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INEQUALITIES AND ASYMPTOTIC EXPANSIONS FOR THE CONSTANTS OF LANDAU AND LEBESGUE

CHAO-PING CHEN* AND JUNESANG CHOI†

ABSTRACT. The constants of Landau and Lebesgue are defined, for all integers $n \geq 0$, in order, by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t,$$

which play important roles in the theories of complex analysis and Fourier series, respectively. We establish new bounds for the Landau constants G_n in terms of the Digamma and Polygamma functions, which improves all of earlier involved results, for example, those by Alzer who provided sharp bounds for G_n in terms of the Digamma function. We also establish inequalities for the Lebesgue constants $L_{n/2}$ and then apply it to derive the asymptotic expansion for $L_{n/2}$.

1. Introduction and Preliminaries

The Landau constants are defined by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1,\,2,\,3,\,\ldots\}\right), \tag{1.1}$$

which play an important role in the theory of complex analysis. More precisely, in 1913, Landau [17] proved that if $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ is an analytic function in the unit disc $\mathcal{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{C}$ being the set of complex numbers, which satisfies $|f(z)| < 1$ for all $z \in \mathcal{D}$, then $|\sum_{k=0}^{n} a_k| \leq G_n$ $\left(n \in \mathbb{N}_0\right)$ whose bounds are seen to be optimal.

The Lebesgue constants are defined by

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t \quad \left(n \in \mathbb{N}_0\right), \tag{1.2}$$

which play an important role in the theory of Fourier series. More precisely, in 1906, Lebesgue [18] proved the following result: Assume a function f is integrable on the interval $[-\pi,\,\pi]$ and $S_n(f,x)$ is the nth partial sum of the Fourier series of f. That is,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos(kt)\ \mathrm{d}t \quad \left(k \in \mathbb{N}_0\right) \quad \text{and} \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\sin(kt)\ \mathrm{d}t \quad \left(k \in \mathbb{N}\right)$$

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*Corresponding Author.

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and

$$S_n(f,x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} \Big(a_k \cos(kx) + b_k \sin(kx)\Big) \quad \big(n \in \mathbb{N}_0\big),$$

where the empty sum is (as usual, throughout this paper) understood to be nil. If $|f(x)| \leq 1$ for all $x \in [-\pi,\, \pi]$, then

$$S_n(f,x) \leq L_n \quad \big(n \in \mathbb{N}_0\big). \tag{1.3}$$

It is noted that L_n is the smallest possible constant for which the inequality (1.3) holds true for all continuous functions f on $[-\pi,\, \pi]$.

Here, in this paper, we aim at presenting new bounds for the Landau constants G_n in terms of the Digamma and Polygamma functions, which improves all of earlier related results, for example, those by Alzer [3] who provided sharp bounds for G_n in terms of the Digamma function. We also establish inequalities for the Lebesgue constants $L_{n/2}$ and then apply it to derive the asymptotic expansion for $L_{n/2}$.

For this purpose, we recall the following functions and Lemmas. The familiar (Euler's) Gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, \mathrm{d}t \quad (\Re(z) > 0), \tag{1.4}$$

which is one of the simplest and most important special functions and has several other important equivalent forms (see, *e.g.*, [25, Section 1.1]), knowledge of whose properties is a prerequisite for the study of many other special functions. The Gamma function $\Gamma(z)$ arises in many areas of mathematics such as applied mathematics as well as mathematical analysis. The origin, history, and development of the Gamma function $\Gamma(z)$ are described very nicely by Davis [9].

The logarithmic derivative of the Gamma function $\Gamma(z)$:

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\{\ln\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t \tag{1.5}$$

is known as the Psi (or Digamma) function. The successive derivatives of the Psi function $\psi(z)$:

$$\psi^{(n)}(z) := \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\psi(z)\} \quad (n \in \mathbb{N}) \tag{1.6}$$

are called the Polygamma functions. In particular, the functions $\psi'(z)$ and $\psi^{(2)}(z)$ are called the Trigamma and Tetragamma functions (see, *e.g.*, [1, p. 260]).

The following lemma is required in the sequel.

Lemma 1.1 ([27])**.** *The following Brouncker's continued fraction formula holds true*

$$\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 = \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\ddots}}}} \qquad \big(n \in \mathbb{N}_0\big). \tag{1.7}$$

Very recently, Granath [14] derived the asymptotic expansions for the Landau constants (1.1) and related inequalities by using Brouncker's continued fraction formula (1.7).

By (1.7), one finds the following inequality [14, pp. 741–742]:

$$\frac{16(19+92n+96n^2+128n^3)}{105+704n+1920n^2+2048n^3+2048n^4} = \frac{4}{1+4n+\frac{1^2}{2+8n+\frac{3^2}{2+8n+\frac{5^2}{2+8n}}}}$$

$$< \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 < \frac{4}{1+4n+\frac{1^2}{2+8n+\frac{3^2}{2+8n}}} = \frac{4(13+32n+64n^2)}{15+92n+192n^2+256n^3} \quad (n \in \mathbb{N}). \tag{1.8}$$

2. The Landau Constants

Landau studied the asymptotic behavior of G_n to show that

$$G_n \sim \frac{1}{\pi}\ln n \quad (n \to \infty). \tag{2.1}$$

Watson [28] continued this investigation to prove the asymptotic formula:

$$G_n = \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)\ , \tag{2.2}$$

where

$$c_0 = \frac{1}{\pi}(\gamma + 4\ln 2) = 1.06627\,58532\ldots. \tag{2.3}$$

Here γ denotes the Euler-Mascheroni constant defined by (see, for details, [12, Section 1.5] and [25, Section 1.2])

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log\, n\right) \cong 0.57721\,56649\ \ldots. \tag{2.4}$$

In what follows, c_0 is referred to the constant in (2.3). Inspired by formula (2.2), Brutman [4] discovered upper and lower bounds for G_n:

$$1 + \frac{1}{\pi}\ln(n+1) \le G_n < c_0 + \frac{1}{\pi}\ln(n+1) \qquad \big(n \in \mathbb{N}_0\big). \tag{2.5}$$

New bounds for G_n were given by Falaleev [10] who proved

$$c_0 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) < G_n \le 1.0976 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) \qquad \big(n \in \mathbb{N}_0\big). \tag{2.6}$$

In fact, Falaleev's approximation

$$G_n \approx c_0 + \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right)$$

is better than Brutman's approximation

$$G_n \approx c_0 + \frac{1}{\pi}\ln(n+1)\ ,$$

since, as $n \to \infty$,

$$G_n - c_0 - \frac{1}{\pi}\ln\left(n+\frac{3}{4}\right) = O\left(\frac{1}{n^2}\right) \quad \text{and} \quad G_n - c_0 - \frac{1}{\pi}\ln(n+1) = O\left(\frac{1}{n}\right)$$

(see, *e.g.*, [5]). Zhao [30, Theorem 1] established the following two-sided inequality:

$$\begin{aligned}\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}&<G_n\\&<\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{3}{128(n+1)^3},\end{aligned}\tag{2.7}$$

which implies Watson's asymptotic formula (2.2).

Another direction for developing the problem of approximation of G_n was initiated by Cvijović and Klinowski [8] who gave some estimates in terms of the Digamma function ψ, namely,

$$c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)<G_n<1.0725+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)\qquad\big(n\in\mathbb{N}_0\big);\tag{2.8}$$

$$0.9883+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)<G_n<c_0+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)\qquad\big(n\in\mathbb{N}_0\big).\tag{2.9}$$

Alzer [3, Theorem 1] established sharp inequalities for G_n in terms of the Digamma function:

$$c_0+\frac{1}{\pi}\psi\left(n+\alpha\right)<G_n\leq c_0+\frac{1}{\pi}\psi\left(n+\beta\right)\qquad\big(n\in\mathbb{N}_0\big)\tag{2.10}$$

with the best possible constants

$$\alpha=\frac{5}{4}\quad\text{and}\quad\beta=\psi^{-1}\big(\pi(1-c_0)\big)=1.26621\ldots\ .$$

Alzer [3, Remark 1] affirmed that, for all $n\in\mathbb{N}$, the upper and lower bounds for G_n given in (2.10) improve the bounds presented in (2.5) to (2.9). In fact, we have

$$G_n=c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+O(n^{-2})\quad(n\to\infty)\tag{2.11}$$

(see, *e.g.*, [5, Theorem 4]).

Very recently, some sharp inequalities and asymptotic expansions for G_n have been established (see, *e.g.*, [5, 6, 7, 14, 20, 21, 22, 23, 24]).

Theorem 2.1 below establishes new asymptotic expansion for the Landau constants G_n in terms of the digamma and polygamma functions.

Theorem 2.1. *The Landau constants G_n has the following asymptotic expansion*

$$G_n\sim c_0+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)+\frac{1}{\pi}\sum_{j=1}^{\infty}c_{2j}\psi^{(2j)}\left(n+\frac{5}{4}\right)\quad(n\to\infty),\tag{2.12}$$

where the coefficients c_{2j} are given by

$$c_{2j}=\sum_{k=0}^{2j}\sum_{m=0}^{k}(-1)^m\binom{2m}{m}^2\binom{2j}{k}\frac{m!S(k,m)}{(2j)!\cdot4^{2j-k+2m}}\qquad(j\in\mathbb{N})\tag{2.13}$$

and $S(k,m)$ denotes Stirling numbers of the second kind. Namely,

$$\begin{aligned}G_n\sim c_0&+\frac{1}{\pi}\psi\left(n+\frac{5}{4}\right)-\frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right)+\frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right)\\&-\frac{173}{47185920\pi}\psi^{(6)}\left(n+\frac{5}{4}\right)+\frac{22931}{338228674560\pi}\psi^{(8)}\left(n+\frac{5}{4}\right)-\cdots.\end{aligned}\tag{2.14}$$

Proof. It is known [22] that, for $n \in \mathbb{N}_0$ and $0 < h < \frac{3}{2}$,

$$G_n = \frac{1}{\pi} \ln(n+h) + c_0 + \frac{1}{\pi} \int_0^\infty \left(\frac{1}{t} - \frac{e^{(h-1/2)t}}{e^t - 1} F(1 - e^{-t}) \right) e^{-(n+h)t} \, \mathrm{d}t, \tag{2.15}$$

where

$$F(x) = \sum_{k=0}^\infty \frac{1}{16^k} \binom{2k}{k}^2 x^k \qquad (|x| < 1).$$

Taking $h = \frac{5}{4}$ in (2.15) yields

$$G_n = \frac{1}{\pi} \ln\left(n + \frac{5}{4}\right) + c_0 + \frac{1}{\pi} \int_0^\infty \left(\frac{1}{t} - \frac{e^{-t/4} F(1 - e^{-t})}{1 - e^{-t}} \right) e^{-(n+5/4)t} \, \mathrm{d}t. \tag{2.16}$$

Using the following known integral representation for $\psi(x)$ (see, *e.g.*, [25, p. 26, Eq. (20)]):

$$\psi(x) = \ln x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-xt} \, \mathrm{d}t \qquad (x > 0),$$

we have

$$\ln\left(n + \frac{5}{4}\right) = \psi\left(n + \frac{5}{4}\right) - \int_0^\infty \left(\frac{1}{t} - \frac{1}{1 - e^{-t}} \right) e^{-(n+5/4)t} \, \mathrm{d}t. \tag{2.17}$$

Plugging (2.17) into (2.16) yields

$$G_n = c_0 + \frac{1}{\pi} \psi\left(n + \frac{5}{4}\right) - \frac{1}{\pi} \int_0^\infty \left(\frac{f(t)}{1 - e^{-t}} \right) e^{-(n+5/4)t} \, \mathrm{d}t, \tag{2.18}$$

where

$$f(t) := F(1 - e^{-t}) e^{-t/4} - 1 \qquad (t > 0).$$

It was shown in [22] that

$$F(1 - e^{-t}) = \sum_{k=0}^\infty \left(\sum_{m=0}^k (-1)^{m+k} \binom{2m}{m}^2 \frac{m! S(k,m)}{16^m} \right) \frac{t^k}{k!},$$

where $S(k,m)$ denotes Stirling numbers of the second kind (see, e.g., [25, Section 1.6]). Hence we have

$$\begin{aligned} f(t) &= \sum_{k=0}^\infty \left(\sum_{m=0}^k (-1)^{m+k} \binom{2m}{m}^2 \frac{m! S(k,m)}{16^m} \right) \frac{t^k}{k!} \sum_{j=0}^\infty \frac{(-1)^j}{4^j j!} t^j - 1 \\ &= \sum_{j=0}^\infty \left(\sum_{k=0}^j \sum_{m=0}^k (-1)^{m+j} \binom{2m}{m}^2 \binom{j}{k} \frac{m! S(k,m)}{j! \cdot 4^{j-k+2m}} \right) t^j - 1 \\ &= -\frac{1}{64} t^2 + \frac{11}{49152} t^4 - \frac{173}{47185920} t^6 + \frac{22931}{338228674560} t^8 - \frac{1319183}{974098582732800} t^{10} \\ &\quad + \frac{233526463}{8229184826926694400} t^{12} - \frac{2673857519}{4356979312001915289600} t^{14} + \cdots, \end{aligned}$$

or

$$f(t) = \sum_{j=1}^\infty c_{2j} t^{2j},$$

where

$$c_{2j} = \sum_{k=0}^{2j} \sum_{m=0}^{k} (-1)^m \binom{2m}{m}^2 \binom{2j}{k} \frac{m! S(k,m)}{(2j)! \cdot 4^{2j-k+2m}} \qquad (j \in \mathbb{N}).$$

In fact, since F is analytic in $(-1,1)$, f is continuous in $(-\ln 2, \infty)$ and analytic around the origin. Note that $F(1-e^{-t}) = F(1-e^t)e^{t/2}$ (see [22]). Thus we find

$$f(t) = F(1-e^{-t})e^{-t/4} - 1 = F(1-e^t)e^{t/2}e^{-t/4} - 1 = F(1-e^t)e^{t/4} - 1 = f(-t),$$

that is, f is an even function. From these facts, we see that $c_{2j-1} = 0$ $(j \in \mathbb{N})$.

It is known (see [1, p. 260]) that

$$\psi^{(j)}(z) = (-1)^{j+1} \int_0^\infty \frac{t^j}{1-e^{-t}} e^{-zt}\, \mathrm{d}t \qquad (\Re(z) > 0;\ j \in \mathbb{N})$$

and

$$\psi^{(j)}(z) = \frac{(-1)^{j-1}(j-1)!}{z^j} + O\left(\frac{1}{z^{j+1}}\right) \qquad (z \to \infty;\ |\arg z| < \pi;\ j \in \mathbb{N}).$$

We find that

$$\psi^{(2N+2)}\left(n + \frac{5}{4}\right) = -\frac{(2N+1)!}{(n+\frac{5}{4})^{2N+2}} + O\left(\frac{1}{(n+\frac{5}{4})^{2N+3}}\right) = O\left(\frac{1}{n^{2N+2}}\right)$$

and

$$O\left(\frac{1}{n^{2N+2}}\right) = O\left(\psi^{(2N+2)}\left(n + \frac{5}{4}\right)\right).$$

Hence, (2.18) implies that, for $n \to \infty$,

$$\begin{aligned} G_n &= c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{\pi}\sum_{j=1}^{N} c_{2j}\psi^{(2j)}\left(n+\frac{5}{4}\right) + O\left(\frac{1}{n^{2N+2}}\right) \\ &= c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{\pi}\sum_{j=1}^{N} c_{2j}\psi^{(2j)}\left(n+\frac{5}{4}\right) + O\left(\psi^{(2N+2)}\left(n+\frac{5}{4}\right)\right), \end{aligned}$$

which is equivalent to (2.12). The proof of Theorem 2.1 is complete. □

The formula (2.14) motivated us to introduce Theorem 2.2, which provides newer bounds for G_n in terms of the digamma and polygamma functions.

Theorem 2.2. *For every $n \in \mathbb{N}$, we have*

$$\begin{aligned} c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) - \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) + \frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right) \\ < G_n < c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) - \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right). \end{aligned} \tag{2.19}$$

Proof. We consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n := G_n - c_0 - \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) - \frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right) \qquad (n \in \mathbb{N}).$$

By applying (2.11) and the asymptotic formulas of $\psi^{(2)}(z)$ and $\psi^{(4)}(z)$ (see, *e.g.*, [1, p. 260]), we conclude that

$$\lim_{n\to\infty} x_n = 0.$$

Applying (1.8) and the representation (see [3, p. 218] and [14, p. 739]):

$$G_n - G_{n-1} = \frac{\big(\Gamma(2n+1)\big)^2}{16^n\big(\Gamma(n+1)\big)^4} = \left[\frac{(2n)!}{4^n(n!)^2}\right]^2 = \frac{1}{\pi}\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 \tag{2.20}$$

and the recurrence formula [1, p. 260]:

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}, \tag{2.21}$$

we have

$$\begin{aligned}\pi(x_n - x_{n-1}) &= \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 - \frac{1}{n+\frac{1}{4}} + \frac{1}{64}\left(\frac{2}{\left(n+\frac{1}{4}\right)^3}\right) - \frac{11}{49152}\left(\frac{24}{\left(n+\frac{1}{4}\right)^5}\right)\\ &< \frac{4(13+32n+64n^2)}{15+92n+192n^2+256n^3} - \frac{4}{4n+1} + \frac{2}{(4n+1)^3} - \frac{11}{2(4n+1)^5}\\ &= -\frac{121}{2(1+4n)^5(64n^2+32n+15)} < 0.\end{aligned}$$

We thus find that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is strictly decreasing for $n \in \mathbb{N}$. This leads us to the following inequality: For every $n \in \mathbb{N}$,

$$\begin{aligned} x_n &= G_n - c_0 - \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) - \frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right)\\ &> \lim_{n\to\infty} x_n = 0 \qquad (n \in \mathbb{N}).\end{aligned} \tag{2.22}$$

We also consider the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by

$$y_n := G_n - c_0 - \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) \qquad (n \in \mathbb{N}).$$

By applying (2.11) and the asymptotic formula of $\psi^{(2)}(z)$ [1, p. 260], we see that

$$\lim_{n\to\infty} y_n = 0.$$

Applying (1.8), (2.20) and (2.21), we have

$$\begin{aligned}\pi(y_n - y_{n-1}) &= \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 - \frac{4}{4n+1} + \frac{2}{(4n+1)^3}\\ &> \frac{16(19+92n+96n^2+128n^3)}{105+704n+1920n^2+2048n^3+2048n^4} - \frac{4}{4n+1} + \frac{2}{(4n+1)^3}\\ &= \frac{2(47+176n+352n^2)}{(105+704n+1920n^2+2048n^3+2048n^4)(1+4n)^3} > 0.\end{aligned}$$

We thus see that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is strictly increasing for $n \in \mathbb{N}$. This leads us to the following inequality:

$$y_n = G_n - c_0 - \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) + \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) < \lim_{n\to\infty} y_n = 0 \qquad (n \in \mathbb{N}). \tag{2.23}$$

Now it is easy to observe that the inequalities in (2.22) and (2.23) prove the first and second inequalities in (2.19). Hence the proof of Theorem 2.2 is complete. □

Following the same method used in the proof of Theorem 2.2, we can prove the following further refined inequalities for G_n than those in (2.19).

Theorem 2.3. *For every $n \in \mathbb{N}$, we have*

$$\begin{aligned} &c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) - \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) + \frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right) \\ &- \frac{173}{47185920\pi}\psi^{(6)}\left(n+\frac{5}{4}\right) + \frac{22931}{338228674560\pi}\psi^{(8)}\left(n+\frac{5}{4}\right) \\ &< G_n < c_0 + \frac{1}{\pi}\psi\left(n+\frac{5}{4}\right) - \frac{1}{64\pi}\psi^{(2)}\left(n+\frac{5}{4}\right) + \frac{11}{49152\pi}\psi^{(4)}\left(n+\frac{5}{4}\right) \\ &- \frac{173}{47185920\pi}\psi^{(6)}\left(n+\frac{5}{4}\right). \end{aligned} \tag{2.24}$$

Remark 2.1. *Computer experiments point out that, for all $n \in \mathbb{N}$, the upper and lower bounds for G_n given in* (2.19) *improve the bounds presented in* (2.10)*. Computer experiments also show that the inequalities in* (2.24) *is sharper than those in* (2.19)*.*

3. The Lebesgue Constants

The Lebesgue constants L_n in (1.2) attracted the attention of several well-known mathematicians, such as Fejér [11], Gronwall [15], Hardy [16], Szegö [26], and Watson [28], who established remarkable properties of these numbers. For instance, they presented monotonicity theorems as well as various series and integral representations for L_n. The following asymptotic formula is due to Watson [28]:

$$L_{n/2} = \frac{4}{\pi^2}\ln(n+1) + c_1 + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty)\ , \tag{3.1}$$

where

$$c_1 := \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\ln k}{4k^2-1} + \frac{4}{\pi^2}(\gamma + 2\ln 2) = 0.98943\,12738\ \ldots, \tag{3.2}$$

γ being the Euler-Mascheroni constant given in (2.4). It should be *remarked* in passing that c_1 in this section has nothing to do with the coefficient $c_1(=0)$ appearing in the proof of Theorem 2.1. Throughout this section, c_1 is referred to the constant in (3.2). Using (3.1) and (3.2), Galkin [13] obtained the following inequalities for $L_{n/2}$:

$$c_1 + \frac{4}{\pi^2}\ln(n+1) < L_{n/2} \le 1 + \frac{4}{\pi^2}\ln(n+1) \qquad \big(n \in \mathbb{N}_0\big); \tag{3.3}$$

$$0.7190 + \frac{4}{\pi^2}\ln(n+2) < L_{n/2} \le c_1 + \frac{4}{\pi^2}\ln(n+2) \qquad \big(n \in \mathbb{N}_0\big). \tag{3.4}$$

Another direction for developing the problem of approximation of $L_{n/2}$ was initiated by Alzer [3, Theorem 4] who gave an estimate in terms of the Digamma function ψ, namely,

$$c_1 + \frac{4}{\pi^2}\psi\left(n+a_1\right) \le L_{n/2} < c_1 + \frac{4}{\pi^2}\psi\left(n+b_1\right) \qquad \big(n \in \mathbb{N}_0\big) \tag{3.5}$$

with the best possible constants

$$a_1 = \psi^{-1}\big(\pi^2(1-c_1)/4\big) = 1.48891\ldots \quad \text{and} \quad b_1 = \frac{3}{2}.$$

Zhao [30] pointed out that the error order of the second inequality in (3.5) to $L_{n/2}$ is just $O(n^{-2})$, but the first one is only $O(n^{-1})$. Thus, the inequality (3.5) does not imply the Watson asymptotic formula (3.1). Zhao [30, Theorem 2] established the following two-sided inequalities:

$$\begin{aligned}\frac{4}{\pi^2}\ln(n+1)+c_1+\frac{12-\pi^2}{18\pi^2(n+1)^2}-\frac{7(720-60\pi^2-\pi^4)}{10800\pi^2(n+1)^4}<L_{n/2}\\ <\frac{4}{\pi^2}\ln(n+1)+c_1+\frac{12-\pi^2}{18\pi^2(n+1)^2}-\frac{7(720-60\pi^2-\pi^4)}{10800\pi^2(n+1)^4}\\ +\frac{30240-2520\pi^2-42\pi^4-\pi^6}{15120\pi^2(n+1)^6}\qquad \big(n\in\mathbb{N}_0\big),\end{aligned}\tag{3.6}$$

which implies Watson's asymptotic formula (3.1).

Secondly we aim at establishing inequalities for $L_{n/2}$ and then using them to derive the asymptotic expansion for $L_{n/2}$.

Theorem 3.1. *For $n\in\mathbb{N}_0$ and $N\in\mathbb{N}_0$, we have*

$$\begin{aligned}\frac{4}{\pi^2}\ln(n+1)+c_1-\frac{8}{\pi^2}\sum_{j=1}^{2N}\left(\frac{B_{2j}(1/2)}{2j}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)k^{2j}}\right)\frac{1}{(n+1)^{2j}}<L_{n/2}\\ <\frac{4}{\pi^2}\ln(n+1)+c_1-\frac{8}{\pi^2}\sum_{j=1}^{2N+1}\left(\frac{B_{2j}(1/2)}{2j}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)k^{2j}}\right)\frac{1}{(n+1)^{2j}},\end{aligned}\tag{3.7}$$

where

$$B_k(1/2)=-\left(1-\frac{1}{2^{k-1}}\right)B_k\quad\big(k\in\mathbb{N}_0\big),$$

B_k being the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\qquad(|z|<2\pi).$$

Proof. By using the Szegö formula (see [26]):

$$L_{n/2}=\frac{16}{\pi^2}\sum_{k=1}^{\infty}\left(\frac{1}{4k^2-1}\sum_{j=1}^{(n+1)k}\frac{1}{2j-1}\right)$$

and the formula [1, p. 258]:

$$\psi\left(n+\frac{1}{2}\right)=-\gamma-2\ln 2+2\sum_{j=1}^{n}\frac{1}{2j-1}$$

as well as the formula:

$$\sum_{k=1}^{\infty}\frac{1}{4k^2-1}=\frac{1}{2},$$

we get

$$L_{n/2}-c_1-\frac{4}{\pi^2}\ln(n+1)=\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left[\psi\left(k(n+1)+\frac{1}{2}\right)-\ln\big(k(n+1)\big)\right].\tag{3.8}$$

It is known (see [2, p. 550]) that, for $x>0$ and $N\in\mathbb{N}_0$,

$$\ln x-\sum_{k=1}^{2N}\frac{B_{2k}(1/2)}{2kx^{2k}}<\psi\left(x+\frac{1}{2}\right)<\ln x-\sum_{k=1}^{2N+1}\frac{B_{2k}(1/2)}{2kx^{2k}}.\tag{3.9}$$

Applying the inequality (3.9) to (3.8) leads to the desired inequality (3.7). This completes the proof of Theorem 3.1. □

Remark 3.1. *We can simplify the coefficients in* Theorem 3.1 *by noting that*

$$\begin{aligned}\frac{B_{2j}(1/2)}{2j}\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)k^{2j}} &= \frac{B_{2j}}{j}(1-2^{2j-1})\sum_{k=1}^{\infty}\frac{1}{(4k^2-1)(2k)^{2j}}\\ &= \frac{B_{2j}}{j}(1-2^{2j-1})\sum_{k=1}^{\infty}\left(\frac{1}{4k^2-1}-\frac{1}{(2k)^2}-\frac{1}{(2k)^4}-\cdots-\frac{1}{(2k)^{2j}}\right)\\ &= \frac{B_{2j}}{2j}(1-2^{2j-1})\left(1+\sum_{k=1}^{j}\frac{(-1)^k}{(2k)!}B_{2k}\pi^{2k}\right)\end{aligned}$$

and using the following formula [1, p. 807]

$$\sum_{m=1}^{\infty}\frac{1}{m^{2n}} = \frac{(2\pi)^{2n}2^{2n}}{2(2n)!}(-1)^{n-1}B_{2n}.$$

The inequality (3.7) *can be rewritten as follows:*

$$\begin{aligned}&\frac{4}{\pi^2}\ln(n+1)+c_1-\sum_{j=1}^{2N}\frac{a_j}{(n+1)^{2j}} < L_{n/2}\\ &\qquad < \frac{4}{\pi^2}\ln(n+1)+c_1-\sum_{j=1}^{2N+1}\frac{a_j}{(n+1)^{2j}} \quad \big(n\in\mathbb{N}_0;\ N\in\mathbb{N}_0\big),\end{aligned}\tag{3.10}$$

where

$$a_j := \frac{8}{\pi^2}\frac{B_{2j}}{2j}(1-2^{2j-1})\left(1+\sum_{k=1}^{j}\frac{(-1)^k}{(2k)!}B_{2k}\pi^{2k}\right).\tag{3.11}$$

Now it is easy to see that, by taking $N = 1$ *in* (3.10)*, we obtain the inequalities* (3.6)*. The formula* (3.11) *shows also how one can arrive at the coefficients appearing in the inequalities* (3.5)*.*

From (3.10) we obtain the following corollary.

Corollary 3.1. *The following asymptotic formula holds:*

$$L_{n/2} = \frac{4}{\pi^2}\ln(n+1)+c_1-\sum_{j=1}^{m}\frac{a_j}{(n+1)^{2j}}+O\left(\frac{1}{(n+1)^{2m+2}}\right) \quad \big(n,\, m\in\mathbb{N}_0\big),\tag{3.12}$$

where a_j *are the coefficients given by* (3.11)*.*

Remark 3.2. *The asymptotic formula* (3.12) *can be found in Wong* [29, pp. 40-42]*. Here we give a different proof of* (3.12) *from that in* [29, pp. 40-42]*.*

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(C.-P. Chen) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City 454003, Henan Province, People's Republic of China
E-mail address: `chenchaoping@sohu.com`

(J. Choi) Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea
E-mail address: `junesang@mail.dongguk.ac.kr`