f(a)+f(b)}{2}. \tag{2.4}$$

Proof. Since $\breve{f}$ is convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$\frac{\breve{f}(x) + \breve{f}(a+b-x)}{2} \ge \breve{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\breve{f}(x) + \breve{f}(a+b-x)}{2} = \frac{1}{2}[f(x) + f(a+b-x)]$$

while

$$\breve{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of $\breve{f}$ we have for any $x \in [a,b]$ that

$$\begin{aligned}\breve{f}(x) &\le \frac{x-a}{b-a}\cdot \breve{f}(b) + \frac{b-x}{b-a}\cdot \breve{f}(a)\\ &= \frac{x-a}{b-a}\cdot\frac{f(a)+f(b)}{2} + \frac{b-x}{b-a}\cdot\frac{f(a)+f(b)}{2}\\ &= \frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the second part of (2.4). □

Remark 1. *If $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$, then we have the bounds*

$$\inf_{x\in[a,b]} \breve{f}(x) = \breve{f}\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{x\in[a,b]} \breve{f}(x) = \breve{f}(a) = \breve{f}(b) = \frac{f(a)+f(b)}{2}.$$

Corollary 1. *If $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$ and $w : [a,b] \to [0,\infty)$ is integrable on $[a,b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt &\le \frac{1}{2}\int_a^b w(t)[f(t) + f(a+b-t)]\,dt \\ &\le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \end{aligned} \tag{2.5}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, i.e. $w(t) = w(a+b-t)$ for almost every $t \in [a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \le \int_a^b w(t) f(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \tag{2.6}$$

Proof. The inequality (2.5) follows by (2.4) written for $x = t$, multiplying by $w(t) \geq 0$ and integrating over t on $[a,b]$.

By changing the variable, we have

$$\int_a^b w(t) f(a+b-t)\,dt = \int_a^b w(a+b-t) f(t)\,dt.$$

Since w is symmetric almost everywhere on $[a,b]$, then

$$\int_a^b w(a+b-t) f(t)\,dt = \int_a^b w(t) f(t)\,dt.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_a^b w(t)\left[f(t)+f(a+b-t)\right]dt \\
&= \frac{1}{2}\left[\int_a^b w(t) f(t)\,dt + \int_a^b w(t) f(a+b-t)\,dt\right] \\
&= \frac{1}{2}\left[\int_a^b w(t) f(t)\,dt + \int_a^b w(t) f(t)\,dt\right] = \int_a^b w(t) f(t)\,dt
\end{aligned}$$

and by (2.5) we get (2.6). □

Remark 2. *The inequality (2.6) was obtained by L. Fejér in 1906 for convex functions f and symmetric weights w. It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions f on the interval $[a,b]$.*

The following result also holds.

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x, y \in [a,b]$ with $x \neq y$ we have the Hermite-Hadamard inequalities*

$$\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{x+y}{2}\right) + f\left(a+b-\frac{x+y}{2}\right)\right] \\
&\le \frac{1}{2(y-x)}\left[\int_x^y f(t)\,dt + \int_{a+b-y}^{a+b-x} f(t)\,dt\right] \\
&\le \frac{1}{4}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right].
\end{aligned} \tag{2.7}$$

Proof. Since $\breve{f}_{[a,b]}$ is convex on $[a,b]$, then $\breve{f}_{[a,b]}$ is also convex on any subinterval $[x,y]$ (or $[y,x]$) where $x, y \in [a,b]$.

By Hermite-Hadamard inequalities for convex functions we have

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)\,dt \le \frac{\breve{f}_{[a,b]}(x)+\breve{f}_{[a,b]}(y)}{2} \tag{2.8}$$

for any $x, y \in [a,b]$ with $x \neq y$.

We have

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right)=\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right],$$

$$\begin{aligned}\int_x^y \breve{f}_{[a,b]}(t)\,dt &= \frac{1}{2}\int_x^y [f(t)+f(a+b-t)]\,dt \\ &= \frac{1}{2}\int_x^y f(t)\,dt+\frac{1}{2}\int_x^y f(a+b-t)\,dt \\ &= \frac{1}{2}\int_x^y f(t)\,dt+\frac{1}{2}\int_{a+b-y}^{a+b-x} f(t)\,dt\end{aligned}$$

and

$$\frac{\breve{f}_{[a,b]}(x)+\breve{f}_{[a,b]}(y)}{2}=\frac{1}{4}[f(x)+f(a+b-x)+f(y)+f(a+b-y)].$$

Then by (2.8) we deduce the desired result (2.7). □

Remark 3. *If we take $x=a$ and $y=b$ in (2.7), then we get (2.2).*

If, for a given $x\in[a,b]$, we take $y=a+b-x$, then from (2.7) we get

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt\le\frac{1}{2}[f(x)+f(a+b-x)], \tag{2.9}$$

where $x\neq\frac{a+b}{2}$, provided that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$.

Integrating this inequality over x we get the following refinement of the first part of (2.2)

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{1}{2(b-a)}\int_a^b\left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt\right]dx \\ &\le \frac{1}{b-a}\int_a^b f(t)\,dt,\end{aligned} \tag{2.10}$$

provided that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$.

When the function is convex, we have the following inequalities as well:

Remark 4. *If $f:[a,b]\to\mathbb{R}$ is convex, then from (2.7) we have the inequalities*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right] \\ &\le \frac{1}{2(y-x)}\left[\int_x^y f(t)\,dt+\int_{a+b-y}^{a+b-x} f(t)\,dt\right] \\ &\le \frac{1}{4}[f(x)+f(a+b-x)+f(y)+f(a+b-y)]\end{aligned} \tag{2.11}$$

for any $x,y\in[a,b]$, $x\neq y$.

If we integrate (2.11) over (x,y) on the square $[a,b]^2$ and divide by $(b-a)^2$, then we get the following refinement of the first Hermite-Hadamard inequality for *convex functions*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy + \int_a^b\int_a^b f\left(a+b-\frac{x+y}{2}\right)dxdy\right] \leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b \frac{1}{y-x}\left[\int_x^y f(t)\,dt + \int_{a+b-y}^{a+b-x} f(t)\,dt\right]dxdy \leq \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.12}$$

We notice that, the second and the third inequalities also hold for the more general case of symmetrized convex functions on the interval $[a,b]$.

A concept of weaker symmetrized convexity can be introduced as follows:

Definition 2. *We say that the function* $f:[a,b]\to\mathbb{R}$ *is weak symmetrized convex (concave) on the interval* $[a,b]$ *if* $\breve{f}$ *is convex (concave) on the interval* $\left[a,\frac{a+b}{2}\right]$.

We denote this class by $WSCon[a,b]$.

It is clear that any symmetrized convex function on $[a,b]$ is weak symmetrized convex on that interval. Also, there are weak symmetrized convex function on $[a,b]$ that are not symmetrized convex on $[a,b]$.

If we consider the function $f_0:[a,b]\to\mathbb{R}$ defined by

$$f_0(t)=\begin{cases} t^2,\ t\in\left[a,\frac{a+b}{2}\right], \\ \\ (a+b-t)^2,\ t\in\left(\frac{a+b}{2},b\right], \end{cases}$$

then we observe that f_0 is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$ but not convex on the whole interval $[a,b]$. We also observe that f_0 is a symmetrical function on $[a,b]$ and then $\breve{f}_0=f_0$. Therefore f_0 is weak symmetrized convex function on $[a,b]$ but not symmetrized convex on that interval.

We have the following strict inclusion

$$SCon[a,b]\subsetneq WSCon[a,b]. \tag{2.13}$$

We also notice that f is weak symmetrized convex function on $[a,b]$ if and only if $\breve{f}$ is convex on the second half of the interval $[a,b]$, namely $\left[\frac{a+b}{2},b\right]$.

Theorem 4. *Assume that* $f:[a,b]\to\mathbb{R}$ *is weak symmetrized convex on the interval* $[a,b]$. *Then for any* $x,y\in\left[a,\frac{a+b}{2}\right]$ $x\neq y$ *we have the Hermite-Hadamard inequalities (2.7).*

In particular, we have

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \tag{2.14}$$

Proof. The first part follows from the proof of Theorem 3 for $x,y\in\left[a,\frac{a+b}{2}\right]$.

The second part follows from the inequality (2.7) by taking $x=a$ and $y=\frac{a+b}{2}$. □

Remark 5. *We observe that if $f:[a,b]\to\mathbb{R}$ is weak symmetrized convex on the interval $[a,b]$, then the inequality (2.9) holds for any $x\in\left[a,\frac{a+b}{2}\right)$ and integrating on $\left[a,\frac{a+b}{2}\right]$ we also have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\int_a^{\frac{a+b}{2}}\left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt\right]dx \le \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{2.15}$$

We can state in general the following result for symmetrized convex functions.

Proposition 1. *Any inequality that holds for convex functions f defined on the interval $[a,b]$ will hold for symmetrized convex functions by replacing f with $\breve{f}_{[a,b]}$ and performing the required calculations.*

We can illustrate this fact with two simple examples.

It is known that, see [19] if $f:[a,b]\to\mathbb{R}$ is differentiable convex on (a,b), then for any $x,y\in(a,b)$ with $x\neq y$ we have

$$0\le\frac{1}{y-x}\int_x^y f(t)-f\left(\frac{x+y}{2}\right)\le\frac{1}{8}\left(f'(y)-f'(x)\right)(y-x). \tag{2.16}$$

Now, if $f:[a,b]\to\mathbb{R}$ is differentiable and symmetrized convex on (a,b), then by writing (2.16) for $\breve{f}_{[a,b]}$ we have

$$0\le\frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)-\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right) \le\frac{1}{8}\left(\left(\breve{f}_{[a,b]}\right)'(y)-\left(\breve{f}_{[a,b]}\right)'(x)\right)(y-x). \tag{2.17}$$

However

$$\frac{1}{y-x}\int_x^y \breve{f}_{[a,b]}(t)=\frac{1}{2(y-x)}\left[\int_x^y f(t)\,dt+\int_{a+b-y}^{a+b-x} f(t)\,dt\right],$$

$$\breve{f}_{[a,b]}\left(\frac{x+y}{2}\right)=\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right]$$

and

$$\left(\breve{f}_{[a,b]}\right)'(y)-\left(\breve{f}_{[a,b]}\right)'(x)=\frac{1}{2}\left(f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right).$$

Then by (2.17) we get

$$\begin{aligned}0&\le\frac{1}{2(y-x)}\left[\int_x^y f(t)\,dt+\int_{a+b-y}^{a+b-x} f(t)\,dt\right]\\&\quad-\frac{1}{2}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right]\\&\le\frac{1}{16}\left[f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right](y-x)\end{aligned} \tag{2.18}$$

that holds for any $x,y\in(a,b)$ with $x\neq y$.

From this inequality, by taking $y=a+b-x$, we get

$$0\leq\frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_{x}^{a+b-x}f(t)\,dt-f\left(\frac{a+b}{2}\right)\leq\frac{1}{4}\left[f'(a+b-x)-f'(x)\right]\left(\frac{a+b}{2}-x\right)\tag{2.19}$$

for any $x\in(a,b)$ with $x\neq\frac{a+b}{2}$.

If $f:[a,b]\to\mathbb{R}$ is differentiable convex on (a,b), then for any $x,y\in(a,b)$ with $x\neq y$ we also have [20]

$$0\leq\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_{x}^{y}f(t)\leq\frac{1}{8}\left(f'(y)-f'(x)\right)(y-x).\tag{2.20}$$

Now, if $f:[a,b]\to\mathbb{R}$ is differentiable and symmetrized convex on (a,b), then by a similar argument as above we have

$$\begin{aligned}0&\leq\frac{1}{4}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right]\\&-\frac{1}{2(y-x)}\left[\int_{x}^{y}f(t)\,dt+\int_{a+b-y}^{a+b-x}f(t)\,dt\right]\\&\leq\frac{1}{16}\left[f'(y)-f'(a+b-y)-f'(x)+f'(a+b-x)\right](y-x)\end{aligned}\tag{2.21}$$

for any $x,y\in(a,b)$ with $x\neq y$.

In particular, we have

$$\begin{aligned}0&\leq\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_{x}^{a+b-x}f(t)\,dt\\&\leq\frac{1}{4}\left[f'(a+b-x)-f'(x)\right]\left(\frac{a+b}{2}-x\right)\end{aligned}\tag{2.22}$$

for any $x\in(a,b)$ with $x\neq\frac{a+b}{2}$.

3. Symmetrized h-Convexity

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 3 ([37]). *We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have*

$$f(tx+(1-t)y)\leq\frac{1}{t}f(x)+\frac{1}{1-t}f(y).\tag{3.1}$$

Some further properties of this class of functions can be found in [27], [28], [30], [43], [46] and [47]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 4 ([30]). *We say that a function $f:I\to\mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x,y\in I$ and $t\in[0,1]$ we have*

$$f(tx+(1-t)y)\leq f(x)+f(y).\tag{3.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\} \tag{3.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [30] and [44] while for quasi convex functions, the reader can consult [29].

Definition 5 ([7])**.** *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx+(1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

For some properties of this class of functions see [1], [2], [7], [8], [25], [26], [38], [40] and [49].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 6 ([52])**.** *Let $h : J \to [0,\infty)$ with h not identical to 0. We say that $f : I \to [0,\infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx+(1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{3.4}$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [52], [6], [41], [50], [48] and [51].

We can introduce now another class of functions.

Definition 7. *We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx+(1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{3.5}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

The following inequality of Hermite-Hadamard type holds [48]

Theorem 5. *Assume that the function $f : I \to [0,\infty)$ is an h-convex function with $h \in L[0,1]$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0,1] \ni t \mapsto f[(1-t)x+ty]$ is Lebesgue integrable on $[0,1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq [f(x)+f(y)]\int_0^1 h(t)\,dt. \tag{3.6}$$

If we write (3.6) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{2}. \tag{3.7}$$

If we write (3.6) for the case of P-type functions $f : I \to [0,\infty)$, i.e., $h(t) = 1, t \in [0,1]$, then we get the inequality

$$\frac{1}{2}f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq f(x)+f(y), \tag{3.8}$$

that has been obtained for functions of real variable in [30].

If f is Breckner s-convex on I, for $s \in (0,1)$, then by taking $h(t) = t^s$ in (3.6) we get

$$2^{s-1}f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{s+1}, \tag{3.9}$$

that was obtained for functions of a real variable in [25].

If $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1)$, then

$$\frac{1}{2^{s+1}}f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du \leq \frac{f(x)+f(y)}{1-s}. \tag{3.10}$$

We notice that for $s = 1$ the first inequality in (3.10) still holds [30], i.e.

$$\frac{1}{4}f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_x^y f(u)\,du. \tag{3.11}$$

We can introduce the following concept generalizing the notion of h-convexity.

Definition 8. *Assume that h is as in Definition 6. We say that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex (concave) on the interval $[a,b]$ if $\breve{f}$ is h-convex (concave) on $[a,b]$.*

Now, if we denote by $Con_h[a,b]$ the closed convex cone of h-convex functions defined on $[a,b]$ and by $SCon_h[a,b]$ the class of h-symmetrized convex, then, as in the previous section, we can conclude in general that

$$Con_h[a,b] \subsetneq SCon_h[a,b]. \tag{3.12}$$

Definition 9. *Assume that h is as in Definition 6. We say that the function $f : [a,b] \to \mathbb{R}$ is h-weak symmetrized convex (concave) on the interval $[a,b]$ if $\breve{f}$ is h-convex (concave) on the interval $\left[a, \frac{a+b}{2}\right]$.*

We denote this class by $WSCon_h[a,b]$. As in the previous section, we can conclude in general that

$$SCon_h[a,b] \subsetneq WSCon_h[a,b]. \tag{3.13}$$

Utilising Theorem 5 and a similar proof to that of Theorem 3, we can state the following result as well:

Theorem 6. *Assume that the function $f : [a,b] \to [0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h integrable on $[0,1]$ and f integrable on $[a,b]$. Then for* *any* $x, y \in [a,b]$ *we have the Hermite-Hadamard inequalities*

$$\begin{aligned}(3.14)\quad & \frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\frac{x+y}{2}\right)+f\left(a+b-\frac{x+y}{2}\right)\right] \\ & \leq \frac{1}{2(y-x)}\left[\int_x^y f(t)\,dt+\int_{a+b-y}^{a+b-x} f(t)\,dt\right] \\ & \leq \frac{1}{2}\left[f(x)+f(a+b-x)+f(y)+f(a+b-y)\right]\int_0^1 h(t)\,dt.\end{aligned}$$

In particular, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq [f(a)+f(b)]\int_0^1 h(t)\,dt. \tag{3.15}$$

Remark 6. *If, for a given* $x \in [a,b]$, *we take* $y = a+b-x$, *then from (3.14) we get*

$$\begin{aligned}(3.16)\quad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) & \leq \frac{1}{2\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt \\ & \leq [f(x)+f(a+b-x)]\int_0^1 h(t)\,dt,\end{aligned}$$

where $x \neq \frac{a+b}{2}$, *provided that* $f : [a,b] \to \mathbb{R}$ *is h-symmetrized convex and integrable on the interval* $[a,b]$.

Integrating on $[a,b]$ *over* x *we get*

$$\begin{aligned}(3.17)\quad & \frac{1}{4h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \\ & \leq \frac{1}{4(b-a)}\int_a^b \left[\frac{1}{\left(\frac{a+b}{2}-x\right)}\int_x^{a+b-x} f(t)\,dt\right] dx \\ & \leq \frac{1}{b-a}\int_a^b f(x)\,dx \int_0^1 h(t)\,dt.\end{aligned}$$

We have the following result as well:

Theorem 7. *Assume that h is as in Definition 6. If the function* $f : [a,b] \to [0,\infty)$ *is h-symmetrized convex on the interval* $[a,b]$, *then we have the bounds*

$$\begin{aligned}(3.18)\quad \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) & \leq \frac{f(x)+f(a+b-x)}{2} \\ & \leq \left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}\end{aligned}$$

for any $x \in [a,b]$.

Proof. Since $\breve{f}$ is h-convex on $[a,b]$ then for any $x \in [a,b]$ we have

$$h\left(\frac{1}{2}\right)\left[\breve{f}(x)+\breve{f}(a+b-x)\right] \geq \breve{f}\left(\frac{a+b}{2}\right)$$

and since

$$\frac{\breve{f}(x)+\breve{f}(a+b-x)}{2} = \frac{1}{2}\left[f(x)+f(a+b-x)\right]$$

while

$$\breve{f}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right),$$

we get the first inequality in (2.4).

Also, by the convexity of $\breve{f}$ we have for any $x\in[a,b]$ that

$$\begin{aligned}\breve{f}(x)&\leq h\left(\frac{x-a}{b-a}\right)\cdot\breve{f}(b)+h\left(\frac{b-x}{b-a}\right)\cdot\breve{f}(a)\\&=h\left(\frac{x-a}{b-a}\right)\cdot\frac{f(a)+f(b)}{2}+h\left(\frac{b-x}{b-a}\right)\cdot\frac{f(a)+f(b)}{2}\\&=\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2},\end{aligned}$$

which proves the second part of (3.18). □

Corollary 2. *Assume that the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h integrable on $[0,1]$ and f integrable on $[a,b]$. If $w:[a,b]\to[0,\infty)$ is integrable on $[a,b]$, then*

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt\\&\leq\frac{1}{2}\int_a^b w(t)\left[f(t)+f(a+b-t)\right]dt\\&\leq\frac{f(a)+f(b)}{2}\int_a^b h\left(\frac{t-a}{b-a}\right)\left[w(t)+w(a+b-t)\right]dt.\end{aligned}\tag{3.19}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, then

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt&\leq\int_a^b w(t)f(t)\,dt\\&\leq\left[f(a)+f(b)\right]\int_a^b h\left(\frac{t-a}{b-a}\right)w(t)\,dt.\end{aligned}\tag{3.20}$$

Proof. From (3.18) we have

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)&\leq\frac{f(t)+f(a+b-t)}{2}\\&\leq\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}\end{aligned}$$

for any $t\in[a,b]$.

Multiplying with $w(t)\geq0$ and integrating over $t\in[a,b]$ we get

$$\begin{aligned}&\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt\\&\leq\frac{1}{2}\int_a^b w(t)\left[f(t)+f(a+b-t)\right]dt\\&\leq\frac{f(a)+f(b)}{2}\int_a^b\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]w(t)\,dt.\end{aligned}\tag{3.21}$$

Observe that, by changing the variable $t = a + b - s,\ s \in [a, b]$, we have

$$\int_a^b h\left(\frac{b-t}{b-a}\right) w(t)\, dt = \int_a^b h\left(\frac{s-a}{b-a}\right) w(a+b-s)\, ds,$$

then we get

$$\begin{aligned} &\int_a^b \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] w(t)\, dt \\ &= \int_a^b h\left(\frac{t-a}{b-a}\right) [w(t) + w(a+b-t)]\, dt \end{aligned}$$

and by (3.21) we obtain the second part of (3.19). □

Utilising the previous examples of h-convex functions the reader may state various inequalities of Hermite-Hadamard type.

For instance, if we assume that the functions $f : [a, b] \to [0, \infty)$ is integrable and of Godunova-Levin type, then for the symmetric weight

$$w : [a, b] \to [0, \infty), w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned} \frac{1}{4} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t)\, dt &\le \int_a^b (t-a)(b-t) f(t)\, dt \\ &\le [f(a) + f(b)](b-a) \int_a^b (b-t)\, dt \end{aligned}$$

and since

$$\int_a^b (t-a)(b-t)\, dt = \frac{1}{6}(b-a)^3, \int_a^b (b-t)\, dt = \frac{1}{2}(b-a)^2,$$

then we get the following inequality of interest:

$$\frac{1}{24} f\left(\frac{a+b}{2}\right)(b-a)^3 \le \int_a^b (t-a)(b-t) f(t)\, dt \le \frac{f(a)+f(b)}{2}(b-a)^3. \tag{3.22}$$

Moreover, if we assume that the function $f : [a, b] \to [0, \infty)$ is integrable and Breckner s-convex with $s \in (0, 1)$, then for the symmetric weight

$$w : [a, b] \to [0, \infty), w(t) = (t-a)(b-t)$$

we have from (3.20) that

$$\begin{aligned} &\frac{1}{2^{1-s}} f\left(\frac{a+b}{2}\right) \int_a^b (t-a)(b-t)\, dt \\ &\le \int_a^b (t-a)(b-t) f(t)\, dt \\ &\le \frac{f(a)+f(b)}{(b-a)^s} \int_a^b (t-a)^{s+1}(b-t)\, dt \end{aligned}$$

and since

$$\int_a^b (t-a)^{s+1}(b-t)\, dt = \frac{(b-a)^{s+3}}{(s+2)(s+3)}$$

then we get the following inequality of interest:

$$\frac{1}{2^{2-s}3} f\left(\frac{a+b}{2}\right)(b-a)^3 \le \int_a^b (t-a)(b-t) f(t)\,dt \le \frac{f(a)+f(b)}{(s+2)(s+3)}(b-a)^3. \tag{3.23}$$

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