Received 19/07/14

ON HADAMARD'S INEQUALITIES FOR THE PRODUCT OF TWO CONVEX MAPPINGS DEFINED IN TOPOLOGICAL GROUPS

M. A. LATIF, S. S. DRAGOMIR[1,2], AND E. MOMONIAT

Abstract. In this paper we study Hermite-Hadamard type inequalities for the product of two midconvex and quasi-midconvex functions and give some applications of our results.

1. Introduction

Let $f : I \longrightarrow \mathbb{R}$ be a cnovex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mappings. The history of this inequality goes back to the papers of Ch. Hermite [9] and J. Hadamard [8] in 1883 and 1893 respectively. This inequality produces some classical inequalities of means for particular choice of the mapping f. The inequality (1.1) attracted a number of mathematicians and is generalized, extended, and refined it in a number of ways (see e.g. [3, 4] and [7]). Also some mappings naturally connected with (1.1) are defined and properties of these mappings are discussed by many mathematicians (see e.g. [5, 6]). We discuss only recent studies in this paper.

A generalization of the left side of (1.1) for convex functions defined on a convex subset of $\mathbb{R}^n$ is the following inequality from [13]

$$f(0) \leq \frac{1}{\mu(X)}\int_X f(x)dx, \tag{1.2}$$

where $X \subset \mathbb{R}^n$ is a convex bounded symmetrical set that is, if $x \in X \Rightarrow -x \in X$, f is a lower semicontinuous convex function $f : X \longrightarrow \mathbb{R}$ and $\mu(X)$ is the volume of the set X.

In [12], Morassaei established Hadamard's inequality for midconvex and quasi-midconvex functions in topological groups and discussed some of the properties of the mapping naturally connected with the Hadamard's inequality for globally midconvex function defined in a topological group. Some of the main results from [12] are stated in the following theorems.

Date: March 9, 2010.

2000 *Mathematics Subject Classification.* Primary 26A51, 22A10; Secondary 26D07, 26D15.

Key words and phrases. convex functions, Hadamard's inequality, midconvex function, quasi-midconvex function, topological groups.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Theorem 1. [12] *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. If $f : \Omega \longrightarrow \mathbb{R}$ is measurable and locally midconvex in a and $f \in L_1(\Omega)$. If $\omega : \Omega \longrightarrow \mathbb{R}$ is non-negative symmetric to a and $\omega \in L_1(\Omega)$ such that $f\omega \in L_1(\Omega)$, then*

$$f(a) \int_\Omega \omega(az)d\mu(z) \leq \int_\Omega f(az)g(az)\omega(az)d\mu(z), \tag{1.3}$$

where μ is the Haar measure.

Theorem 2. [12] *Suppose that G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$ and $e \in \Omega$. Let f be measurable and quasi-midconvex real-valued function on Ω such that $f \in L_2(\Omega)$. If $\omega : \Omega \to \mathbb{R}$ is a non-negative and symmetric to a and $\omega \in L_2(\Omega)$, then*

$$f(a) \int_\Omega \omega(az)d\mu(z) \leq \int_\Omega f(az)\omega(az)d\mu(z) + I(a), \tag{1.4}$$

where

$$I(a) = \frac{1}{2} \int_\Omega \left| f(az) - f(az^{-1}) \right| \omega(az)d\mu(z).$$

Furthermore, $I(a)$ satisfies the following inequality

$$0 \leq I(a) \tag{1.5}$$

$$\leq \min \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} \left(\int_\Omega f^2(az)d\mu(z) - \int_\Omega f^2(az)f(az^{-1})d\mu(z) \right)^{\frac{1}{2}} \left(\int_\Omega \omega^2(az)d\mu(z) \right)^{\frac{1}{2}}, \\ \int_\Omega |f(az)| \, \omega(az)d\mu(z). \end{array} \right\}$$

Theorem 3. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. If f are measurable real valued P-function on Ω such that $f \in L_1(\Omega)$. If $\omega : \Omega \to \mathbb{R}$ is non-negative symmetric to a, $\omega \in L_1(\Omega)$ and $f\omega \in L_1(\Omega)$, then*

$$f(a) \int_\Omega \omega(az)\, d\mu(z) \leq 2 \int_\Omega f(az)g(az)\omega(az)\, d\mu(z). \tag{1.6}$$

We give a result similar to (1.2) for the product of two convex functions defined on a convex bounded symmetrical subset X of $\mathbb{R}^n$. We will also give our results for the product of two midconvex and qusi-midconvex mappings defined in a topological groups in Section 3. Applications of the obtained results are given as well in Section 3.

2. A Secondary Result

Theorem 4. *Let f, g be two convex functions defined on a convex bounded symmetrical subset X of $\mathbb{R}^n$, we have*

$$\begin{aligned} f(0)g(0) &\leq \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(x)g(-x)]\, dx \\ &= \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(-x)g(x)]\, dx. \end{aligned} \tag{2.1}$$

Proof. Consider the transformation of the $\mathbb{R}^n$ in itself defined by

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ h = (h_1, ..., h_n)$$

and

$$h_i\left(x_1,...,x_n\right)=-x_i,\ i=1,2,...,n.$$

Then $h(X)=X$ and since

$$\frac{D\left(h_1,...,h_n\right)}{D\left(x_1,...,x_n\right)}=\begin{vmatrix}-1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 0 & \cdots & 0\\ 0 & 0 & -1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & -1\end{vmatrix}=(-1)^n.$$

Thus we have, by the change of variables that

$$\begin{aligned}&\int_X f\left(x_1,...,x_n\right)g\left(x_1,...,x_n\right)dx_1...dx_n\\ &=\int_X f\left(h_1\left(x_1,...,x_n\right),...,h_n\left(x_1,...,x_n\right)\right)g\left(h_1\left(x_1,...,x_n\right),...,h_n\left(x_1,...,x_n\right)\right)\times\\ &\left|\frac{D\left(h_1,...,h_n\right)}{D\left(x_1,...,x_n\right)}\right|dx_1...dx_n\\ &=\int_X f\left(-x_1,...,-x_n\right)g\left(-x_1,...,-x_n\right)dx_1...dx_n\end{aligned}$$

and

$$\begin{aligned}&\int_X f\left(x_1,...,x_n\right)g\left(-x_1,...,-x_n\right)dx_1...dx_n\\ &=\int_X f\left(h_1\left(x_1,...,x_n\right),...,h_n\left(x_1,...,x_n\right)\right)g\left(-h_1\left(x_1,...,x_n\right),...,-h_n\left(x_1,...,x_n\right)\right)\times\\ &\left|\frac{D\left(h_1,...,h_n\right)}{D\left(x_1,...,x_n\right)}\right|dx_1...dx_n\\ &=\int_X f\left(-x_1,...,-x_n\right)g\left(x_1,...,x_n\right)dx_1...dx_n.\end{aligned}$$

Now by the convexity of f and g on X, we get that

$$\begin{aligned}&f(0,...,0)g(0,...,0)\\ &=f\left(\frac{x_1-x_2}{2},...,\frac{x_n-x_n}{2}\right)g\left(\frac{x_1-x_2}{2},...,\frac{x_n-x_n}{2}\right)\\ &=f\left(\frac{\left(x_1,...,x_n\right)+\left(-x_1,...,-x_n\right)}{2}\right)g\left(\frac{\left(x_1,...,x_n\right)+\left(-x_1,...,-x_n\right)}{2}\right)\\ &\le\frac{1}{4}\left[f\left(x_1,...,x_n\right)+f\left(-x_1,...,-x_n\right)\right]\left[g\left(x_1,...,x_n\right)+g\left(-x_1,...,-x_n\right)\right]\\ &=\frac{1}{4}\left[f\left(x_1,...,x_n\right)g\left(x_1,...,x_n\right)+f\left(-x_1,...,-x_n\right)g\left(-x_1,...,-x_n\right)\right.\\ &\left.+f\left(-x_1,...,-x_n\right)g\left(x_1,...,x_n\right)+f\left(x_1,...,x_n\right)g\left(-x_1,...,-x_n\right)\right]\end{aligned}$$

which gives by integration on X that

$$\begin{aligned}(2.2)\quad & \int_X f(0,...,0)g(0,...,0)dx_1...dx_n \\ & \leq \frac{1}{4}\left[2\int_X f\left(x_1,...,x_n\right)g\left(x_1,...,x_n\right)dx_1...dx_n \right. \\ & + \int_X f\left(x_1,...,x_n\right)g\left(-x_1,...,-x_n\right)dx_1...dx_n \\ & \left. + \int_X f\left(-x_1,...,-x_n\right)g\left(x_1,...,x_n\right)dx_1...dx_n\right].\end{aligned}$$

Hence (2.1) follows from (2.2). This completes the proof of the Theorem. □

3. Main Results

Now we prove Hadamard's type inequalities for product of midconvex and quasi-convex functions defined in a topological groups. Before we proceed to prove our results we give some definitions from [1, 11] and [12].

We recall that for a group $(G, \cdot, e)$, a topology on G is compatible with the group structure when the maps $G \times G \to G : (x, y) \mapsto xy$ (multiplication) and $G \to G : x \mapsto x^{-1}$ (inverse) are continuous. A group together with a topology compatible with its group structure is a topological group.

A Haar measure on G is a measure $\mu : \Sigma \to [0, \infty)$, with a σ-algebra containing all Borel subsets of G, such that $\mu(G) = 1$ and $\mu(\gamma S) = \mu(S)$ for all $\gamma \in G$, $S \in \Sigma$, where $\gamma S = \{\gamma\alpha : \alpha \in S\}$.

Definition 1. [1] *Let G be a topological group, Ω a non-empty open subset of G and f a real-valued function on Ω. We say that f is globally (right) midconvex if*

$$2f(a) \leq f(az) + f(az^{-1})$$

for all a, $z \in G$ such that $a, az, az^{-1} \in \Omega$. We say that f is locally (right) midconvex in $a \in \Omega$ if there exists an open symmetric set $V = V^{-1}$ from e such that

$$2f(a) \leq f(az) + f(az^{-1})$$

for all $z \in G$ such that $az, az^{-1} \in \Omega$.

Definition 2. [11] *Let G be a topological group, Ω a non-empty open subset of G and f a real-valued function on Ω. The mapping f is called quasi-(right) midconvex, if*

$$f(az) \leq \max\{f(a), f(az^2)\}$$

for every $a, z \in G$ so that $a, az, az^2 \in \Omega$. Note that a is midpoint of az^{-1} and az, and az is midpoint of a and az^2.

Definition 3. [12, Definition 1, page 4] *Let Ω be an open subset of topological group G, and $a \in G$. Ω is said to be symmetric relative to a, if $a^{-1}\Omega$ is symmetric and $e \in a^{-1}\Omega$.*

Definition 4. [12, Definition 2, Page 4] *Let G be a topological group and $\Omega \subset G$ an open set. A function $\omega : \Omega \longrightarrow R$ is called symmetric relative to $a \in G$, if $\forall z \in G$; $az, az^{-1} \in \Omega$ and $\omega(az) = \omega(az^{-1})$.*

We now give our main result.

Theorem 5. *Suppose that G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let $f,\ g : \Omega \longrightarrow \mathbb{R}_+$ be measurable and locally midconvex in a and $f, g \in L_1(\Omega)$. If $\omega : \Omega \longrightarrow \mathbb{R}$ is non-negative symmetric to a and $\omega \in L_1(\Omega)$ such that $fg\omega \in L_1(\Omega)$, we have*

$$
\begin{aligned}
(3.1)\quad & f(a)g(a)\int_{\Omega}\omega(az)d\mu(z)\\
&\leq \frac{1}{2}\left[\int_{\Omega} f(az)g(az)\omega(az)d\mu(z) + \int_{\Omega} f(az)g(az^{-1})\omega(az)d\mu(z)\right]\\
&\quad = \frac{1}{2}\left[\int_{\Omega} f(az)g(az)\omega(az)d\mu(z) + \int_{\Omega} f(az^{-1})g(az)\omega(az)d\mu(z)\right],
\end{aligned}
$$

where μ is the Haar measure.

Proof. Since f and g are midconvex in a, therefore we have

$$2f(a) \leq f(az) + f(az^{-1})$$

and

$$2g(a) \leq g(az) + g(az^{-1}).$$

for any $z \in \Omega$. From these inequalities we get that

$$
\begin{aligned}
&4f(a)g(a)\\
&\leq f(az)g(az) + f(az^{-1})g(az)\\
&+ f(az)g(az^{-1}) + f(az^{-1})(g(az^{-1}).
\end{aligned}
$$

Since ω is non-negative and symmetric relative to a, we have

$$
\begin{aligned}
&4f(a)g(a)\omega(az)\\
&\leq f(az)g(az)\omega(az) + f(az^{-1})g(az)\omega(az^{-1})\\
&+ f(az)g(az^{-1})\omega(az) + f(az^{-1})(g(az^{-1})\omega(az^{-1}).
\end{aligned}
$$

Integrating this inequality over Ω, we get that

$$
\begin{aligned}
&4f(a)g(a)\int_{\Omega}\omega(az)d\mu(z)\\
&\leq \int_{\Omega} f(az)g(az)\omega(az)d\mu(z)+\int_{\Omega} f(az^{-1})g(az)\omega(az^{-1})d\mu(z)\\
&+\int_{\Omega} f(az)g(az^{-1})\omega(az)d\mu(z)+\int_{\Omega} f(az^{-1})g(az^{-1})\omega(az^{-1})d\mu(z)\\
&=\int_{a^{-1}\Omega} f(z)g(z)\omega(z)d\mu(z)+\int_{a^{-1}\Omega} f(z^{-1})g(z)\omega(z^{-1})d\mu(z)\\
&+\int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z)d\mu(z)+\int_{a^{-1}\Omega} f(z^{-1})g(z^{-1})\omega(z^{-1})d\mu(z)\\
&=\int_{G} f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_{G} f(z^{-1})g(z)\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&+\int_{G} f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_{G} f(z^{-1})g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&=\int_{G} f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_{G} f(z^{-1})g(z)\omega(z^{-1})\chi_{a^{-1}\Omega}\left(z^{-1}\right)d\mu(z)\\
&+\int_{G} f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)+\int_{G} f(z^{-1})g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}\left(z^{-1}\right)d\mu(z).
\end{aligned}
$$

That is

$$
\begin{aligned}
(3.2)\quad & f(a)g(a)\int_{\Omega}\omega(az)d\mu(z)\\
&\leq 2\int_{G} f(z)g(z)\omega(z)\chi_{a^{-1}\Omega}d\mu(z)+2\int_{G} f(z)g(z^{-1})\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&\quad =2\int_{a^{-1}\Omega} f(z)g(z)\omega(z)d\mu(z)+2\int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z)d\mu(z)\\
&\qquad =\frac{1}{2}\left[\int_{\Omega} f(az)g(az)\omega(az)d\mu(z)+\int_{\Omega} f(az)g(az^{-1})\omega(az)d\mu(z)\right].
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{\Omega} f(az)g(az^{-1})\omega(az)d\mu(z) &= \int_{\Omega} f(az)g(az^{-1})\omega(az^{-1})d\mu(z)\\
&=\int_{a^{-1}\Omega} f(z)g(z^{-1})\omega(z^{-1})d\mu(z)\\
&=\int_{G} f(z)g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&=\int_{G} f(z)g(z^{-1})\omega(z^{-1})\chi_{a^{-1}\Omega}\left(z^{-1}\right)d\mu(z)\\
&=\int_{G} f(z^{-1})g(z)\omega(z)\chi_{a^{-1}\Omega}(z)\,d\mu(z)\\
&=\int_{a^{-1}\Omega} f(z^{-1})g(z)\omega(z)\,d\mu(z)\\
&=\int_{\Omega} f(az^{-1})g(az)\omega(az)d\mu(z).
\end{aligned}
$$

Thus, we also have

$$f(a)g(a)\int_{\Omega}\omega(az)d\mu(z) \leq \frac{1}{2}\left[\int_{\Omega}f(az)g(az)\omega(az)d\mu(z)+\int_{\Omega}f(az^{-1})g(az)\omega(az)d\mu(z)\right]. \tag{3.3}$$

Consequently the inequality (3.1) follows from (3.2) and (3.3). This completes the proof of the theorem. □

Remark 1. *If we take $a=e$ and $\omega\equiv 1$ on Ω in Theorem 5, we have*

$$\begin{aligned} f(e)g(e) &\leq \frac{1}{2\mu(\Omega)}\left[\int_{\Omega}f(z)g(z)\omega(z)d\mu(z)+\int_{\Omega}f(z)g(z^{-1})\omega(z)d\mu(z)\right] \\ &= \frac{1}{2\mu(\Omega)}\left[\int_{\Omega}f(z)g(z)\omega(z)d\mu(z)+\int_{\Omega}f(z^{-1})g(z)\omega(z)d\mu(z)\right] \end{aligned} \tag{3.4}$$

which is similar to (2.1).

Remark 2. *If $g\equiv 1$ on Ω in Theorem 5, we have*

$$f(a)\int_{\Omega}\omega(az)d\mu(z) \leq \int_{\Omega}f(az)g(az)\omega(az)d\mu(z) \tag{3.5}$$

which is a similar result as proved in [12, Theorem 1].

Remark 3. *If we take $a=e$, $\omega\equiv 1$ and $g\equiv 1$ on Ω in Theorem 5, we have*

$$f(e) \leq \frac{1}{\mu(\Omega)}\int_{\Omega}f(z)d\mu(z) \tag{3.6}$$

which is the same result as proved in [12, Remark1].

Theorem 6. *Suppose that G be a locally compact group and $\Omega\subset G$ an open symmetric set relative to $a\in G$ with $0<\mu(\Omega)<\infty$ and $e\in\Omega$. Let f and g be measurable and quasi-midconvex non-negative real-valued functions on Ω such that $fg\in L_2(\Omega)$. If $\omega:\Omega\to\mathbb{R}$ is a non-negative and symmetric to a and $\omega\in L_2(\Omega)$, we have*

$$\begin{aligned} f(a)g(a)\int_{\Omega}\omega(az)d\mu(z) &\leq \frac{1}{2}\int_{\Omega}f(az)g(az)\omega(az)d\mu(z)+\frac{1}{2}\int_{\Omega}f(az)g(az^{-1})\omega(az)d\mu(z)+I(a) \\ &= \frac{1}{2}\int_{\Omega}f(az)g(az)\omega(az)d\mu(z)+\frac{1}{2}\int_{\Omega}f(az^{-1})g(az)\omega(az)d\mu(z)+I(a), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} I(a) &= \frac{1}{2}\int_{\Omega}f(az)\left|g(az)-g(az^{-1})\right|\omega(az)d\mu(z) \\ &+\frac{1}{2}\int_{\Omega}\left|f(az)-f(az^{-1})\right|g(az)\omega(az)d\mu(z) \\ &+\frac{1}{4}\int_{\Omega}\left|f(az)-f(az^{-1})\right|\left|g(az)-g(az^{-1})\right|\omega(az)d\mu(z). \end{aligned}$$

Furthermore, $I(a)$ satisfies the following inequality

$$(3.8)\quad 0 \le I(a)$$
$$\le \min\left\{\begin{array}{l} \frac{1}{2}\left(\int_\Omega \left(f(az^{-1})\left(g(az)-g(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\ +\frac{1}{2}\left(\int_\Omega \left(g(az^{-1})\left(f(az)-f(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\ +\frac{1}{4}\left(\int_\Omega \left(\left(f(az^{-1})-f(az)\right)\left(g(az)-g(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}, \\ \frac{1}{\sqrt{2}}\left(\int_\Omega f^2(az)d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega g^2(az)\omega^2(az)d\mu(z)\right. \\ \left.-\int_\Omega g(az)g(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}\left(\int_\Omega g^2(az)d\mu(z)\right)^{\frac{1}{2}} \\ \times\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z)-\int_\Omega f(az)f(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\ +\frac{1}{2}\left(\int_\Omega f^2(az)d\mu(z)-\int_\Omega f(az)f^{-1}(az)d\mu(z)\right)^{\frac{1}{2}} \\ \times\left(\int_\Omega g^2(az)\omega^2(az)d\mu(z)-\int_\Omega g(az)g(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}, \\ \frac{1}{\sqrt{2}}\left(\int_\Omega g^2(az)d\mu(z)-\int_\Omega g^2(az)g(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\ \times\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}\left(\int_\Omega f^2(az)d\mu(z)-\int_\Omega f^2(az)f(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\ \times\left(\int_\Omega g^2(az)\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}+\frac{1}{2}\left(\int_\Omega g^2(az)d\mu(z)-\int_\Omega g^2(az)g(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\ \times\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z)-\int_\Omega f^2(az)f(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}, \\ \frac{3}{2}\int_\Omega f(az)g(az)\omega(az)d\mu(z)+\frac{3}{2}\int_\Omega f(az)g(az^{-1})\omega(az)d\mu(z). \end{array}\right\}$$

Proof. Since Ω is symmetric set relative to a, thus for any $z \in G$ and by the quasi-midconvexity of f and g, we have

$$f(a) = \max\{f(az), f(az^{-1})\} = \frac{f(az)+f(az^{-1})+\left|f(az)-f(az^{-1})\right|}{2}$$

and

$$g(a) = \max\{g(az), g(az^{-1})\} = \frac{g(az)+g(az^{-1})+\left|g(az)-g(az^{-1})\right|}{2}.$$

Now by the non-negativity of f and g, we get

$$(3.9)\quad \begin{aligned} f(a)g(a) \le \frac{1}{4}\Big[&f(az)g(az)+f(az)g(az^{-1})+f(az)\left|g(az)-g(az^{-1})\right| \\ &+f(az^{-1})g(az)+f(az^{-1})g(az^{-1})+f(az^{-1})\left|g(az)-g(az^{-1})\right| \\ &+\left|f(az)-f(az^{-1})\right|g(az)+\left|f(az)-f(az^{-1})\right|g(az^{-1}) \\ &+\left|f(az)-f(az^{-1})\right|\left|g(az)-g(az^{-1})\right|\Big]. \end{aligned}$$

Since ω is non-negative and symmetric relative to a, we have from (3.9) that

$$\begin{aligned} &f(a)g(a)\int_\Omega \omega(az)d\mu(z) \\ &\le \frac{1}{2}\int_\Omega f(az)g(az)\omega(az)d\mu(z)+\frac{1}{2}\int_\Omega f(az)g(az^{-1})\omega(az)d\mu(z)+I(a) \\ &= \frac{1}{2}\int_\Omega f(az)g(az)\omega(az)d\mu(z)+\frac{1}{2}\int_\Omega f(az^{-1})g(az)\omega(az)d\mu(z)+I(a). \end{aligned}$$

Hence (3.7) is proved, where

$$
\begin{aligned}
(3.10)\quad I(a) = \frac{1}{2}\int_{\Omega} f(az)\left|g(az)-g(az^{-1})\right|\omega(az)d\mu(z) \\
+\frac{1}{2}\int_{\Omega}\left|f(az)-f(az^{-1})\right|g(az)\omega(az)d\mu(z) \\
+\frac{1}{4}\int_{\Omega}\left|f(az)-f(az^{-1})\right|\left|g(az)-g(az^{-1})\right|\omega(az)d\mu(z).
\end{aligned}
$$

Now by Cauchy-Schwartz inequality, we observe from (3.10) that

(3.11)

$$
\begin{aligned}
0 \le I(a) \le \frac{1}{2}\left(\int_{\Omega}\left(f(az)\left(g(az)-g(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
+\frac{1}{2}\left(\int_{\Omega}\left(g(az)\left(f(az)-f(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
+\frac{1}{4}\left(\int_{\Omega}\left((f(az^{-1})-f(az))\left(g(az)-g(az^{-1})\right)\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Again by Cauchy-Schwartz inequality, we have from (3.10) the following inequality

$$
\begin{aligned}
0 \le I(a) \le \frac{1}{2}\left(\int_{\Omega} f^2(az)d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\left(g(az)-g(az^{-1})\right)^2\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
+\frac{1}{2}\left(\int_{\Omega} g^2(az)d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\left(f(az)-f(az^{-1})\right)^2\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
+\frac{1}{4}\left(\int_{\Omega}\left(f(az)-f(az^{-1})\right)^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega}\left(g(az)-g(az^{-1})\right)^2\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
(3.12)\quad 0 \le I(a) \le \frac{1}{\sqrt{2}}\left(\int_{\Omega} f^2(az)d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\Omega} g^2(az)\omega^2(az)d\mu(z)\right. \\
\left.-\int_{\Omega} g(az)g(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}\left(\int_{\Omega} g^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
\times\left(\int_{\Omega} f^2(az)\omega^2(az)d\mu(z)-\int_{\Omega} f(az)f(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} \\
+\frac{1}{2}\left(\int_{\Omega} f^2(az)d\mu(z)-\int_{\Omega} f(az)f^{-1}(az)d\mu(z)\right)^{\frac{1}{2}} \\
\times\left(\int_{\Omega} g^2(az)\omega^2(az)d\mu(z)-\int_{\Omega} g(az)g(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using Cauchy-Schwartz inequality again, we have from (3.10) the following inequality

$$
\begin{aligned}
(3.13)\quad 0 \le I(a) \le{}& \frac{1}{\sqrt{2}}\left(\int_\Omega g^2(az)d\mu(z) - \int_\Omega g^2(az)g(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\
&\times\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} + \frac{1}{\sqrt{2}}\left(\int_\Omega f^2(az)d\mu(z) - \int_\Omega f^2(az)f(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\
&\times\left(\int_\Omega g^2(az)\omega^2(az)d\mu(z)\right)^{\frac{1}{2}} + \frac{1}{2}\left(\int_\Omega g^2(az)d\mu(z) - \int_\Omega g^2(az)g(az^{-1})d\mu(z)\right)^{\frac{1}{2}} \\
&\times\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z) - \int_\Omega f^2(az)f(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}.
\end{aligned}
$$

Lastly, by using the properties of absolute value, we have

$$
(3.14)\quad 0 \le I(a) \le \frac{3}{2}\int_\Omega f(az)g(az)\omega(az)d\mu(z) + \frac{3}{2}\int_\Omega f(az)g(az^{-1})\omega(az)d\mu(z).
$$

The inequality (3.8) follows from (3.11)-(3.14). This completes the proof of the theorem. □

Corollary 1. *Suppose the assumptions of Theorem 6 are satisfied and if $g \equiv 1$ on Ω in Theorem 6, we have*

$$
(3.15)\qquad f(a)\int_\Omega \omega(az)d\mu(z) \le \int_\Omega f(az)\omega(az)d\mu(z) + I(a),
$$

where

$$
I(a) = \frac{1}{2}\int_\Omega \left|f(az) - f(az^{-1})\right|\omega(az)d\mu(z).
$$

Furthermore, $I(a)$ satisfies the following inequality

$$
(3.16)\quad 0 \le I(a)
$$
$$
\le \min\left\{\begin{array}{l}
\frac{1}{\sqrt{2}}\sqrt{\mu(\Omega)}\left(\int_\Omega f^2(az)\omega^2(az)d\mu(z) - \int_\Omega f(az)f(az^{-1})\omega^2(az)d\mu(z)\right)^{\frac{1}{2}}, \\
\frac{1}{\sqrt{2}}\left(\int_\Omega f^2(az)d\mu(z) - \int_\Omega f^2(az)f(az^{-1})d\mu(z)\right)^{\frac{1}{2}}\left(\int_\Omega \omega^2(az)d\mu(z)\right)^{\frac{1}{2}}, \\
\int_\Omega f(az)\omega(az)d\mu(z).
\end{array}\right\}
$$

Definition 5. [12] *The function $f : \Omega \to \mathbb{R}$ is said to be a P-function in Ω, if*

$$
f(a) \le f(az) + f(az^{-1})
$$

for all $a \in \Omega$ and $z \in G$ such that $az, az^{-1} \in \Omega$.

Theorem 7. *Let G be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. If f, g are measurable non-negative real valued P-functions on Ω such that $fg \in L_1(\Omega)$. If $\omega : \Omega \to \mathbb{R}$ is non-negative* *symmetric to* a, $\omega \in L_1(\Omega)$ *and* $fg\omega \in L_1(\Omega)$, *we have*

$$
\begin{aligned}
(3.17)\quad & f(a)\, g(a) \int_{\Omega} \omega(az)\, d\mu(z) \\
& \leq 2 \int_{\Omega} f(az) g(az) \omega(az)\, d\mu(z) + 2 \int_{\Omega} f(az) g(az^{-1}) \omega(az)\, d\mu(z) \\
& \qquad = 2 \int_{\Omega} f(az) g(az) \omega(az)\, d\mu(z) + 2 \int_{\Omega} f(az^{-1}) g(az) \omega(az)\, d\mu(z).
\end{aligned}
$$

Proof. Since f and g are P-functions and ω is non-negative and symmetric to a, we have

$$
\begin{aligned}
f(a)\, g(a)\, \omega(az) &\leq \left(f(az) + f(az^{-1}) \right) \left(g(az) + g(az^{-1}) \right) \omega(az) \\
&= f(az) g(az) \omega(az) + f(az^{-1}) g(az^{-1}) \omega(az) \\
&+ f(az) g(az^{-1}) \omega(az) + f(az^{-1}) g(az) \omega(az).
\end{aligned}
$$

Integrating this inequality on Ω, we get

$$
\begin{aligned}
& f(a)\, g(a) \int_{\Omega} \omega(az)\, d\mu(z) \\
& \leq \int_{\Omega} f(az) g(az) \omega(az)\, d\mu(z) + \int_{\Omega} f(az^{-1}) g(az^{-1}) \omega(az)\, d\mu(z) \\
& + \int_{\Omega} f(az) g(az^{-1}) \omega(az)\, d\mu(z) + \int_{\Omega} f(az^{-1}) g(az) \omega(az)\, d\mu(z) \\
& = 2 \int_{\Omega} f(az) g(az) \omega(az)\, d\mu(z) + 2 \int_{\Omega} f(az) g(az^{-1}) \omega(az)\, d\mu(z) \\
& = 2 \int_{\Omega} f(az) g(az) \omega(az)\, d\mu(z) + 2 \int_{\Omega} f(az^{-1}) g(az) \omega(az)\, d\mu(z).
\end{aligned}
$$

Hence the proof of the theorem is completed. □

Corollary 2. *Suppose that the assumptions of Theorem 7 are satisfied and if* $g \equiv 1$ *on* Ω, *we have*

$$
(3.18)\qquad f(a) \int_{\Omega} \omega(az)\, d\mu(z) \leq 4 \int_{\Omega} f(az) \omega(az)\, d\mu(z).
$$

Corollary 3. *If we take* $a = e$ *and* $\omega \equiv 1$ *on* Ω *in Corollary 2, we have*

$$
(3.19)\qquad f(e) \leq \frac{4}{\mu(\Omega)} \int_{\Omega} f(z) d\mu(z).
$$

Some of the applications of our results are given in the following remarks.

Remark 4. *Set* $G = \mathbb{R}$. *Since* $\mathbb{R}$ *is an abelian additive group, thus, for all* $a, z \in \mathbb{R}$, $a - z$ *and* $a + z$ *are points for which* a *is the midpoint. Now, if* $a - z = y$ *and* $a + z = x$, *then* $a = \frac{x+y}{2}$. *If we take* $\Omega = [-b, b]$, *we get* $a = 0$ *and* $y = -x$. *Hence* *from Theorem 5, we have*

$$\begin{aligned} f(0)g(0)\int_{-b}^{b}\omega(x)dx &\leq \frac{1}{2}\left[\int_{-b}^{b} f(x)g(x)\omega(x)dx + \int_{-b}^{b} f(x)g(-x)\omega(x)dx\right] \\ &= \frac{1}{2}\left[\int_{-b}^{b} f(x)g(x)\omega(x)dx + \int_{-b}^{b} f(-x)g(x)\omega(x)dx\right]. \end{aligned} \tag{3.20}$$

If $\omega(x) \equiv 1$ *for all* $x \in [-b, b]$ *in (3.20), we obtain*

$$\begin{aligned} f(0)g(0) &\leq \frac{1}{4b}\left[\int_{-b}^{b} f(x)g(x)dx + \int_{-b}^{b} f(x)g(-x)dx\right] \\ &= \frac{1}{4b}\left[\int_{-b}^{b} f(x)g(x)dx + \int_{-b}^{b} f(-x)g(x)dx\right]. \end{aligned} \tag{3.21}$$

Remark 5. *If in the Theorem 5,* $G = \mathbb{R}^n$ *with an additive operation and* $\Omega = X$ *is an open bounded symmetric and convex subset of* $\mathbb{R}^n$ *, then the result of Theorem 4 holds.*

References

[1] A. Chademan and F. Mirzapour, Midconvex functions in locally compact groups, Proc. Amer. Math. Soc., 127, 10 (1999), 2961-2968.

[2] K. Chandrasekharan, A course on topological groups, Volume 9 of Texts and readings in mathematics, Hindustan Book Agency, New Delhi, 1996, 117 pp.

[3] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequalities, Bull. Austral. Math. Soc., 57, (1998), 377-385.

[4] S. S. Dragomir and C. E. M. Pearce, Select topics on Hermite-Hadamard inequalities and applications, Melbourne and Adelaide, 2000.

[5] S. S. Dragomir, A mapping in connection to Hadamard's inequality, An Ostro. Akad. Wiss. Math.-Natur (Wien), 128 (1991), 17-20.

[6] S. S. Dragomir, Two mappings in connection to Hadamard's inequality, J. Math. Anal. Appl., 167 (1992), 49–56.

[7] S. S. Dragomir, On Hadamard's inequality for convex functions, Math. Balkanica, 6 (1992), 215-222.

[8] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.

[9] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.

[10] P. J. Higgins, An Introduction to Topological Groups, London Mathematical Society Lecture Note Series, No. 15. Cambridge University Press, London-New York, 1974, 109 pp.

[11] F. Mirzapour and A. Morassaei, Quasi-convex functions in topological groups, Int. J. Appl. Math., 16, 4 (2004), 281–290.

[12] A. Morassaei, On Hadamard's inequality for convex mappings defined in topological groups and connected results, J. Math, Inequal., Vol. 4, No. 3, 2010, 445-452.

[13] A. M. Rubinov, Abstract convexity and global optimization, Kluwer Academic Publishers, Dordrecht, 2000.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

E-mail address: m_amer_latif@hotmail.com

[1] School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC8001, Australia

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: sever.dragomir@vu.edu.au

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: Ebrahim.Momoniat@wits.ac.za