Received 12/01/14

ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR n-TIMES DIFFERENTIABLE LOG-PREINVEX FUNCTIONS

M. A. LATIF AND S. S. DRAGOMIR[1,2]

Abstract. In this paper, new Hermite-Hadamard type inequalities for n-times differentiable log-preinvex functions are established. The established results generalize some of those results proved in recent papers for differentiable log-prinvex functions and differentiable log-convex functions.

1. Introduction

It is well known in mathematics literature that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. The inequalities (1.1), are know as Hermite-Hadamard inequalities, a result first noticed by Ch. Hermite in 1883 and rediscovered ten years later by J. Hadamard. Since the discovery of (1.1) in 1883, Hermite-Hadamard inequality (see [10]) has been considered the most useful inequality in mathematical analysis. Some of the classical inequalities for means can be derived from (1.1) for particular choices of the function f. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see [4]-[7], [9], [11]-[15], [25], [27]-[30], [32, 33] and the references therein.

In recent years, many mathematicians generalized the classical convexity in many ways and some of those are given as follows.

Definition 1. [36] *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ if*

$$u + t\eta(v, u) \in K, \forall u, v \in K, t \in [0, 1].$$

The invex set K is also called an η-connected set.

Definition 2. [36] *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \leq (1-t) f(u) + tf(v)$$

for all $u, v \in K$ and $t \in [0, 1]$. The function f is said to be preconcave if and only if $-f$ is preinvex.

Date: November 15, 2012.

2000 *Mathematics Subject Classification.* 26D15, 26D99.

Key words and phrases. Hermite-Hadamard's inequality, invex set, log-preinvex function, Hölder's inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

It is to be noted that every preinvex function is convex with respect to the map $\eta(x,y)=x-y$ but the converse is not true see for instance [34].

Definition 3. [36] *Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$. A function $f:K\to\mathbb{R}$ is said to be prequasi-invex with respect to η, if*

$$f(u+t\eta(v,u))(1-t)\leq\max\{f(u),f(v)\},\forall u,v\in K,t\in[0,1].$$

Definition 4. [21] *Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$. A function $f:K\to(0,\infty)$ is said to be logarithmic preinvex with respect to η, if*

$$f(u+t\eta(v,u))(1-t)\leq(f(u))^{1-t}(f(v))^{t},\forall u,v\in K,t\in[0,1].$$

It is clear from the arithmetic-geometric mean inequality that if $f:K\to(0,\infty)$ is logarithmic preinvex, we have

$$\begin{aligned}f(u+t\eta(v,u))&\leq(f(u))^{1-t}(f(v))^{t}\\&\leq(1-t)f(u)+tf(v)\\&\leq\max\{f(u),f(v)\},\end{aligned}$$

$\forall u,v\in K,t\in[0,1]$.

Most recently, Noor [20] has obtained the following Hermite-Hadamard inequalities for the preinvex and log-preinvex functions.

Theorem 1. [20] *Let $f:[a,a+\eta(b,a)]\to(0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b\in K\circ$ with $a<a+\eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.2}$$

Theorem 2. [20] *Let $f:[a,a+\eta(b,a)]\to(0,\infty)$ be a log-preinvex function. Then*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\leq\frac{f(a)+f(b)}{\log f(a)-\log f(b)}.$$

The other results connected with (1.2) in which two log-preinvex functions are involved can be found in [24].

For log-preinvex functions, following Hermite-Hadamard type inequalities were also proved in [31].

Theorem 3. [31] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function. If $|f|$ is log-preinvex on K, for every $a,b\in K$ with $\eta(b,a)>0$, we have the inequality*

$$\begin{aligned}&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx-f\left(a+\frac{1}{2}\eta(b,a)\right)\right|\\&\qquad\leq\eta(b,a)\left[\frac{\left(\left|f'(b)\right|\right)^{1/2}-\left(\left|f'(a)\right|\right)^{1/2}}{\log(|f'(b)|)-\log(|f'(a)|)}\right]^2.\end{aligned} \tag{1.3}$$

Theorem 4. [31] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f|^q$, $q > 1$, $q \in \mathbb{R}$, is a log-preinvex on K, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality*

$$\left| f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{\eta(b,a)\sqrt{|f'(a)|}}{2^{1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(\left|f'(b)\right|\right)^{q/2} - \left(\left|f'(a)\right|\right)^{q/2}}{(\log(|f'(b)|) - \log(|f'(a)|))} \right]^{1/q}, \quad (1.4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For more results on Hermite-Hadamard type inequalities for preinvex functions and n-times differentiable preinvex functions, we refer the readers to the recent works of Sarikaya et. al , [31] and Latif [16].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for n-times differentiable log-preinvex functions which generalize those results established for differentiable log-preinvex functions given in [31].

2. Main Results

In order to prove our main results, we need the following two lemmas:

Lemma 1. [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, for every $a, b \in K$ with $\eta(b,a) > 0$, the equality holds:*

$$-\frac{f(a) + f(a + \eta(b,a))}{2} + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a + \eta(b,a)) = \frac{(-1)^{n-1}(\eta(b,a))^n}{2n!} \int_0^1 t^{n-1}(n - 2t) f^{(n)}(a + t\eta(b,a))dt, \quad (2.1)$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Lemma 2. [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$, for every $a, b \in K$ with $\eta(b,a) > 0$, the equality holds:*

$$\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx = \frac{(-1)^{n+1}(\eta(b,a))^n}{n!} \int_0^1 K_n(t) f^{(n)}(a + t\eta(b,a))dt, \quad (2.2)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

The following useful result will also help us establishing our results:

Lemma 3. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}. \tag{2.3}$$

Proof. For $n = 0$, we have

$$\int_0^1 \mu^t dt = \frac{\mu - 1}{\ln \mu},$$

which coincides with the right hand side of (2.3) for $n = 0$.

For $n = 1$, we have

$$\int_0^1 t\mu^t dt = \frac{\mu}{\ln \mu} - \frac{\mu}{(\ln \mu)^2} + \frac{1}{(\ln \mu)^2},$$

and it coincides with the right hand side of (2.3) for $n = 1$.

Suppose (2.3) is true for $n - 1$, i.e.

$$\int_0^1 t^{n-1} \mu^t dt = \frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}}. \tag{2.4}$$

Now by integration by parts and using (2.4), we have

$$\begin{aligned}
\int_0^1 t^n \mu^t dt &= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \int_0^1 t^{n-1} \mu^t dt \\
&== \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \left[\frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}} \right] \\
&= \frac{\mu}{\ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-1-k)! (\ln \mu)^{k++2}} \\
&= \frac{n!\mu}{n! \ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}} \\
&= \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^{\frac{1}{2}} t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (n-k)! (\ln \mu)^{k+1}}. \tag{2.5}$$

Proof. It follows from Lemma 3 after making use of the substitution $t = \frac{u}{2}$. □

Lemma 5. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_{\frac{1}{2}}^{1} (1-t)^n \mu^t dt = \frac{n!\mu}{(\ln \mu)^{n+1}} - n!\mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}. \tag{2.6}$$

Proof. It follows from Lemma 4 after making the substitution $1-t=u$. □

Lemma 6. [35] *For $\alpha > 0$ and $\mu > 0$, we have*

$$I(\alpha,\mu) := \int_0^1 t^{\alpha-1}\mu^t dt = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln \mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

Moreover, it holds

$$\left| I(\alpha,\mu) - \mu \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(\ln \mu)^{k-1}}{(\alpha)_k} \right| \leq \frac{|\ln \mu|}{\alpha\sqrt{2\pi(m-1)}} \left(\frac{|\ln \mu| e}{m-1} \right)^{m-1}.$$

We are now ready to give our first result.

Theorem 5. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, $q \geq 1$, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality*

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a)) \right|$$
$$\leq \frac{(\eta(b,a))^n}{2n!} \left(\frac{n-1}{n+1} \right)^{1-1/q} [E_1(n,q)]^{1/q}, \tag{2.7}$$

where

$$E_1(n,q) = \frac{(-1)^n n! \left\{ q \left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right) \right] + 2 \right\} \left|f^{(n)}(a)\right|^q}{q^{n+1} \left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right) \right]^{n+1}}$$
$$- \frac{2\left|f^{(n)}(b)\right|^q}{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right) \right]}$$
$$- n! \left|f^{(n)}(b)\right|^q \sum_{k=1}^{n} \frac{(-1)^k \left\{ q \left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right) \right] + 2 \right\}}{(n-k)! q^{k+1} \left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right) \right]^{k+1}}.$$

Proof. Suppose $n \geq 2$. Let $a, b \in K$. Since K is an invex set with respect to η, for every $t \in [0,1]$ we have $a + t\eta(b,a) \in K$. By log-preinvexity of $\left|f^{(n)}\right|^q$, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\leq\frac{(\eta(b,a))^n}{2n!}$$
$$\times\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{1/q}$$
$$\leq\frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(\int_0^1 t^{n-1}(n-2t)\left(\left(\left|f^{(n)}(a)\right|\right)^{q(1-t)}\left(\left|f^{(n)}(b)\right|\right)^{qt}\right)dt\right)^{1/q}$$
$$=\frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(n\int_0^1 t^{n-1}\mu^t dt-2\int_0^1 t^n\mu^t dt\right)^{1/q},\quad(2.8)$$

where $\mu=\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}$.
By Lemma 3, we have

$$n\int_0^1 t^{n-1}\mu^t dt-2\int_0^1 t^n\mu^t dt$$
$$=\frac{(-1)^n n!}{(\ln\mu)^n}-n!\mu\sum_{k=1}^{n}\frac{(-1)^k}{(n-k)!(\ln\mu)^k}-\frac{2(-1)^{n+1}n!}{(\ln\mu)^{n+1}}-2n!\mu\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}$$
$$=\frac{(-1)^n n![\ln\mu+2]-2\mu(\ln\mu)^n}{(\ln\mu)^{n+1}}-n!\mu\sum_{k=1}^{n}\frac{(-1)^k[\ln\mu+2]}{(n-k)!(\ln\mu)^{k+1}}.\quad(2.9)$$

Applying (2.9) in (2.8) and replacing $\mu=\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q}$, we get the desired inequality (2.7). This completes the proof of the theorem □

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied and if $q=1$, we have the inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\leq\frac{(\eta(b,a))^n}{2n!}E_1(n,1),\quad(2.10)$$

where

$$E_1(n,1)=\frac{(-1)^n n!\left\{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]+2\right\}\left|f^{(n)}(a)\right|}{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}}$$
$$-\frac{2\left|f^{(n)}(b)\right|}{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]}$$
$$-n!\left|f^{(n)}(b)\right|\sum_{k=1}^{n}\frac{(-1)^k\left\{\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]+2\right\}}{(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}.$$

Corollary 2. *Under the assumptions of Theorem 5, if $n = 2$, we have the inequality*

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{(\eta(b,a))^2}{4} \left(\frac{1}{3}\right)^{1-1/q} \left[E_1(2,q)\right]^{1/q}, \quad (2.11)$$

where

$$E_1(2,q) = \frac{2\left\{q\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] + 2\right\}\left|f''(a)\right|^q}{q^3\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3} + \frac{2\left\{q\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] - 2\right\}\left|f''(b)\right|^q}{q^3\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3}.$$

Corollary 3. *Under the assumptions of Theorem 5, if $n = 2$ and $q = 1$, we have the inequality*

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{(\eta(b,a))^2}{4} \left[E_1(2,1)\right], \quad (2.12)$$

where

$$E_1(2,1) = \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] + 2\right\}\left|f''(a)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3} + \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] - 2\right\}\left|f''(b)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3}.$$

Remark 1. *If $\eta(b,a) = b - a$ in the inequalities (2.11) and (2.12), one can get inequalities for the bounds of the difference between middle and the right most terms in the Hermite-Hadamard inequalities (1.1) in terms of second order derivatives for log-convex functions.*

Theorem 6. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a + \eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$, $q > 1$, is log-preinvex on K for*

$n \in \mathbb{N}$, $n \geq 2$, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a)) \right|$$
$$\leq \frac{q(\eta(b,a))^n \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)} \right] \left(\left| f^{(n)}(a) \right|^{1-q} \right) \left(\left| f^{(n)}(b) \right|^q \right)}{2^{2-1/q} n!}$$
$$\times \left(\frac{q-1}{2q-1} \right)^{1-1/q} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\left[\ln\left(\left| f^{(n)}(b) \right| \right) - \ln\left(\left| f^{(n)}(a) \right| \right) \right]^{k-1}}{(q(n-1)+1)_k}. \quad (2.13)$$

Proof. By log-preinvexity of $\left| f^{(n)} \right|^q$, Lemma 1 and Hölder inequality, we have

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a)) \right| \leq \frac{(\eta(b,a))^n}{2n!}$$
$$\times \left(\int_0^1 (n-2t)^{q/(q-1)}\,dt \right)^{1-1/q} \left(\int_0^1 t^{q(n-1)} \left| f^{(n)}(a+t\eta(b,a)) \right|^q dt \right)^{1/q}$$
$$\leq \frac{(\eta(b,a))^n \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)} \right]}{2^{2-1/q} n!} \left(\frac{q-1}{2q-1} \right)^{1-1/q}$$
$$\times \left(\int_0^1 t^{q(n-1)} \left(\left(\left| f^{(n)}(a) \right| \right)^{q(1-t)} \left(\left| f^{(n)}(b) \right| \right)^{qt} \right) dt \right)^{1/q}$$
$$= \frac{(\eta(b,a))^n \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)} \right] \left| f^{(n)}(a) \right|}{2^{2-1/q} n!}$$
$$\times \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\int_0^1 t^{q(n-1)} \mu^t\,dt \right)^{1/q}, \quad (2.14)$$

where $\mu = \frac{\left| f^{(n)}(b) \right|^q}{\left| f^{(n)}(a) \right|^q}$. Applying Lemma 6 to the last integral in the inequality (2.14) and simplifying, we get the required inequality (2.13). □

Corollary 4. *Suppose the assumptions of Theorem 6 are satisfied and $n = 2$. Then*

$$\left| \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right|$$
$$\leq \frac{(\eta(b,a))^2 \left[2^{\left(2q^2-1\right)/q(q-1)} \right] \left(\left| f''(a) \right|^{1-q} \right) \left(\left| f''(b) \right|^q \right)}{8}$$
$$\times \left(\frac{q-1}{2q-1} \right)^{1-1/q} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q \left[\ln\left(\left| f''(b) \right| \right) - \ln\left(\left| f''(a) \right| \right) \right]^{k-1}}{(q+1)_k}. \quad (2.15)$$

Corollary 5. *If $\eta(b,a)=b-a$ in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2\left[2^{\left(2q^2-1\right)/q(q-1)}\right]\left(\left|f''(a)\right|^{1-q}\right)\left(\left|f''(b)\right|^{q}\right)}{8} \times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q\left[\ln\left(\left|f''(b)\right|\right)-\ln\left(\left|f''(a)\right|\right)\right]^{k-1}}{(q+1)_k}. \quad (2.16)$$

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable log-preinvex functions.

Theorem 7. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 1$. If $\left|f^{(n)}\right|^q$ is log-preinvex on K for $n\in\mathbb{N}$, $n\geq 1$, $q\geq 1$, for every $a,b\in K$ with $\eta(b,a)>0$ we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|(n!)^{1-1/q}}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left\{\left[E_2(n,q)\right]^{1/q}+\left[E_3(n,q)\right]^{1/q}\right\}, \quad (2.17)$$

where

$$E_2(n,q)=\frac{(-1)^{n+1}}{q^{n+1}\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}} +\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{(-1)^k}{q^{k+1}2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}$$

and

$$E_3(n,q)=\frac{\left|f^{(n)}(b)\right|^q}{q^{n+1}\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}\left|f^{(n)}(a)\right|^q} -\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{1}{q^{k+1}2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}.$$

Proof. Suppose $n \geq 1$. By using Lemma 2 and the log-preinvexity of $\left|f^{(n)}\right|$ on K for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right] (\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{n!}\left[\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a+t\eta(b,a))\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(a+t\eta(b,a))\right| dt\right]$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}(a)\right|}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}} t^n \mu^t dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt\right)^{1/q}\right], \quad (2.18)$$

where $\mu = \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^q$. Applying Lemma 4 and Lemma 6 to the integrals in the inequality (2.18) and replacing $\mu = \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^q$, we get the desired inequality (2.17). This completes the proof of the theorem. □

Corollary 6. *Suppose the assumptions of Theorem 7 are fulfilled and if $q = 1$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right] (\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq (\eta(b,a))^n \left|f^{(n)}(a)\right| \left\{\left[E_2(n,1)\right] + \left[E_3(n,1)\right]\right\}, \quad (2.19)$$

where

$$E_2(n,1) = \frac{(-1)^{n+1}}{\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}} + \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}$$

and

$$E_3(n,1) = \frac{\left|f^{(n)}(b)\right|}{\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}\left|f^{(n)}(a)\right|} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}.$$

Corollary 7. [31] *If we take $n=1$ in Corollary 6, we have*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq \eta(b,a)\left[\frac{\left(\left|f'(b)\right|\right)^{1/2}-\left(\left|f'(a)\right|\right)^{1/2}}{\ln\left(\left|f'(b)\right|\right)-\ln\left(\left|f'(a)\right|\right)}\right]^2. \quad (2.20)$$

Corollary 8. [31] *If $\eta(b,a)=b-a$ in Corollary 7, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left[\frac{\left(\left|f'(b)\right|\right)^{1/2}-\left(\left|f'(a)\right|\right)^{1/2}}{\ln\left(\left|f'(b)\right|\right)-\ln\left(\left|f'(a)\right|\right)}\right]^2. \quad (2.21)$$

Theorem 8. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq1$. If $\left|f^{(n)}\right|^q$ is preinvex on K for $n\in\mathbb{N}$, $n\geq1$, $q\in\mathbb{R}$, $q>1$, for every $a,b\in K$ with $\eta(b,a)>0$ we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n\left[\sqrt{\left|f^{(n)}(a)\right|}+\sqrt{\left|f^{(n)}(b)\right|}\right]}{2^{n+1/p}(np+1)^{1/p}q^{1/q}n!}\left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2}-\left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)}\right]^{1/q}, \quad (2.22)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, the Hölder integral inequality and log-preinvexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt}dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(\eta(b,a))^n\left[\sqrt{\left|f^{(n)}(a)\right|}+\sqrt{\left|f^{(n)}(b)\right|}\right]}{2^{n+1/p}(np+1)^{1/p}q^{1/q}n!}\left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2}-\left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)}\right]^{1/q}. \quad (2.23)$$

Which is the required inequality. This completes the proof of the theorem. □

Corollary 9. *Under the assumptions of Theorem 8, if $n = 1$, we have the inequality*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)\left[\sqrt{|f'(a)|}+\sqrt{|f'(b)|}\right]}{2^{1+1/p}(p+1)^{1/p}q^{1/q}}\left[\frac{\left(\left|f'(b)\right|\right)^{q/2}-\left(\left|f'(a)\right|\right)^{q/2}}{(\ln(|f'(b)|)-\ln(|f'(a)|))}\right]^{1/q}, \quad (2.24)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 10. *If we take $\eta(b,a) = b - a$ in (2.24), we get the inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)\left[\sqrt{|f'(a)|}+\sqrt{|f'(b)|}\right]}{2^{1+1/p}(p+1)^{1/p}q^{1/q}}\left[\frac{\left(\left|f'(b)\right|\right)^{q/2}-\left(\left|f'(a)\right|\right)^{q/2}}{(\log(|f'(b)|)-\log(|f'(a)|))}\right]^{1/q}. \quad (2.25)$$

Remark 2. *Inequalities (2.24) and (2.25) are the corrected inequalities that are given in Theorem 4 and its related corollary from* [31].

References

[1] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.

[2] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, J. Inequal. Appl. 2012, 2012:247 doi:10.1186/1029-242X-2012-247.

[3] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[4] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.

[5] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[6] S. S. Dragomir and B. Mond, Integral inequalities of Hadamard type for log-convex functions, Demonst. Math., 31 (1998), 354–364.

[7] Wei-Dong Jiang, Da-Wei Niu, Yun Hua, and Feng Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012; Available online at http://dx.doi.org/10.1524/anly.2012.1161.

[8] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[9] Shu-Hong, Bo-Yan Xi and Feng Qi, Some new inequalities of Hermite-Hadamard type for n-times differentiable functions which are m-convex, Analysis (Munich) 32 (2012), no. 3, 247-262; Available online at http://dx.doi.org/10.1524/anly.2012.1167.

[10] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.

[11] Dah-Yang Hwang, Some inequalities for n-time differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343

[12] I. Iscan, Ostrowski type inequalites for functions whose derivatives are preinvex, arXiv:1204.2010v1.

[13] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.
[14] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.
[15] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable mappings and applications, Computers and Math. with Appl., 55 (2008), 485-493.
[16] M. A. Latif, On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications, Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 325–343.
[17] M. A. Latif, Some inequalities for differenitable prequasiinvex functions with applications, Konuralp Journal of Mathematics Volume 1, No. 2 pp. 17-29 (2013).
[18] M. A. Latif and S. S. Dragomir, Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications, J. Inequal Appl. 2013, 2013:575 doi:10.1186/1029-242X-2013-575.
[19] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.
[20] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2 (2007), 126-131.
[21] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323–330.
[22] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463–475.
[23] M. A. Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl.,11(2006),165-171
[24] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-6, Article 75.
[25] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.
[26] R. Pini, Invexity and generalized Convexity, Optimization 22 (1991) 513-525.
[27] M. Z. Sarikaya, A. Saglam and H. Yıldırım, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, International Journal of Open Problems in Computer Science and Mathematics (IJOPCM), 5(3), 2012.
[28] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, Journal of Mathematical Inequalities, Volume 2, Number 3 (2008), 335-341.
[29] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, 852 (2010).
[30] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.
[31] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Haermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, Contemporary Analysis and Applied Mathematics, Vol.1, No.2, 237-252, 2013.
[32] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On some new inequalities of Hadamard type involving h-convex functions, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.
[33] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[34] T. Weir, and B. Mond, Preinvex functions in multiple objective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
[35] J. Wang, J. Deng, M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press)
[36] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001) 229-241.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: m_amer_latif@hotmail.com

[1] School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC8001, Australia

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: sever.dragomir@vu.edu.au