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ON A GEOMETRIC INEQUALITY OF KLAMKIN

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Abstract. Several sharpened versions and a reversed result of a Klamkin's geometric inequality are established. As a consequence, the classical fundamental triangle inequality is derived by one of our results.

1. Introduction

In 1975, Klamkin first gave important "the polar moment of inertia inequality" in [10] (see also [21, p.278]). As a consequence, the following compact geometric inequality is given:

For any triangle ABC and a point P in the plane

$$aR_1^2 + bR_2^2 + cR_3^2 \geq abc, \tag{1.1}$$

where a, b, c are the side lengths BC, CA, AB and R_1, R_2, R_3 the distances from P to the vertices A, B, C, respectively. Equality in (1.1) holds if and only if P is the incenter of triangle ABC.

At almost the same time, G.Bennett [6] and M.S.Klamkin [11] himself obtained the following generalization:

$$aR_1D_1 + bR_2D_2 + cR_3D_3 \geq abc, \tag{1.2}$$

where D_1, D_2, D_3 are the distances from another point Q in the plane to the vertices A, B, C respectively. Equality in (1.2) holds if and only if P and Q are isogonal conjugates with respect to the triangle ABC.

In [13], the author of this paper further generalized an equivalent form of (1.2) to the convex polygon (see also [16]). In a recent paper [14], we also prove that the following extension of (1.1):

$$aR_1^2 + bR_2^2 + cR_3^2 \leq a^2R_1 + b^2R_2 + c^2R_3 \tag{1.3}$$

holds for any interior P of ABC. Equality in (1.3) occurs if and only if P coincide with one of the vertices of triangle ABC. The author also conjectured that

$$aR_1^k + bR_2^k + cR_3^k \leq a^kR_1 + b^kR_2 + c^kR_3 \tag{1.4}$$

holds for any interior P with $k > 1$.

In this paper, we study the equivalent form of Klamkin's inequality (1.1):

$$\frac{R_1^2}{bc} + \frac{R_2^2}{ca} + \frac{R_3^2}{ab} \geq 1. \tag{1.5}$$

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We will establish several sharpened versions and a reversed result of this inequality.

In what follows, we will continuously use the above symbols. Also, we denote the distances from P to the sidelines BC, CA, AB by r_1, r_2, r_3 respectively, and denote the area, the semi-perimeter, the circumradius and the inradius of triangle ABC by S, s, R, r respectively. In addition, we also often make use of the symbols $\sum$ (cyclic sum) and $\prod$ (cyclic product). For examples:

$$\sum yza^2 = yza^2 + zxb^2 + xyc^2,$$

$$\sum \frac{R_1^2 - r_1^2}{bc} = \frac{R_1^2 - r_1^2}{bc} + \frac{R_2^2 - r_2^2}{ca} + \frac{R_3^2 - r_3^2}{ab},$$

$$\prod (b + c - a) = (b + c - a)(c + a - b)(a + b - c).$$

2. **Several sharpened inequalities**

We begin with an inconspicuous inequality first given by L.Carlitz and M.S.K.lamkin in [7], which states that for interior point P of triangle ABC the following inequality holds:

$$ar_1^2 + br_2^2 + cr_3^2 \geq 2sr^2, \tag{2.1}$$

with equality if and only if P is the incenter of triangle ABC.

In fact, by using Cauchy inequality and the following obvious inequality:

$$ar_1 + br_2 + cr_3 \geq 2S, \tag{2.2}$$

it is easily proven that (2.1) holds for any point P in the plane. M.S.K.lamkin also gave the exponential generalization of (2.1), see [7] or [21,p.301-302].

Dividing both sides of (2.1) by abc and using formula $abc = 4Rrs$, we get the equivalent inequality:

$$\frac{r_1^2}{bc} + \frac{r_2^2}{ca} + \frac{r_3^2}{ab} \geq \frac{r}{2R}, \tag{2.3}$$

with equality if and only if P is the incenter of triangle ABC.

If we add inequalities (1.5) and (2.3), then

$$\frac{R_1^2 + r_1^2}{bc} + \frac{R_2^2 + r_2^2}{ca} + \frac{R_3^2 + r_3^2}{ab} \geq 1 + \frac{r}{2R}. \tag{2.4}$$

Considering sharpened versions of this inequality, we find that

Theorem 2.1. *For any point P in the plane of triangle ABC, we have*

$$\frac{R_1^2 + r_1^2 - (r_2 - r_3)^2}{bc} + \frac{R_2^2 + r_2^2 - (r_3 - r_1)^2}{ca} + \frac{R_3^2 + r_3^2 - (r_1 - r_2)^2}{ab}$$
$$\geq 1 + \frac{r}{2R}, \tag{2.5}$$

with equality holding if and only if P is an interior point of triangle ABC and lies on the segment which is intersected by the line IO and $\triangle ABC$, where I and O are the incenter and circumcenter of $\triangle ABC$ respectively.

If P lies inside triangle ABC, then inequality (2.5) actually sharpens Klamkin's inequality (1.5). We shall prove this fact after proving the theorem. Theorem 2.1 has an interesting application, we put it in the last section.

In order to prove Theorem 2.1, we first give an identity which involving directed distances (cf.[17]).

Lemma 2.1. *Let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be directed distances from a point P in the plane of triangle ABC to the sidelines BC, CA, AB, then*

$$\sum \frac{R_1^2 + \vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} = 1 + \frac{r}{2R} + \frac{\left[\sum (b-c)(b+c-a)\vec{r_1}\right]^2}{8Rs^2r^3}. \tag{2.6}$$

Proof. Let (x, y, z) be the barycentric coordinates of point P with respect to the triangle ABC $(x+y+z \neq 0)$. By the following known formula (see, e.g. [21, p.278]):

$$(x+y+z)^2 R_1^2 = (x+y+z)(yc^2 + zb^2) - (yza^2 + zxb^2 + xyc^2), \tag{2.7}$$

we have

$$\sum \frac{R_1^2}{bc} = \frac{\sum x \sum a(yc^2 + zb^2) - \sum a \sum yza^2}{abc(\sum x)^2}$$

from which, by a simple calculation, we obtain

$$\sum \frac{R_1^2}{bc} = 1 + \frac{H_1}{abc\left(\sum x\right)^2}, \tag{2.8}$$

where

$$H_1 = \sum bc(b+c-a)x^2 - \sum a(c+a-b)(a+b-c)yz. \tag{2.9}$$

On the other hand, by the known formula:

$$\vec{r_1} = \frac{2xS}{a(x+y+z)} \tag{2.10}$$

and the equivalent form of Heron's formula

$$16S^2 = \sum a \prod (b+c-a), \tag{2.11}$$

we easily obtain

$$\sum \frac{(\vec{r_2} - \vec{r_3})^2}{bc} = \frac{H_2 \sum a \prod (b+c-a)}{4(abc)^3 \left(\sum x\right)^2}, \tag{2.12}$$

where

$$H_2 = \sum a^3 (yc - zb)^2. \tag{2.13}$$

Also, using (2.10) and the following known identity in $\triangle ABC$:

$$\frac{\prod (b+c-a)}{abc} = \frac{2r}{R}, \tag{2.14}$$

one has

$$\begin{aligned}
&\sum \frac{r_1^2}{bc} - \frac{r}{2R} \\
&= \frac{4S^2 \sum bcx^2}{(abc)^2 \left(\sum x\right)^2} - \frac{\prod(b+c-a)}{4abc} \\
&= \frac{16S^2 \sum bcx^2 - abc \prod(b+c-a) \left(\sum x\right)^2}{4(abc)^2 \left(\sum x\right)^2} \\
&= \frac{\prod(b+c-a) \left[\sum a \sum bcx^2 - abc \left(\sum x\right)^2\right]}{4(abc)^2 \left(\sum x\right)^2}.
\end{aligned}$$

And then

$$\sum \frac{r_1^2}{bc} - \frac{r}{2R} = \frac{H_3 \prod(b+c-a)}{4(abc)^2 \left(\sum x\right)^2}, \tag{2.15}$$

where

$$H_3 = \sum bc(b+c)x^2 - 2abc \sum yz. \tag{2.16}$$

Note that $r_1^2 = \vec{r_1}^2$ etc., from (2.8) and (2.15), we get

$$\frac{r}{2R} - \sum \frac{\vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} = \frac{(H_2 \sum a - abcH_3) \prod(b+c-a)}{4(abc)^3 \left(\sum x\right)^2}.$$

We further obtain by using (2.13) and (2.16) that

$$\frac{r}{2R} - \sum \frac{\vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} = \frac{H_4 \prod(b+c-a)}{4(abc)^3 \left(\sum x\right)^2}, \tag{2.17}$$

where

$$H_4 = \sum b^2c^2(b+c)^2x^2 - 2abc \sum a(a^2+ab+ac-bc)yz. \tag{2.18}$$

Finally, from (2.8) and (2.17), one has

$$\begin{aligned}
&\sum \frac{R_1^2 + \vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} \\
&= \sum \frac{R_1^2}{bc} + \sum \frac{\vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} \\
&= 1 + \frac{H_1}{abc \left(\sum x\right)^2} + \frac{r}{2R} - \frac{H_4 \prod(b+c-a)}{4(abc)^3 \left(\sum x\right)^2} \\
&= 1 + \frac{r}{2R} + \frac{4(abc)^2 H_1 - H_4 \prod(b+c-a)}{4(abc)^3 \left(\sum x\right)^2}.
\end{aligned}$$

But, it is easy to check the following identity:

$$4(abc)^2 H_1 - H_4 \prod(b+c-a) = \sum a \left[\sum bc(b-c)(b+c-a)x\right]^2. \tag{2.19}$$

Therefore we have the following identity:

$$\begin{aligned}&\sum \frac{R_1^2 + \vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} \\ &= 1 + \frac{r}{2R} + \frac{\left[\sum bc(b-c)(b+c-a)x\right]^2 \sum a}{4(abc)^3 \left(\sum x\right)^2}.\end{aligned} \tag{2.20}$$

Using (2.10) and relations $\sum a = 2s, S = rs, abc = 4Rrs$ in $\triangle ABC$, we immediately obtain identity (2.6) from (2.20). This completes the proof of Lemma 2.1.

We now prove Theorem 2.1.

Proof of Theorem 2.1 By Lemma 2.1, we see that for any point P in the plane

$$\sum \frac{R_1^2 + \vec{r_1}^2 - (\vec{r_2} - \vec{r_3})^2}{bc} \geq 1 + \frac{r}{2R}, \tag{2.21}$$

with equality holding if and only if

$$\sum (b-c)(b+c-a)\vec{r_1} = 0. \tag{2.22}$$

Since $r_1 = |\vec{r_1}| \geq \vec{r_1}, r_2 = |\vec{r_2}| \geq \vec{r_2}, r_3 = |\vec{r_3}| \geq \vec{r_3}$, then we have

$$\sum \frac{(\vec{r_2} - \vec{r_3})^2}{bc} \geq \sum \frac{(r_2 - r_3)^2}{bc}. \tag{2.23}$$

Equality holds if and only if $r_1 = \vec{r_1}, r_2 = \vec{r_2}, r_3 = \vec{r_3}$, i.e., P lies inside of $\triangle ABC$.

According to inequalities (2.21) and (2.23), we conclude that the inequality (2.5) holds for any point in the plane. Also, identity (2.22) implies by (2.10) that

$$\sum bc(b-c)(b+c-a)x = 0, \tag{2.24}$$

which expresses a line passing through point (x, y, z). Since $x = a, y = b, z = c$ satisfy (2.24), then the incenter $I(a, b, c)$ of $\triangle ABC$ lies on the line. On the other hand, it is easy to check the following identity:

$$\sum bc(b-c)(b+c-a)a_0 = 0, \tag{2.25}$$

where $a_0 = a^2(b^2+c^2-a^2)$ etc. We hence conclude again that the circumcenter $O(a_0, b_0, c_0)$ of $\triangle ABC$ lies on the above line. Therefore, (2.24) expresses the line IO and there is equality in (2.21) only if P lies on this line. Thus, noting the equality condition of (2.23), we deduce that the equality in (2.5) holds if and only if P lies inside $\triangle ABC$ and on the line IO at the same time. This completes the proof of Theorem 2.1. □

Remark 2.1. For any interior point P of triangle ABC, we have the following inequality:

$$\frac{r_1^2 - (r_2 - r_3)^2}{bc} + \frac{r_2^2 - (r_3 - r_1)^2}{ca} + \frac{r_3^2 - (r_1 - r_2)^2}{ab} \leq \frac{r}{2R}, \tag{2.26}$$

which shows that inequality (2.5) improves Klamkin's inequality (1.5) if P lies inside triangle ABC. This inequality can be proven as follows:

When P lies inside triangle ABC, from (2.17) we have

$$\frac{r}{2R}-\sum\frac{r_1^2-(r_2-r_3)^2}{bc}=\frac{H_4\prod(b+c-a)}{4(abc)^3\left(\sum x\right)^2}. \tag{2.27}$$

Hence, we need to show that $H_4\geq 0$, namely,

$$\sum b^2c^2(b+c)^2x^2-2abc\sum a(a^2+ab+ac-bc)yz\geq 0. \tag{2.28}$$

Replacing in this inequality x,y,z by xa,yb,zc respectively and then dividing both sides by $(abc)^2$, we know it is equivalent to

$$\sum(b+c)^2x^2-2\sum yz(a^2+ab+ac-bc)\geq 0. \tag{2.29}$$

However, we recall the following classical Wolstenholme's inequality (see [26, p.69] or [21, p.421]):

$$\sum a^2x^2-\sum yz(b^2+c^2-a^2)\geq 0, \tag{2.30}$$

with equality if and only if $x=y=z$. If we apply (2.30) to the triangle with side lengths $b+c,c+a$ and $a+b$, inequality (2.29) follows at once and then (2.26) is proved.

Incidentally, we have known that inequality (2.26) is not valid when P lies outside triangle ABC.

Remark 2.2. Inequality (2.21) is equivalent to

$$\sum\frac{R_1^2+(r-\vec{r_1})^2-(\vec{r_2}-\vec{r_3})^2}{bc}\geq 1. \tag{2.31}$$

Since we have the following the identity:

$$\sum\frac{\vec{r_1}^2-(r-\vec{r_1})^2}{bc}=\frac{r}{2R}, \tag{2.32}$$

which is easily obtained by the identity $\sum a\vec{r_1}=2S$ and the relation $\sum\frac{1}{bc}=\frac{1}{2Rr}$ in $\triangle ABC$.

In particular, when P lies inside ABC, (2.31) reduces to

$$\sum\frac{R_1^2+(r-r_1)^2-(r_2-r_3)^2}{bc}\geq 1, \tag{2.33}$$

which is equivalent with inequality (2.5). We conjecture here that (2.33) holds strictly for an external point P of ABC.

Klamkin's inequality (1.5) and previous inequality (2.3) also motivates the author to find the following inequality:

Theorem 2.2. *For any point P in the plane of triangle ABC, we have*

$$\frac{R_1^2-r_1^2}{bc}+\frac{R_2^2-r_2^2}{ca}+\frac{R_3^2-r_3^2}{ab}\geq 1-\frac{r}{2R}, \tag{2.34}$$

with equality holding if and only if P is the incenter of triangle ABC.

It is interesting to compare inequality (2.4) with (2.34). Obviously, inequality (2.3) shows that (2.34) improves Klamkin's inequality (1.5).

In order to prove Theorem 2.2, we need the following known lemma (for the proofs, see e.g.[17], [18]).

Lemma 2.2 *Let* $p_1, p_2, p_3, q_1, q_2, q_3$ *be real numbers such that* $p_1 > 0, p_2 > 0, p_3 > 0, 4p_2p_1 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0,$ *and*

$$D_0 \equiv 4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geq 0. \tag{2.35}$$

Then the ternary quadratic inequality:

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy \tag{2.36}$$

holds for all real numbers x, y, z. *If* $x, y, z \neq 0$, *then the equality in (2.36) holds if and only if* $D_0 = 0$ *and* $(2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z$.

Proof of Theorem 2.2. From previous identities (2.8) and (2.15), we get

$$\sum \frac{R_1^2 - r_1^2}{bc} = 1 - \frac{r}{2R} + \frac{4abcH_1 - \prod(b+c-a)H_3}{4(abc)^2\left(\sum x\right)^2}.$$

Hence, to prove inequality (2.34), we have to prove that

$$4abcH_1 - \prod(b+c-a)H_3 \geq 0 \tag{2.37}$$

Substituting (2.9) and (2.16) into the above and arranging, we know it is equivalent to

$$\sum bc[ab + ac + (b-c)^2](b+c-a)^2x^2$$
$$-2abc\sum(c+a-b)(a+b-c)(3a-b-c)yz \geq 0. \tag{2.38}$$

Replacing x, y, z by $x/(b+c-a), y/(c+a-b), z/(a+b-c)$ respectively, we see that (2.38) is equivalent to the following quadratic inequality required to prove:

$$m_1x^2 + m_2y^2 + m_3z^2 - n_1yz - n_2zx - n_3xy \geq 0, \tag{2.39}$$

where

$$m_1 = bc\left[a(b+c) + (b-c)^2\right],$$
$$m_2 = ca\left[b(c+a) + (c-a)^2\right],$$
$$m_3 = ab\left[c(a+b) + (a-b)^2\right],$$

$$n_1 = 2abc(3a-b-c),$$
$$n_2 = 2abc(3b-c-a),$$
$$n_3 = 2abc(3c-a-b).$$

With the help of the famous mathematical software Maple (we used Maple 15), it is not difficult to check the following identities:

$$4m_2m_3 - n_1^2 = 4bc(a+b+c)a^3(b+c-a)^2, \tag{2.40}$$
$$4m_1m_2m_3 - (n_1n_2n_3 + m_1n_1^2 + m_2n_2^2 + m_3n_3^2) = 0, \tag{2.41}$$
$$2m_1n_1 + n_2n_3 = 4a(a+b+c)(c+a-b)(a+b-c)b^2c^2. \tag{2.42}$$

From (2.40), we see that $4m_2m_3 - n_1^2 > 0$ and its two analogues hold too. Thus, by Lemma 2.2 and (2.41), we conclude that inequality (2.39) holds for any real numbers x, y, z. Also by Lemma 2.2, (2.42) and its two analogues, we know that the equality in (2.39) holds if and only if $x : y : z = a(b+c-a) : b(c+a-b) : c(a+b-c)$. Further, the equality in (2.38) holds only when $x : y : z = a : b : c$, which means that the equality in (2.34) holds if and only if the barycentric coordinates of P is (a, b, c), i.e., P coincide with the incentre of $\triangle ABC$. This completes the proof of Theorem 2.2. □

Remark 2.3. The previous inequality (2.3) can be strengthened to the following:

$$\frac{r_1^2-(r-r_1)^2}{bc}+\frac{r_2^2-(r-r_2)^2}{ca}+\frac{r_3^2-(r-r_3)^2}{bc}\geq\frac{r}{2R}. \tag{2.43}$$

Since we have

$$\sum\frac{r_1^2-(r-r_1)^2}{bc}=\sum\frac{-r^2+2rr_1}{bc}=-r^2\sum\frac{1}{bc}+\frac{2r}{abc}\sum ar_1$$
$$\geq -r^2\frac{\sum a}{abc}+\frac{2r}{abc}\cdot 2S=\frac{r}{R}-\frac{r}{2R}=\frac{r}{2R},$$

where we used inequality (2.2) and the relations $\sum a = 2s, abc = 4Rrs$ in triangle ABC.

Adding (2.34) and (2.43) gives

$$\frac{R_1^2-(r-r_1)^2}{bc}+\frac{R_2^2-(r-r_2)^2}{ca}+\frac{R_3^2-(r-r_3)^2}{bc}\geq 1, \tag{2.44}$$

which is an obvious improvement of Klamkin's inequality (1.5). Indeed, (2.44) is equivalent with (2.34) when P lies inside triangle ABC.

Adding inequalities of Theorem 2.1 and 2.2 and then dividing both sides by $\frac{1}{2}$, we immediately obtain the following inequality:

Theorem 2.3. *For any point P in the plane of triangle ABC, we have*

$$\frac{R_1^2-\frac{1}{2}(r_2-r_3)^2}{bc}+\frac{R_2^2-\frac{1}{2}(r_3-r_1)^2}{ca}+\frac{R_3^2-\frac{1}{2}(r_1-r_2)^2}{ab}\geq 1, \tag{2.45}$$

with equality holding if and only if P is the incenter of triangle ABC.

Inequality (2.45) is an obvious sharpened version of Klamkin's inequality (1.5). In the next section (see Remark 3.4 below), we will give another direct proof based on an identity.

Remark 2.4 The constants $\frac{1}{2}$ in inequality (2.45) is the best possible, i.e., it cannot be replaced by a larger constant. This fact can be proven as follows: Suppose that the following inequality holds:

$$\sum\frac{R_1^2-k(r_2-r_3)^2}{bc}\geq 1, \tag{2.46}$$

where $k > 0$ is a constant. We denote the corresponding medians and the altitudes of ABC by m_a, m_b, m_c and h_a, h_b, h_c respectively. If we let P coincide with the centroid of ABC, then $R_1 = \frac{2}{3}m_a, r_1 = \frac{1}{3}h_a$ etc. Moreover, it follows from (2.46) that

$$\sum\frac{4m_a^2-k(h_b-h_c)^2}{bc}\geq 9, \tag{2.47}$$

which is equivalent to

$$4\sum m_a^2 - 9abc \geq k\sum k(h_b - h_c)^2. \tag{2.48}$$

Now, we consider an isosceles triangle with sides $b = c = 1$ and $a = 2-2t(0 < t < 1)$. Then, it is easy to obtain that

$$S = (1-t)\sqrt{2t-t^2},\ m_a = h_a = \sqrt{t(2-t)},\ m_b = m_c = \frac{1}{2}\sqrt{8t^2-16t+9}.$$

Further, we easily get

$$4\sum m_a^2 - 9abc = 2t(2t-1)^2,\ \sum k(h_b-h_c)^2 = 2t(2-t)(2t-1)^2.$$

In this case, it follows from (2.48) that $k \leq \frac{1}{2-t}$. Hence, by letting $t \to 0$ we then conclude that $k \leq \frac{1}{2}$, which means that $k = \frac{1}{2}$ is the best possible for inequality (2.46).

3. A reversed inequality for the acute triangle

For the acute triangle ABC and its interior point P, we find the following reversed inequality similar to (2.45):

Theorem 3.1. *For any interior point P of the acute triangle ABC, we have*

$$\frac{R_1^2-(r_2-r_3)^2}{bc} + \frac{R_2^2-(r_3-r_1)^2}{ca} + \frac{R_3^2-(r_1-r_2)^2}{ab} \leq 1, \tag{3.1}$$

with equality holding if and only if P is the incenter of the acute triangle ABC.

We gave a complicated proof of Theorem 2.4 at first (we used Lemma 2.2 above). A simple proof is obtained after finding the following identity which may be of independent interest.

Lemma 3.1 *Let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be directed distances from a point P in the plane of triangle ABC to the sidelines BC, CA, AB, then*

$$\sum \frac{R_1^2}{bc} = 1 + \frac{1}{2s}\sum \frac{(\vec{r_2}-\vec{r_3})^2}{s-a}. \tag{3.2}$$

Proof. Applying the formulae (2.10), (2.11) and $s = (a+b+c)/2$, one has

$$\begin{aligned}&\sum \frac{(\vec{r_2}-\vec{r_3})^2}{s-a}\\ &= \frac{4S^2}{\left(\sum x\right)^2}\sum \frac{1}{s-a}\left(\frac{y}{b}-\frac{z}{c}\right)^2\\ &= \frac{4s}{\left(\sum x\right)^2}\sum \frac{(s-b)(s-c)(yc-zb)^2}{b^2c^2}.\end{aligned}$$

Hence,

$$\sum \frac{(\vec{r_2}-\vec{r_3})^2}{s-a} = \frac{H_5\sum a}{2(abc)^2\left(\sum x\right)^2}, \tag{3.3}$$

where

$$H_5=\sum(c+a-b)(a+b-c)a^2(yc-zb)^2. \tag{3.4}$$

By (3.3) and previous identity (2.8), we get

$$\sum\frac{R_1^2}{bc}-\frac{1}{2s}\sum\frac{(\vec{r_2}-\vec{r_3})^2}{s-a}-1=\frac{2abcH_1-H_5}{2(abc)^2\left(\sum x\right)^2}.$$

But, by using (2.9) and (3.4), it is easy to check that

$$2abcH_1-H_5=0. \tag{3.5}$$

Thus, the desired identity is proved. □

Remark 3.1. By identity (3.2) and the known identity (see [21, p.280]):

$$aR_1^2+bR_2^2+cR_3^2=abc+2sPI^2, \tag{3.6}$$

where I is the the incenter of triangle ABC, we obtain new identity:

$$\sum\frac{(\vec{r_2}-\vec{r_3})^2}{s-a}=\frac{4s^2}{abc}PI^2. \tag{3.7}$$

Also, by the identity

$$8(s-a)s^2-4abc=2bc(b+c-a)+(a+b+c)(b^2+c^2-a^2), \tag{3.8}$$

we see that inequality $2(s-a)s^2>2abc$ is valid for the acute triangle ABC. Thus, by (3.7), we obtain the following interesting inequality:

$$(\vec{r_2}-\vec{r_3})^2+(\vec{r_3}-\vec{r_1})^2+(\vec{r_1}-\vec{r_2})^2\geq 2PI^2, \tag{3.9}$$

which holds for the acute triangle ABC. Particularly, the following inequality:

$$(r_2-r_3)^2+(r_2-r_3)^2+(r_2-r_3)^2\geq 2PI^2 \tag{3.10}$$

holds for any interior P of the acute $\triangle ABC$.

We now prove Theorem 3.1

Proof of Theorem 3.1. When P lies inside triangle ABC, the identity (3.2) becomes

$$\sum\frac{R_1^2}{bc}=1+\frac{1}{2s}\sum\frac{(r_2-r_3)^2}{s-a}. \tag{3.11}$$

Consequently, we have

$$\begin{aligned}&\sum\frac{R_1^2-(r_2-r_3)^2}{bc}-1\\&=\frac{1}{2s}\sum\frac{(r_2-r_3)^2}{s-a}-\sum\frac{(r_2-r_3)^2}{bc}\\&=\sum\frac{bc-2s(s-a)}{2sbc(s-a)}(r_2-r_3)^2\\&=-\frac{1}{4s}\sum\frac{(b^2+c^2-a^2)(r_2-r_3)^2}{bc(s-a)}\leq 0,\end{aligned}$$

where we used $s=(a+b+c)/2$ and the inequality $b^2+c^2-a^2>0$ in the acute triangle ABC. Thus, inequality (3.1) is proved and it is easily seen that the equality in (3.1) holds if and only if P is the incenter of the triangle ABC. Theorem 3.1 is proved.

Remark 3.2 In the same way as in Remark 2.4, we can prove that the best possible constant k such that (2.46) holds reversely for the acute triangle ABC is that $k = 1$.

Remark 3.3 When P lies inside triangle ABC, it is easy to prove the following identity:

$$R_1^2 - (r_2 - r_3)^2 = \frac{r_2 r_3}{\sin^2 \frac{A}{2}} + \frac{(r_2 - r_3)^2 \cos^2 A}{\sin^2 A}, \tag{3.12}$$

where A is the angle of ABC. Thus, for any interior point P, we have the strict inequality

$$R_1^2 - (r_2 - r_3)^2 > 0. \tag{3.13}$$

Hence, we can apply Cauchy inequality to inequality (3.1) and obtain the following interesting inequality

$$\sqrt{R_1^2 - (r_2 - r_3)^2} + \sqrt{R_2^2 - (r_3 - r_1)^2} + \sqrt{R_3^2 - (r_1 - r_2)^2} \leq \sqrt{bc + ca + ab}, \tag{3.14}$$

which holds for interior point P of the acute triangle ABC.

Remark 3.4 According to Lemma 3.1, we obtain a direct proof of inequality (2.45, is as follows: For any point P in the plane, we obviously have the following inequality similar to (2.23):

$$\sum \frac{(\vec{r_2} - \vec{r_3})^2}{s - a} \geq \sum \frac{(r_2 - r_3)^2}{s - a}, \tag{3.15}$$

with equality if and only if P lies inside $\triangle ABC$. Thus, it follows from (3.2) that

$$\sum \frac{R_1^2}{bc} \geq 1 + \frac{1}{2s} \sum \frac{(r_2 - r_3)^2}{s - a}. \tag{3.16}$$

Hence, we have

$$\begin{aligned}
&\sum \frac{R_1^2 - \frac{1}{2}(r_2 - r_3)^2}{bc} - 1 \\
&= \sum \frac{R_1^2}{bc} - 1 - \frac{1}{2} \sum \frac{(r_2 - r_3)^2}{bc} \\
&\geq \frac{1}{2s} \sum \frac{(r_2 - r_3)^2}{s - a} - \frac{1}{2} \sum \frac{(r_2 - r_3)^2}{bc} \\
&= \frac{1}{2} \sum \left[\frac{1}{s(s - a)} - \frac{1}{bc} \right] (r_2 - r_3)^2 \\
&= \frac{2}{s} \sum \frac{(s - b)(s - c)}{bc(s - a)} (r_2 - r_3)^2 \geq 0.
\end{aligned}$$

Thus, inequality (2.45) is proved.

4. An application of Theorem 2.1

In this section, we give an interesting application of Theorem 2.1. Namely, we apply Theorem 2.1 to derive the fundamental triangle inequality which involves the semi-perimeter, the circumradius and the inradius of a triangle.

Theorem 4.1 *In any triangle ABC, we have*

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + r(4R + r)^3 \le 0, \tag{4.1}$$

with equality holding if and only if the triangle ABC is isosceles.

Proof of Theorem 4.1. In Theorem 2.1, we let point P coincide with the centroid G of $\triangle ABC$, then $R_1 = \frac{2}{3}m_a, r_1 = \frac{1}{3}h_a$, etc. (the symbols m_a, h_a are the same as in (2.47)), then it follows from (2.5) that

$$\sum \frac{4m_a^2 + h_a^2 - (h_b - h_c)^2}{bc} \ge 9 + \frac{9r}{2R}. \tag{4.2}$$

By the relation $bc = 2Rh_a$ in $\triangle ABC$, we get

$$\sum \frac{h_a^2 - (h_b - h_c)^2}{bc} = \frac{1}{4R^2} \sum \frac{b^2c^2 - a^2(b-c)^2}{bc},$$

and then

$$\sum \frac{h_a^2 - (h_b - h_c)^2}{bc} = \frac{abc \sum bc - \sum a^3(b-c)^2}{4abcR^2}. \tag{4.3}$$

Again, we have the following identity:

$$\sum a^3(b-c)^2 = \sum a \left(\sum bc\right)^2 - 2abc\left(\sum a\right)^2 - abc\sum bc - 2abc\sum a^2, \tag{4.4}$$

which is readily checked by expanding. Using $\sum a = 2s$ and the following known identities (see e.g. [14, p.52]):

$$abc = 4Rrs, \tag{4.5}$$

$$\sum bc = s^2 + 4Rr + s^2, \tag{4.6}$$

$$\sum a^2 = 2(s^2 - 4Rr - r^2), \tag{4.7}$$

we further obtain

$$\sum a^3(b-c)^2 = 2s^5 - (36Rr - 4r^2)s^3 + (80R^2 + 28Rr + 2r^2)r^2 s. \tag{4.8}$$

From (4.3), by using (4.5), (4.6) and (4.8), we obtain

$$\sum \frac{h_a^2 - (h_b - h_c)^2}{bc} = -\frac{s^4 - 2r(10R - r)s^2 + (4R + r)(8R + r)r^2}{8rR^3}. \tag{4.9}$$

Finally, using (4.9) and the known identity (see [21, p.211]):

$$\sum \frac{m_a^2}{bc} = \frac{s^2 + 2Rr + 5r^2}{8Rr}, \tag{4.10}$$

we easily obtain the following identity:

$$\sum \frac{4m_a^2 + h_a^2 - (h_b - h_c)^2}{bc} - \frac{9r}{2R} - 9$$
$$= \frac{-s^4 + 2(2R^2 + 10Rr - r^2)s^2 - r(4R + r)^3}{8rR^3}, \tag{4.11}$$

which together with (4.2) give the fundamental triangle inequality (4.1).

Now we determine the equality condition of (4.1). According to Theorem 2.1, there is equality in (4.2) if and only if the centroid $G(1, 1, 1)$, the incenter

$I(a,b,c)$ and the circumcenter $O(a_0,b_0,c_0)$ are collinear, where $a_0 = a^2(b^2 + c^2 - a^2)$ etc., which implies that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a_0 & b_0 & c_0 \end{vmatrix} = 0. \tag{4.12}$$

But, it is not difficult to check that the value of the determinant is equal to $-\prod(b-c)\left(\sum a\right)^2$. Thus, by (4.12), we conclude that the triangle ABC must be isosceles. i.e., the equality in (4.2) and then the one in (4.1) hold if and only if $\triangle ABC$ is isosceles. This completes the proof of Theorem 4.1. □

The fundamental triangle inequality (3.16) was first proved by Sonday [4, 13.8] in 1891. Blundon [2] proved the following double inequalities from (3.16)

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2 - 2Rr} \\ &\le s^2 \le 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr} \end{aligned} \tag{4.13}$$

are the strongest possible inequalities of the form $q(R,r) \le s^2 \le Q(R,r)$, where $q(R,r)$ and $Q(R,r)$ are functions of R,r. It has the following important consequences, i.e, the Gerretsen's inequalities (see [9], [21, p.45]):

$$16Rr - 5r^2 \le s^2 \le 4R^2 + 4Rr + 3r^2. \tag{4.14}$$

These inequalities has drawn a large number of research papers involving its new proofs, various generalizations , variations and applications etc. For more details we refer the reader to the monograph [21] and papers [1, 8, 12, 15, 19, 20], [21]-[25] and [27, 28].

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