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Bouniakowsky and the logarithmic mean inequalities

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Abstract

We point out that the logarithmic inequalities $G < L < A$ discovered by B.C. Carlson (left side) and B. Ostle and H.L. Terwilliger (right side) was proved in fact by V. Bouniakowsky in 1859. More direct and simpler proof will be offered, too.

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1 Introduction

The logarithmic mean $L(a, b)$ of two positive real numbers a and b is defined by

$$L = L(a, b) = \frac{b - a}{\ln b - \ln a} \text{ for } a \neq b, \; L(a, a) = a \tag{1}$$

Let $A := A(a,b) = \dfrac{a+b}{2}$ and $G := G(a,b) = \sqrt{ab}$ denote the arithmetic, resp. geometric means of a and b.

One of the basic inequalities connecting the above means is the following:

$$G < L < A, \text{ for } a \neq b \tag{2}$$

Up to now, the left side of (2) was attributed to B.C. Carlson [2], while the right side to B. Ostle and H.L. Terwilliger [4].

While reading the original paper by V. Bouniakowsky, written in French, and published in 1859 we learned that both inequalities were proved in fact by him. His proof is based on (now called as the "Cauchy-Bouniakowsky inequality") the following: If f and g are real integrable functions, then:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right) \tag{3}$$

Bouniakowsky considered the case when f and g are continuous functions, in which case there is equality in (3) only if $f(x) = kg(x)$, where $k \in \mathbb{R}$ is a constant, for all $x \in [a,b]$.

We note that in the general case (but this will not be used here) the equality case is satisfied only when $f(x) = kg(x)$ a.e. in $[a,b]$ (i.e. for almost every $x \in [a,b]$).

2 Bouniakowsky's proof

The proof gives the logarithmic inequalities (2), without the introduction of the mean L.

Let $F : [a,b] \to \mathbb{R}$ be a strictly positive function. Applying (3) to $f(x) = \sqrt{F(x)}$ and $g(x) = \dfrac{1}{\sqrt{F(x)}}$, we get:

$$\int_a^b F(x)dx \int_a^b \frac{dx}{F(x)} \geq (b-a)^2 \tag{4}$$

This is inequality (D) in Bouniakowsky's paper (with slightly changed notations. He used f in place of F here, and $a = x_0$, $b = X$).

Now, letting $F(x) = \frac{1}{x}$ in (4) with $b > a > 0$ we get the strict inequality (as $F(x)$ is not a constant)

$$(\ln b - \ln a)\left(\frac{b^2 - a^2}{2}\right) > (b-a)^2, \text{ for } b > a > 0$$

which gives immediately the right side of (2).

Let now $F(x) = e^x$ in (4). We can deduce

$$(e^b - e^a)(e^{-a} - e^{-b}) > (b-a)^2,$$

or after some transformations, as

$$\frac{e^b - e^a}{b-a} > e^{\frac{b+a}{2}}, \ b > a \tag{5}$$

As here a, b are arbitrary real numbers, one may select $a = \ln x$, $b = \ln y$, with $y > x > 0$. One obtains from (5) the inequality

$$\frac{y-x}{\ln y - \ln x} > \sqrt{xy},$$

which is in fact the left side inequality of (2) ($x \neq y$ being arbitrary positive real numbers)

3 A simplified proof

A somewhat simpler and more direct proof may be obtained in the following way: Put $g(x) = 1$ in (3), resulting:

$$\left(\int_a^b f(x)dx\right)^2 \leq (b-a)\int_a^b f^2(x)dx, \tag{6}$$

where f is a continuous function. There is equality only if f is a constant.

Applying (6) for $f(x) = \frac{1}{x}$, we get

$$(\ln b - \ln a)^2 < (b-a)\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{(b-a)^2}{ab},$$

so the left side of (2) follows at once.

Let now $f(x) = \frac{1}{\sqrt{x}}$ in (6). One obtains

$$4(\sqrt{b} - \sqrt{a})^2 < (b-a)(\ln b - \ln a) \tag{7}$$

By replacing $a = u^2$, $b = v^2$ we get from (7):

$$4(v-u) < 2(v^2 - u^2)(\ln v - \ln u),$$

so

$$\frac{v-u}{\ln v - \ln u} < \frac{v+u}{2},$$

i.e. the right side of (2).

Remarks. Inequality (2) and the logarithmic mean has surprising applications in certain problems of physics, probability and statistics, etc. See e.g. [3], [5]. The author has applied the left side of (2) in the proof of a complicated inequality with applications in the theory of quasiconformal mappings and norm inequalities for vector functions (see [6]). Another application in the proof of a conjecture on prime numbers is presented in [7].

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