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A correction to the paper *Some new inequalities for the Gamma, Beta and Zeta functions* which appeared in JIPAM

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Abstract

A new proof of an inequality which involves a positive linear operator acting on the composition of two continuous functions is presented. This proof replaces a previous one which was found to be incorrect.

Note. It has been observed that, although the Theorem and its dependent inequalities are correct in the the paper of the title [1], the proof of the Theorem contains errors. For example, on line 6 of page 3 it is "compact" that is needed rather than "closed." Also, the extreme points of the set S are not limited to the set $\{Ae_0 + Be_1\}$. Now, it may be possible to correct the errors in that paper but we believe it to be preferable to present an entirely different – and shorter – treatment of these matters. The proof given below is a generalisation of that in [2]. We now proceed in this direction, referring to [2] whenever necessary.

1. Introduction

Let f and g be continuous and strictly increasing functions defined in $[0,\infty)$. Let $\frac{f}{g}$ be strictly increasing there also and suppose that each of $f(x)$ and $g(x) \to 0$ as $x \to 0$. $L_{a,b}$ or simply L will denote a positive linear functional defined on $C[a,b]$ where $0 \le a < b$ and F is a function defined on the ranges of f and g.

Theorem. (a) *If F' is increasing then*

$$L[F(f)] \ge L[F(\varphi)]$$

(b) *If F' is decreasing then*

$$L[F(f)] \le L[F(\varphi)]$$

where φ is defined by

$$\varphi = g\frac{L(f)}{L(g)}\,.$$

2. Proofs

We shall need the following lemma, whose proof appears in [2].

Lemma. *Let $p, q \in C[a,b]$ and let L be a positive linear functional as above. Suppose that $L(p) = 0$ and that $p(x)$ changes sign once, from negative to positive, in the interval. Suppose that q is increasing there. Then*

$$L(pq) \geq 0 .$$

(*If q is decreasing then this inequality is to be reversed.*)

An observation:

$$L(f-\varphi) = L(f - g\frac{L(f)}{L(g)}) = 0 ,$$

so that neither of the following is possible

$$f(x) - \varphi(x) > 0 , \quad f(x) - \varphi(x) < 0 \quad \text{throughout } [a,b].$$

Since $f(x) - \varphi(x)$ is increasing in the interval it changes sign once there, from negative to positive. From this observation and the Lemma above we can now prove the Theorem.

Proof of the Theorem. We shall prove the part (a) only.

We write

$$[F(f(x)) - F(\varphi(x))] = [f(x) - \varphi(x)]Q(x)$$

where

$$Q(x) = \frac{1}{[f(x) - \varphi(x)]} \int_{\varphi(x)}^{f(x)} F'(t)dt .$$

We see that Q is increasing since $Q(x)$ is the average of the increasing function F' over the interval $(f(x), \varphi(x))$ [or $(\varphi(x), f(x))$] each of whose end-points moves to the right with increasing x. (Q is to be defined at the zero of $f(x) - \varphi(x)$ by continuity.) So taking $p = f - \varphi$ and $q = Q$ we get the result from the lemma above.

3. Examples

In [1] we obtained four examples from the Theorem above and these derivations remain unchanged. However, for illustrative purposes we repeat here the details of the first two. Some preparation is needed.

To apply the Theorem we shall define

$$F(u) = u^{\alpha} , \quad f(x) = x^{\beta} \quad \text{and} \quad g(x) = x^{\delta} \quad \text{with} \quad \beta > \delta > 0 .$$

We note that F' is increasing if $\alpha \notin (0,1)$ and decreasing if $\alpha \in (0,1)$, and that the inequalities of the Theorem, namely

$$L[F(f)] \gtrless L[F(\varphi)]$$

become (using incorrect, but simpler, notation)

$$\frac{[L(x^{\delta})]^{\alpha}}{[L(x^{\alpha\delta})]} \gtrless \frac{[L(x^{\beta})]^{\alpha}}{[L(x^{\alpha\beta})]} \quad \text{with } \gtrless \text{according as } \alpha \notin (0,1),\ \alpha \in (0,1)\,. \tag{1}$$

Note: The phrase "with $\gtrless$ according as $\alpha \notin$(0,1), $\alpha \in$(0,1)" will be understood in all that follows.

The first example in [1]

Let $L_{a,b}(\equiv L)$, defined on $C[a,b]$, be the functional

$$L_{a,b}(w) = \int_a^b w(x)e^{-x}dx \quad \text{where } 0 \leq a < b\,.$$

Writing this functional into (1), and then letting $a \to 0$ and $b \to \infty$ we get

$$\frac{[\Gamma(1+\delta)]^{\alpha}}{\Gamma(1+\alpha\delta)} \gtrless \frac{[\Gamma(1+\beta)]^{\alpha}}{\Gamma(1+\alpha\beta)} \quad \text{with } \alpha \text{ as above, } \ \beta > \delta > 0 \ \text{ and } \ \alpha\beta > -1,\ \alpha\delta > -1\,.$$

The second example in [1]

This result can be obtained using the functional

$$\int_a^b w(x)(1-x)^{\zeta-1}dx \qquad (\zeta > 0) \qquad : \ w \in [a,b]$$

in a similar way. Writing this into (1) and then letting $a \to 0$ and $b \to 1$ we get the result

$$\frac{[B(1+\delta,\zeta)]^{\alpha}}{B(1+\alpha\delta,\zeta)} \gtrless \frac{[B(1+\beta,\zeta)]^{\alpha}}{B(1+\alpha\beta,\zeta)} \quad : \ \zeta > 0$$

with α as above, $\beta > \delta > 0$ and $\alpha\beta > -1$, $\alpha\delta > -1$.

Repetition of the third and fourth examples in [1] would require considerably more space so we leave it to the interested reader to refer to the paper [1] itself.

References

1. A.McD. Mercer, Some new inequalities for the Gamma, Beta and Zeta functions. *J. Ineq. Pure. App. Math.* **7**(1) (2006) Art. 29.

2. A.McD. Mercer, A generalization of Andersson's Inequality. *J. Ineq. Pure. App. Math.* **6**(2) (2005) Art. 57.