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PERTURBED COMPANIONS OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS (I)

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Abstract. Perturbed companions of Ostrowski's inequality for absolutely continuous functions whose derivatives are either bounded or of bounded variation and applications are given.

1. Introduction

In [16] we established the following companion of Ostrowski inequality [26] for Lebesgue *sup-norm*:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_\infty[a,b]$, then we have the inequalities*

$$\left|\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.1}$$
$$\le\frac{1}{b-a}\left[\frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty}+\left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty}+\frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}\right]$$
$$\le\begin{cases}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}\\[2ex] \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{2\alpha}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}}\\ \quad\times\left[\|f'\|^{\beta}_{[a,x],\infty}+\|f'\|^{\beta}_{[x,a+b-x],\infty}+\|f'\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}(b-a)\\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2,\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}\\ \quad\times\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,a+b-x],\infty}+\|f'\|_{[a+b-x,b],\infty}\right](b-a)\end{cases}$$

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for any $x \in \left[a, \frac{a+b}{2}\right]$, *where*

$$\|g\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |g(s)| .$$

The inequality (1.1), the first inequality in (1.1) and the constant $\frac{1}{8}$ *are sharp.*

If in Theorem 1 we choose $x = a$, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} \tag{1.2}$$

with $\frac{1}{4}$ as a sharp constant (see for example [20, p. 25]).

If in the same theorem we now choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned} \left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ &\le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} \end{aligned} \tag{1.3}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [15] by a different argument.

It is natural to consider the following corollary.

Corollary 1. *With the assumptions in Theorem 1, one has the inequality:*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty} . \tag{1.4}$$

The constant $\frac{1}{8}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

In the same paper [16] we established the corresponding inequalities for Lebesgue p-norms with $p \ge 1$ as well as have provided some applications for cumulative distribution functions and some quadrature rules.

For a monograph devoted to Ostrowski type inequalities, see [20].

For research papers on Ostrowski's inequality see [1]-[19], [21]-[23] and [25].

Motivated by the above results, we investigate in this paper some perturbed versions of the inequality (1.1). Applications for cumulative distribution function are provided as well.

2. Some Identities

The following identity holds.

Lemma 1. *Assume that $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$. Then we have the equality*

$$\begin{aligned}&\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{2}\left(x-a\right)^{2}\frac{\lambda_{3}\left(x\right)-\lambda_{1}\left(x\right)}{b-a}-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt \\ &=\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\left[f'\left(t\right)-\lambda_{1}\left(x\right)\right]dt \\ &+\frac{1}{b-a}\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'\left(t\right)-\lambda_{2}\left(x\right)\right]dt \\ &+\frac{1}{b-a}\int_{a+b-x}^{b}\left(t-b\right)\left[f'\left(t\right)-\lambda_{3}\left(x\right)\right]dt,\end{aligned}\tag{2.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ and $\lambda_{j}\left(x\right),\ j=1,2,3$ complex numbers.

Proof. Using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned}&\int_{a}^{x}\left(t-a\right)\left[f'\left(t\right)-\lambda_{1}\left(x\right)\right]dt \\ &=\int_{a}^{x}\left(t-a\right)f'\left(t\right)dt-\lambda_{1}\left(x\right)\int_{a}^{x}\left(t-a\right)dt \\ &=\left(x-a\right)f\left(x\right)-\int_{a}^{x}f\left(t\right)dt-\frac{1}{2}\lambda_{1}\left(x\right)\left(x-a\right)^{2},\end{aligned}$$

$$\begin{aligned}&\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'\left(t\right)-\lambda_{2}\left(x\right)\right]dt \\ &=\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)f'\left(t\right)dt-\lambda_{2}\left(x\right)\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\ &=f\left(a+b-x\right)\left(\frac{a+b}{2}-x\right)-f\left(x\right)\left(x-\frac{a+b}{2}\right)-\int_{x}^{a+b-x}f\left(t\right)dt \\ &-\lambda_{2}\left(x\right)\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt \\ &=f\left(a+b-x\right)\left(\frac{a+b}{2}-x\right)-f\left(x\right)\left(x-\frac{a+b}{2}\right)-\int_{x}^{a+b-x}f\left(t\right)dt,\end{aligned}$$

since, by symmetry

$$\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)dt=0$$

and

$$\begin{aligned}&\int_{a+b-x}^{b}\left(t-b\right)\left[f'\left(t\right)-\lambda_{3}\left(x\right)\right]dt \\ &=\int_{a+b-x}^{b}\left(t-b\right)f'\left(t\right)dt-\lambda_{3}\left(x\right)\int_{a+b-x}^{b}\left(t-b\right)dt \\ &=\left(x-a\right)f\left(a+b-x\right)-\int_{a+b-x}^{b}f\left(t\right)dt+\frac{1}{2}\lambda_{3}\left(x\right)\left(x-a\right)^{2}.\end{aligned}$$

Summing the above equalities, we deduce

$$\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt$$
$$+\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_3(x)\right]dt$$
$$=(b-a)\frac{f(x)+f(a+b-x)}{2}-\int_a^b f(t)\,dt+\frac{1}{2}\left[\lambda_3(x)-\lambda_1(x)\right](x-a)^2,$$

which is equivalent with the desired identity (2.1). □

The following particular cases are of interest:

Corollary 2. *With the assumption of Lemma 1 we have the equalities*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt, \tag{2.2}$$

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.3}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_3\right]dt,$$

and

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.4}$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\left[f'(t)-\lambda_3\right]dt$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

Corollary 3. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$. *Then we have the equality*

$$\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.5}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2(x)\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[f'(t)-\lambda_1(x)\right]dt,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2$ *complex numbers.*

Remark 1. *We get from (2.3) the following particular case:*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)\left[f'(t)-\lambda_1\right]dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_1\right]dt, \tag{2.6}$$

for any $\lambda_1 \in \mathbb{C}$, *while from (2.4) we get*

$$\begin{aligned}\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt &= \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt\\ &+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt\\ &+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (t-b)\left[f'(t)-\lambda_1\right]dt\end{aligned} \tag{2.7}$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

3. Inequalities for Bounded Derivatives

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C} \mid \operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C} \mid \left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for a.e. } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \text{ for a.e. } t \in [a,b]\}. \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}. \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{3.4}$$

Theorem 2. *Assume that $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function on $[a,b]$ and $x \in \left[a, \frac{a+b}{2}\right]$. If there exists the complex numbers $\gamma_j(x) \neq \Gamma_j(x)$, $j = 1,2,3$ such that*

$$f' \in \bar{\Delta}_{[a,x]}(\gamma_1(x), \Gamma_1(x)) \cap \bar{\Delta}_{[x,a+b-x]}(\gamma_2(x), \Gamma_2(x)) \cap \bar{\Delta}_{[a+b-x,b]}(\gamma_3(x), \Gamma_3(x)), \tag{3.5}$$

then we have the inequality

$$\begin{aligned} &\left|\frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{4}(x-a)^2 \frac{\gamma_3(x) + \Gamma_3(x) - \gamma_1(x) - \Gamma_1(x)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\leq \frac{1}{4(b-a)}\left[|\Gamma_1(x) - \gamma_1(x)|(x-a)^2 + 2|\Gamma_2(x) - \gamma_2(x)|\left(\frac{a+b}{2} - x\right)^2 + |\Gamma_3(x) - \gamma_3(x)|(x-a)^2\right]. \end{aligned} \tag{3.6}$$

Proof. Taking the modulus in the equality (2.1) written for

$$\begin{aligned} \lambda_1(x) &= \frac{\gamma_1(x) + \Gamma_1(x)}{2}, \lambda_2(x) = \frac{\gamma_2(x) + \Gamma_2(x)}{2}, \\ \lambda_3(x) &= \frac{\gamma_3(x) + \Gamma_3(x)}{2} \end{aligned}$$

and utilizing the condition (3.5) we have

$$\begin{aligned}&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{4}(x-a)^2\frac{\gamma_3(x)+\Gamma_3(x)-\gamma_1(x)-\Gamma_1(x)}{b-a}\right.\\&\left.-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\leq\frac{1}{b-a}\int_a^x(t-a)\left|f'(t)-\frac{\gamma_1(x)+\Gamma_1(x)}{2}\right|dt\\&+\frac{1}{b-a}\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'(t)-\frac{\gamma_2(x)+\Gamma_2(x)}{2}\right|dt\\&+\frac{1}{b-a}\int_{a+b-x}^b(b-t)\left|f'(t)-\frac{\gamma_3(x)+\Gamma_3(x)}{2}\right|dt\\&\leq\frac{1}{4(b-a)}\left|\Gamma_1(x)-\gamma_1(x)\right|(x-a)^2\\&+\frac{2}{4(b-a)}\left|\Gamma_2(x)-\gamma_2(x)\right|\left(\frac{a+b}{2}-x\right)^2\\&+\frac{1}{4(b-a)}\left|\Gamma_3(x)-\gamma_3(x)\right|(x-a)^2\end{aligned}$$

and the inequality (3.6) is proved. □

Corollary 5. *Assume that* $f:[a,b]\to\mathbb{C}$ *is an absolutely continuous function on* $[a,b]$ *and* $x\in\left[a,\frac{a+b}{2}\right]$. *If there exists the complex numbers* $\gamma_j(x)\neq\Gamma_j(x)$, $j=1,2$ *such that*

$$f'\in\bar{\Delta}_{[a,x]}(\gamma_1(x),\Gamma_1(x))\cap\bar{\Delta}_{[x,a+b-x]}(\gamma_2(x),\Gamma_2(x))\cap\bar{\Delta}_{[a+b-x,b]}(\gamma_1(x),\Gamma_1(x)),\tag{3.7}$$

then we have the inequality

$$\begin{aligned}&\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\leq\frac{1}{2(b-a)}\left[\left|\Gamma_1(x)-\gamma_1(x)\right|(x-a)^2+\left|\Gamma_2(x)-\gamma_2(x)\right|\left(\frac{a+b}{2}-x\right)^2\right].\end{aligned}\tag{3.8}$$

Remark 2. *Assume that* $f:[a,b]\to\mathbb{C}$ *is an absolutely continuous function on* $[a,b]$.

If there exists the complex numbers $\gamma_2\neq\Gamma_2$ *such that* $f'\in\bar{\Delta}_{[a,b]}(\gamma_2,\Gamma_2)$, *then*

$$\left|\frac{1}{2}\left[f(a)+f(b)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}(b-a)\left|\Gamma_2-\gamma_2\right|.\tag{3.9}$$

If there exists the complex numbers $\gamma_j\neq\Gamma_j$, $j=1,3$ *such that*

$$f'\in\bar{\Delta}_{\left[a,\frac{a+b}{2}\right]}(\gamma_1,\Gamma_1)\cap\bar{\Delta}_{\left[\frac{a+b}{2},b\right]}(\gamma_3,\Gamma_3),$$

then we have the inequality

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{16}(b-a)(\Gamma_3+\gamma_3-\Gamma_1-\gamma_1)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)\left[|\Gamma_1-\gamma_1|+|\Gamma_3-\gamma_3|\right]. \tag{3.10}$$

In particular, if $f'\in\bar{\Delta}_{[a,b]}(\gamma_1,\Gamma_1)$ *then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)|\Gamma_1-\gamma_1|. \tag{3.11}$$

If there exists the complex numbers $\gamma_j\neq\Gamma_j$, $j=1,2,3$ *such that*

$$f'\in\bar{\Delta}_{\left[a,\frac{3a+b}{4}\right]}(\gamma_1,\Gamma_1)\cap\bar{\Delta}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}(\gamma_2,\Gamma_2)\cap\bar{\Delta}_{\left[\frac{a+3b}{4},b\right]}(\gamma_3,\Gamma_3),$$

then

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{64}(b-a)(\Gamma_3+\gamma_3-\Gamma_1-\gamma_1) -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{64}(b-a)\left[|\Gamma_1-\gamma_1|+2|\Gamma_2-\gamma_2|+|\Gamma_3-\gamma_3|\right]. \tag{3.12}$$

In particular, if $\gamma_3=\gamma_1$ *and* $\Gamma_3=\Gamma_1$, *then*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{32}(b-a)\left[|\Gamma_1-\gamma_1|+|\Gamma_2-\gamma_2|\right], \tag{3.13}$$

provided

$$f'\in\bar{\Delta}_{\left[a,\frac{3a+b}{4}\right]}(\gamma_1,\Gamma_1)\cap\bar{\Delta}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}(\gamma_2,\Gamma_2)\cap\bar{\Delta}_{\left[\frac{a+3b}{4},b\right]}(\gamma_1,\Gamma_1).$$

Moreover, if $f'\in\bar{\Delta}_{[a,b]}(\gamma_1,\Gamma_1)$ *then*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{16}(b-a)|\Gamma_1-\gamma_1|. \tag{3.14}$$

The case of real-valued functions is of interest.

Remark 3. *If the function* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous and if there exists the constants* $l<L$ *such that* $l\le f'(t)\le L$ *for almost every* $t\in[a,b]$, *then we have the inequalities*

$$\left|\frac{1}{2}[f(a)+f(b)]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(L-l), \tag{3.15}$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{8}(b-a)(L-l) \tag{3.16}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{16}(b-a)(L-l). \tag{3.17}$$

These results improve the corresponding inequalities from Introduction.

4. Inequalities for Derivatives of Bounded Variation

Assume that $f: I \to \mathbb{C}$ is an absolutely continuous function on $[a,b] \subset \mathring{I}$, the interior of I. Then from (2.1) we have for $\lambda_1(x) = f'(a)$, $\lambda_2(x) = \frac{f'(x)+f'(a+b-x)}{2}$ and $\lambda_3(x) = f'(b)$ the equality

$$\begin{aligned}\frac{1}{2}[f(x)+f(a+b-x)] &+ \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)[f'(t)-f'(a)]\,dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\frac{f'(x)+f'(a+b-x)}{2}\right]dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)[f'(t)-f'(b)]\,dt,\end{aligned} \tag{4.1}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

We can state the following result.

Theorem 3. *Assume that $f: I \to \mathbb{C}$ is an absolutely continuous function on $[a,b] \subset \mathring{I}$. If the derivative f' is of bounded variation on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{f'(b)-f'(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ &\le \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t(f')\,dt + \frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b(f')\,dt \\ &\le \frac{1}{2(b-a)}\left[(x-a)^2\bigvee_a^x(f') + \left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f') + (x-a)^2\bigvee_{a+b-x}^b(f')\right] \\ &\le \begin{cases}\frac{1}{2(b-a)}\max\left\{(x-a)^2, \left(x-\frac{a+b}{2}\right)^2\right\}\bigvee_a^b(f') \\ \frac{1}{2(b-a)}\max\left\{\bigvee_a^x(f'), \bigvee_x^{a+b-x}(f'), \bigvee_{a+b-x}^b(f')\right\}\left[2(x-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\end{cases}\end{aligned} \tag{4.2}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. If we take the modulus in (4.1) we get

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{2}\left(x-a\right)^{2}\frac{f'\left(b\right)-f'\left(a\right)}{b-a}-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right| \\
&\leq\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\left|f'\left(t\right)-f'\left(a\right)\right|dt \\
&+\frac{1}{b-a}\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'\left(t\right)-\frac{f'\left(x\right)+f'\left(a+b-x\right)}{2}\right|dt \\
&+\frac{1}{b-a}\int_{a+b-x}^{b}\left(b-t\right)\left|f'\left(t\right)-f'\left(b\right)\right|dt:=K,
\end{aligned} \tag{4.3}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Let $x\in\left(a,\frac{a+b}{2}\right)$. Since f' is of bounded variation on $[a,b]$, then

$$\left|f'\left(t\right)-f'\left(a\right)\right|\leq\bigvee_{a}^{t}\left(f'\right),$$

for any $t\in[a,x]$ and

$$\begin{aligned}
&\left|f'\left(t\right)-\frac{f'\left(x\right)+f'\left(a+b-x\right)}{2}\right| \\
&=\left|\frac{f'\left(t\right)-f'\left(x\right)+f'\left(t\right)-f'\left(a+b-x\right)}{2}\right| \\
&\leq\frac{1}{2}\left[\left|f'\left(t\right)-f'\left(x\right)\right|+\left|f'\left(a+b-x\right)-f'\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_{x}^{a+b-x}\left(f'\right)
\end{aligned}$$

for any $t\in[x,a+b-x]$.

We also have

$$\left|f'\left(t\right)-f'\left(b\right)\right|\leq\bigvee_{t}^{b}\left(f'\right),\ t\in[a+b-x,b].$$

Then we get

$$\begin{aligned}
K&\leq\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\bigvee_{a}^{t}\left(f'\right)dt+\frac{1}{2\left(b-a\right)}\bigvee_{x}^{a+b-x}\left(f'\right)\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\
&+\frac{1}{b-a}\int_{a+b-x}^{b}\left(b-t\right)\bigvee_{t}^{b}\left(f'\right)dt \\
&\leq\frac{1}{b-a}\bigvee_{a}^{x}\left(f'\right)\int_{a}^{x}\left(t-a\right)dt+\frac{1}{2\left(b-a\right)}\bigvee_{x}^{a+b-x}\left(f'\right)\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|dt \\
&+\frac{1}{b-a}\bigvee_{a+b-x}^{b}\left(f'\right)\int_{a+b-x}^{b}\left(b-t\right)dt
\end{aligned}$$

$$= \frac{1}{2(b-a)}(x-a)^2 \bigvee_a^x (f') + \frac{1}{2(b-a)}\left(x - \frac{a+b}{2}\right)^2 \bigvee_x^{a+b-x} (f')$$
$$+ \frac{1}{2(b-a)}(x-a)^2 \bigvee_{a+b-x}^b (f'),$$

which proves the first two inequalities in (4.2).

The last part is obvious by the maximum properties. □

Corollary 6. *With the assumptions of Theorem 3 we have*

$$\left|\frac{1}{2}[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)\bigvee_a^b (f'), \tag{4.4}$$

$$\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)[f'(b)-f'(a)] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{4.5}$$
$$\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\bigvee_a^t (f')\,dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (b-t)\bigvee_t^b (f')\,dt$$
$$\le \frac{1}{8}(b-a)\bigvee_a^b (f')$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)[f'(b)-f'(a)] \right. \tag{4.6}$$
$$\left. - \frac{1}{b-a}\int_a^b f(t)\,dt\right|$$
$$\le \frac{1}{b-a}\int_a^{\frac{3a+b}{4}} (t-a)\bigvee_a^t (f')\,dt + \frac{1}{32}(b-a)\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} (f')$$
$$+ \frac{1}{b-a}\int_{\frac{a+3b}{4}}^b (b-t)\bigvee_t^b (f')\,dt$$
$$\le \frac{1}{32}(b-a)\bigvee_a^b (f').$$

5. Applications for PDF

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the *probability density function* (PDF) $f:[a,b]\to[0,\infty)$ and with the *cumulative distribution function* (CDF) $F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt$. We know that F is monotonic nondecreasing and absolutely continuous on $[a,b]$, $F' = f$ almost everywhere on $[a,b]$ and $F(a) = 0$, $F(b) = \int_a^b f(t)\,dt = 1$.

Assume that $g:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ and there exists the constants $m < M$ such that

$$m \le g'(t) \le M \text{ for almost every } t \in [a,b]$$

then, by Corollary 5, we have the inequality

$$\left|\frac{1}{2}\left[g(x)+g(a+b-x)\right]-\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}(M-m)\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a) \tag{5.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proposition 2. *Let X be a random variable taking values in the finite interval $[a,b]$, with PDF $f:[a,b]\to[0,\infty)$ and with CDF $F(x)=\Pr(X\leq x)=\int_a^x f(t)\,dt$. If there exists the constants $m<M$ such that*

$$m\leq f(t)\leq M \text{ for almost every } t\in[a,b]$$

then,

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]-\frac{b-E(X)}{b-a}\right| \leq \frac{1}{2}(M-m)\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a) \tag{5.2}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$, where $E(X)=\int_a^b t\,dF(t)$ is the expectation of X.

Proof. Follows from (5.1) for $g=F$ and by taking into account that

$$\int_a^b F(t)\,dt=b-E(X).$$

□

Corollary 7. *With the assumptions in Proposition 2, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|\leq\frac{1}{16}(M-m)(b-a) \tag{5.3}$$

Utilising Theorem 3 we can also state:

Proposition 3. *If PDF* $f:[a,b]\to[0,\infty)$ *is of bounded variation on* $[a,b]$, *then*

$$\left|\frac{1}{2}\left[F(x)+F(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f(b)-f(a)}{b-a}-\frac{b-E(X)}{b-a}\right| \tag{5.4}$$

$$\le \frac{1}{b-a}\int_a^x (t-a)\bigvee_a^t(f)\,dt+\frac{1}{2(b-a)}\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f)$$

$$+\frac{1}{b-a}\int_{a+b-x}^b (b-t)\bigvee_t^b(f)\,dt$$

$$\le \frac{1}{2(b-a)}$$

$$\times\left[(x-a)^2\bigvee_a^x(f)+\left(x-\frac{a+b}{2}\right)^2\bigvee_x^{a+b-x}(f)+(x-a)^2\bigvee_{a+b-x}^b(f)\right]$$

$$\le \frac{1}{2(b-a)}$$

$$\times\begin{cases}\max\left\{(x-a)^2,\left(x-\frac{a+b}{2}\right)^2\right\}\bigvee_a^b(f), \\ \\ \max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}\left[2(x-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Finally we have:

Corollary 8. *With the assumptions in Proposition 3, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)\left[f(b)-f(a)\right]\right. \tag{5.5}$$

$$\left.-\frac{b-E(X)}{b-a}\right|$$

$$\le \frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\bigvee_a^t(f)\,dt+\frac{1}{32}(b-a)\bigvee_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}(f)$$

$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b(b-t)\bigvee_t^b(f)\,dt$$

$$\le \frac{1}{32}(b-a)\bigvee_a^b(f).$$

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