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A note on the logarithmic mean

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Abstract

We show that by integrating some simple inequalities, the logarithmic inequalities $G < L < A$ follows.

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1 Introduction

The logarithmic mean $L(a, b)$ of two positive real numbers a and b is defined by

$$L = L(a, b) = \frac{b - a}{\ln b - \ln a} \text{ for } a \neq b, L(a, a) = a. \tag{1}$$

Let $A := A(a, b) = \frac{a + b}{2}$ and $G := G(a, b) = \sqrt{ab}$ denote the arithmetic, resp. geometric means of a and b.

One of the basic inequalities connecting the above means is the following:

$$G < L < A, \text{ for } a \neq b. \tag{2}$$

Among the first discoveries of this inequality, we quote B. Ostle and H.L. Terwilliger [2] (right side of (2)) and B.C. Carlson [1] (left side of (2)). See also [4], [5] for other references.

Inequality (2) has been rediscovered and reproved many time (see e.g. [3], [4], [5], [6]).

The aim of this note is to offer a new proof of this inequality. The method is based on two simple algebraic inequalities and Riemann integration.

2 The proof

Lemma. *For all $t > 1$ one has*

$$\frac{4}{(t+1)^2} < \frac{1}{t} < \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}. \tag{3}$$

Proof. The left side of (3) holds true, being equivalent to $(t-1)^2 > 0$, while the right side, after reducing with $\sqrt{t}$ to $\frac{1}{2\sqrt{t}} < \frac{1}{2}$, or $t > 1$.

Now, let $b > a > 0$, and integrate both sides of inequalities (3) on $[1, b/a]$. As

$$\int_1^{b/a} \frac{2}{(t+1)^2} dt = \frac{b-a}{b+a} \text{ and } \int_1^{b/a} \frac{1}{2\sqrt{t}} dt = \sqrt{\frac{b}{a}} - 1,$$

$$\int_1^{b/a} \frac{1}{2t\sqrt{t}} dt = -\sqrt{\frac{a}{b}} + 1,$$

we get:

$$2\left(\frac{b-a}{b+a}\right) < \ln b - \ln a < \sqrt{\frac{b}{a}} - \sqrt{\frac{a}{b}} = \frac{b-a}{\sqrt{ab}}. \tag{4}$$

Relation (4) implies immediately (2) for $b > a$. Since $L(a,b) = L(b,a)$, (2) follows for all $a \neq b$.

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