

**INEQUALITIES, ASYMPTOTIC EXPANSIONS AND COMPLETELY
MONOTONIC FUNCTIONS RELATED TO THE GAMMA FUNCTION**

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ABSTRACT. In this paper, we present some completely monotonic functions and asymptotic expansions related to the gamma function. Based on the obtained expansions, we provide new bounds for $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$ and $\Gamma(x+\frac{1}{2})$.

1. INTRODUCTION

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1, 2, 3, \dots\}). \quad (1.1)$$

Dubourdieu [12, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a, \infty)$, then strict inequality holds true in (1.1). See also [15] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$ (see [36, p. 161]). The main properties of completely monotonic functions are given in [36, Chapter IV]. We also refer to [4], where an extensive list of references on completely monotonic functions can be found.

Euler's gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas. The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \log \Gamma(x) = \int_1^x \psi(t) dt$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(x)$:

$$\psi^{(n)}(x) := \frac{d^n}{dx^n} \{\psi(x)\}, \quad n \in \mathbb{N}$$

are called the polygamma functions.

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2. LEMMAS

The Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are defined by the generating functions

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

The rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called Bernoulli and Euler numbers, respectively.

It follows from Problem 154 in Part I, Chapter 4, of [27] that

$$\sum_{j=1}^{2m} \frac{B_{2j}}{(2j)!} t^{2j} < \frac{t}{e^t-1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{(2j)!} t^{2j} \quad (2.1)$$

for $t > 0$ and $m \in \mathbb{N}_0$. The inequality (2.1) can be also found in [16, 28].

Lemma 1 presents an analogue of the inequality (2.1).

Lemma 1. *For $x > 0$ and $m \in \mathbb{N}$,*

$$\sum_{j=2}^{2m+1} \frac{(1-2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} < \frac{2}{e^x+1} - 1 + \frac{x}{2} < \sum_{j=2}^{2m} \frac{(1-2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!}, \quad (2.2)$$

where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers.

Proof. The noted Boole's summation formula (see [33, p. 17]) states for $k \in \mathbb{N}$ that

$$f(1) = \frac{1}{2} \sum_{j=0}^{k-1} \frac{E_j(1)}{j!} \left(f^{(j)}(1) + f^{(j)}(0) \right) + \frac{1}{2(k-1)!} \int_0^1 f^{(k)}(t) E_{k-1}(t) dt,$$

which can be written for $m \in \mathbb{N}$ as

$$\begin{aligned} f(1) - f(0) &= \sum_{j=1}^m \frac{E_{2j-1}(1)}{(2j-1)!} \left(f^{(2j-1)}(1) + f^{(2j-1)}(0) \right) \\ &\quad + \frac{1}{(2m-1)!} \int_0^1 f^{(2m)}(t) E_{2m-1}(t) dt. \end{aligned} \quad (2.3)$$

Applying formula (2.3) to $f(t) = e^{xt}$, we obtain

$$-\frac{2}{e^x+1} + 1 - \frac{x}{2} = \sum_{j=2}^m \frac{E_{2j-1}(1)}{(2j-1)!} x^{2j-1} + \frac{x}{e^x+1} \frac{x^{2m-1}}{(2m-1)!} \int_0^1 e^{xt} E_{2m-1}(t) dt. \quad (2.4)$$

It is well known (see [1, p. 804]) that

$$E_{2m+1}(1-t) = -E_{2m+1}(t) \quad \text{and} \quad E_{2m+1}\left(\frac{1}{2}\right) = 0.$$

Noting that

$$E_{4m-1}(t) > 0, \quad E_{4m+1}(t) < 0 \quad \text{for} \quad 0 < t < 1/2, \quad m = 1, 2, \dots,$$

we imply for $x > 0$ that

$$\int_0^1 e^{xt} E_{4m-1}(t) dt = \int_0^{1/2} (e^{xt} - e^{x(1-t)}) E_{4m-1}(t) dt < 0$$

and

$$\int_0^1 e^{xt} E_{4m+1}(t) dt = \int_0^{1/2} (e^{xt} - e^{x(1-t)}) E_{4m+1}(t) dt > 0.$$

Combining these with (2.4), we immediately obtain that for $x > 0$ and $m \in \mathbb{N}$,

$$-\sum_{j=2}^{2m+1} \frac{E_{2j-1}(1)}{(2j-1)!} x^{2j-1} < \frac{2}{e^x + 1} - 1 + \frac{x}{2} < -\sum_{j=2}^{2m} \frac{E_{2j-1}(1)}{(2j-1)!} x^{2j-1}. \quad (2.5)$$

Noting that

$$E_n(1) = \frac{2(2^{n+1} - 1)}{n+1} B_{n+1}, \quad n \in \mathbb{N},$$

the inequality (2.5) can be written as (2.2). The proof of Theorem 1 is complete. \square

The inequality (2.2) can be written for $x > 0$ and $m \in \mathbb{N}_0$ as

$$\sum_{j=1}^{2m+1} \frac{(1-2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} < \frac{2}{e^x + 1} - 1 < \sum_{j=1}^{2m} \frac{(1-2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!}, \quad (2.6)$$

i.e.,

$$(-1)^{m+1} \left(\frac{2}{e^x + 1} - 1 - \sum_{j=1}^m \frac{(1-2^{2j})B_{2j}}{j} \frac{x^{2j-1}}{(2j-1)!} \right) > 0. \quad (2.7)$$

Lemma 2 ([9]). *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(1 + \sum_{j=1}^{\infty} \frac{b_j}{x^j} \right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (2.8)$$

where the coefficients $b_j = b_j(\ell, r)$ ($j \in \mathbb{N}$) are given by

$$b_j = b_j(\ell, r) = \sum \frac{r^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{B_2}{1 \cdot 2} \right)^{k_1} \left(\frac{B_3}{2 \cdot 3} \right)^{k_2} \dots \left(\frac{B_{j+1}}{j(j+1)} \right)^{k_j}, \quad (2.9)$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (2+\ell)k_2 + \dots + (j+\ell)k_j = j.$$

Lemma 3. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e} \right)^x \left(1 + \sum_{j=1}^{\infty} \frac{c_j}{x^j} \right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (2.10)$$

where the coefficients $c_j = c_j(\ell, r)$ ($j \in \mathbb{N}$) are given by

$$c_j = \sum \frac{(-r)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{(1-2^{-1})B_2}{1 \cdot 2} \right)^{k_1} \left(\frac{(1-2^{-3})B_4}{3 \cdot 4} \right)^{k_2} \dots \left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)} \right)^{k_j}, \quad (2.11)$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \dots + (2j+\ell-1)k_j = j.$$

Proof. The following asymptotic expansion can be found [21, p. 32]

$$\ln \Gamma \left(x + \frac{1}{2} \right) \sim x \ln x - x + \ln \sqrt{2\pi} + \sum_{j=1}^{\infty} \frac{B_{2j}(\frac{1}{2})}{2j(2j-1)} x^{1-2j}, \quad x \rightarrow \infty \quad (2.12)$$

It is well-known (see [1, p. 805]) that

$$B_n(\frac{1}{2}) = -(1-2^{1-n})B_n, \quad n \in \mathbb{N}_0,$$

and then the expansion (2.12) can be rewritten as

$$\frac{\Gamma(x + \frac{1}{2})}{\sqrt{2\pi} (x/e)^x} = \exp \left(\sum_{k=1}^m \frac{-(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k-1}} + \mathcal{R}_m(x) \right), \quad x \rightarrow \infty, \quad (2.13)$$

where $\mathcal{R}_m(x) = O(1/x^{2m+1})$. Further, we have

$$\begin{aligned} & \left(\frac{\Gamma(x + \frac{1}{2})}{\sqrt{2\pi} (x/e)^x} \right)^{r/x^\ell} = e^{r\mathcal{R}_m(x)/x^\ell} \exp \left(\sum_{k=1}^m \frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}} \right) \\ & = e^{r\mathcal{R}_m(x)/x^\ell} \prod_{k=1}^m \left[1 + \left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}} \right) + \frac{1}{2!} \left(\frac{-r(1-2^{1-2k})B_{2k}}{2k(2k-1)x^{2k+\ell-1}} \right)^2 + \dots \right] \\ & = e^{r\mathcal{R}_m(x)/x^\ell} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{1}{k_1!k_2!\dots k_m!} \\ & \quad \times \left(\frac{-r(1-2^{-1})B_2}{1 \cdot 2} \right)^{k_1} \left(\frac{-r(1-2^{-3})B_4}{3 \cdot 4} \right)^{k_2} \dots \left(\frac{-r(1-2^{1-2m})B_{2m}}{(2m-1)(2m)} \right)^{k_m} \\ & \quad \times \frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\dots+(2m+\ell-1)k_m}}. \end{aligned} \quad (2.14)$$

On the other hand, from (2.13) it follows that for any positive integer m ,

$$\left(\frac{\Gamma(x + \frac{1}{2})}{\sqrt{2\pi} (x/e)^x} \right)^{r/x^\ell} = 1 + \sum_{j=1}^m \frac{c_j}{x^j} + O(1/x^{m+1}) \quad (2.15)$$

for some real numbers c_1, \dots, c_m .

Equating the coefficients by the equal powers of x in (2.14) and (2.15), we see that

$$c_j = \sum \frac{(-r)^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{(1-2^{-1})B_2}{1 \cdot 2} \right)^{k_1} \left(\frac{(1-2^{-3})B_4}{3 \cdot 4} \right)^{k_2} \dots \left(\frac{(1-2^{1-2j})B_{2j}}{(2j-1)(2j)} \right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \dots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 3. \square

Lemma 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \left(1 + \sum_{j=1}^{\infty} \frac{p_j}{x^j} \right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (2.16)$$

where the coefficients $p_j = p_j(\ell, r)$ ($j \in \mathbb{N}$) are given by

$$p_j = \sum \frac{r^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{(2^2-1)B_2}{1 \cdot 1 \cdot 2^2} \right)^{k_1} \left(\frac{(2^4-1)B_4}{2 \cdot 3 \cdot 2^4} \right)^{k_2} \dots \left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}} \right)^{k_j}, \quad (2.17)$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \dots + (2j+\ell-1)k_j = j.$$

Proof. From (3.7) we obtain the following asymptotic expansion:

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \exp \left(\sum_{j=1}^{\infty} \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right), \quad x \rightarrow \infty. \quad (2.18)$$

Write (2.18) as

$$\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \sim \exp \left(\sum_{k=1}^m \frac{(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k-1}} + R_m(x) \right), \quad x \rightarrow \infty, \quad (2.19)$$

where $R_m(x) = O(1/x^{2m+1})$. Further, we have

$$\begin{aligned} & \left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \right)^{r/x^\ell} = e^{rR_m(x)/x^\ell} \exp \left(\sum_{k=1}^m \frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}} \right) \\ & = e^{rR_m(x)/x^\ell} \prod_{k=1}^m \left[1 + \left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}} \right) + \frac{1}{2!} \left(\frac{r(2^{2k}-1)B_{2k}}{k(2k-1)2^{2k}x^{2k+\ell-1}} \right)^2 + \dots \right] \\ & = e^{rR_m(x)/x^\ell} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{1}{k_1!k_2!\dots k_m!} \\ & \quad \times \left(\frac{r(2^2-1)B_2}{1 \cdot 1 \cdot 2^2} \right)^{k_1} \left(\frac{r(2^4-1)B_4}{2 \cdot 3 \cdot 2^4} \right)^{k_2} \dots \left(\frac{r(2^{2m}-1)B_{2m}}{m(2m-1)2^{2m}} \right)^{k_m} \\ & \quad \times \frac{1}{x^{(1+\ell)k_1+(3+\ell)k_2+\dots+(2m+\ell-1)k_m}}. \end{aligned} \quad (2.20)$$

On the other hand, from (2.19) it follows that for any positive integer m ,

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \right)^{r/x^\ell} = 1 + \sum_{j=1}^m \frac{p_j}{x^j} + O(1/x^{m+1}) \quad (2.21)$$

for some real numbers p_1, \dots, p_m .

Equating the coefficients by the equal powers of x in (2.20) and (2.21), we see that

$$p_j = \sum \frac{r^{k_1+k_2+\dots+k_j}}{k_1!k_2!\dots k_j!} \left(\frac{(2^2-1)B_2}{1 \cdot 1 \cdot 2^2} \right)^{k_1} \left(\frac{(2^4-1)B_4}{2 \cdot 3 \cdot 2^4} \right)^{k_2} \dots \left(\frac{(2^{2j}-1)B_{2j}}{j(2j-1)2^{2j}} \right)^{k_j},$$

summed over all nonnegative integers k_j satisfying the equation

$$(1+\ell)k_1 + (3+\ell)k_2 + \dots + (2j+\ell-1)k_j = j.$$

This completes the proof of Lemma 4. \square

Lemma 5. For $t \neq 0$,

$$\frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4 - \frac{127}{172800}t^6 < \frac{\cosh t}{\cosh(2t) - 1} < \frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4. \quad (2.22)$$

Proof. We only prove the second inequality in (2.22). The proof of the first inequality in (2.22) is analogous. By using the power series expansion of $\cosh t$, we find that

$$\left(\frac{1}{2t^2} + \frac{1}{12} - \frac{7}{240}t^2 + \frac{31}{6048}t^4 \right) (\cosh(2t) - 1) - \cosh t = \sum_{n=4}^{\infty} \frac{\alpha_n}{(2n+2)!} t^{2n}$$

with

$$\begin{aligned} \alpha_n &= 2^{2n-4} \left(\frac{104}{3} + \frac{10513}{1260}n + \frac{11117}{1890}n^2 - \frac{1577}{1260}n^3 - \frac{4303}{1890}n^4 - \frac{31}{63}n^5 + \frac{62}{189}n^6 \right) \\ &\quad - 2(n+1)(2n+1) \\ &= 2^{2n-4} \left(339 + \frac{1005649}{1260}(n-4) + \frac{1355243}{1890}(n-4)^2 + \frac{382391}{1260}(n-4)^3 \right. \\ &\quad \left. + \frac{125897}{1890}(n-4)^4 + \frac{155}{21}(n-4)^5 + \frac{62}{189}(n-4)^6 \right) - 2(n+1)(2n+1) \\ &> 2^{2n-4} \cdot 339 - 2(n+1)(2n+1) \\ &> 0 \quad \text{for } n \geq 4. \end{aligned}$$

Hence, the second inequality in (2.22) holds. This completes the proof of Lemma 5. \square

3. COMPLETELY MONOTONIC FUNCTIONS

It is known in [33, p. 64] that

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \sum_{j=1}^n \frac{B_{2j}}{(2j)!} t^{2j} + (-1)^n t^{2n+2} \nu_n(t), \quad n \geq 0, \quad (3.1)$$

where

$$\nu_n(t) = \sum_{k=1}^{\infty} \frac{2}{(t^2 + 4\pi^2 k^2)(2\pi k)^{2n}} > 0.$$

It is easy to see that (3.1) implies (2.1).

The noted Binet's first formula [32, p. 16] states that

$$\ln \Gamma(x) = \left(x - \frac{1}{2} \right) \ln x - x + \ln \sqrt{2\pi} + \int_0^{\infty} \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) \frac{e^{-xt}}{t^2} dt, \quad x > 0. \quad (3.2)$$

Combining (3.1) with (3.2), Xu and Han [34] deduced in 2009 that for every $m \in \mathbb{N}_0$, the function

$$R_m(x) = (-1)^m \left[\ln \Gamma(x) - \left(x - \frac{1}{2} \right) \ln x + x - \ln \sqrt{2\pi} - \sum_{j=1}^m \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right] \quad (3.3)$$

is completely monotonic on $(0, \infty)$.

For $m = 0$, complete monotonicity property of $R_m(x)$ was proved by Muldoon [26]. Alzer [2] first proved in 1997 that $R_m(x)$ is completely monotonic on $(0, \infty)$. In 2006, Koumandos [16]

proved double inequality (2.1), and then used (2.1) and (3.2) to give the proof of complete monotonicity property of $R_m(x)$. In 2009, Koumandos and Pedersen [17, Theorem 2.1] strengthened this result.

Based on the inequality (2.2), in this section we prove that for every $m \in \mathbb{N}_0$, the function

$$F_m(x) = (-1)^m \left[\ln \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right) - \frac{1}{2} \ln x - \sum_{j=1}^m \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right] \quad (3.4)$$

is completely monotonic on $(0, \infty)$. This result is similar to complete monotonicity property of $R_m(x)$ in (3.3).

Theorem 1. *For every $m \in \mathbb{N}_0$, the function $F_m(x)$, defined by (3.4), is completely monotonic on $(0, \infty)$.*

Proof. The logarithm of the gamma function has the following integral representation (see [1, p. 258])

$$\ln \Gamma(x) = \int_0^\infty \left[(x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} \right] \frac{dt}{t}, \quad x > 0. \quad (3.5)$$

By using (3.5) and the following representations

$$\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \quad x > 0$$

in [1, p. 230, 5.1.32] and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt, \quad x > 0 \quad \text{and} \quad r > 0$$

in [1, p. 255, 6.1.1], we find that

$$\begin{aligned} F_m(x) &= (-1)^m \left[\int_0^\infty \left(-\frac{1}{e^{t/2} + 1} + \frac{1}{2} \right) \frac{e^{-xt}}{t} dt \right. \\ &\quad \left. - \sum_{j=1}^m \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)!} \int_0^\infty t^{2j-2} e^{-xt} dt \right] \\ &= \frac{1}{2} \int_0^\infty (-1)^{m+1} \lambda_m(t) \frac{e^{-xt}}{t} dt, \end{aligned} \quad (3.6)$$

where

$$\lambda_m(t) = \frac{2}{e^{t/2} + 1} - 1 - \sum_{j=1}^m \frac{(1 - 2^{2j})B_{2j}}{j(2j-1)!} \left(\frac{t}{2} \right)^{2j-1}.$$

By (2.7), we have $(-1)^{m+1} \lambda_m(t) > 0$ for $t > 0$ and $m \in \mathbb{N}_0$. From (3.6) we obtain that for every $m \in \mathbb{N}_0$,

$$(-1)^n F_m^{(n)}(x) = \frac{1}{2} \int_0^\infty (-1)^{m+1} \lambda_m(t) t^{n-1} e^{-xt} dt > 0$$

for $x > 0$ and $n \in \mathbb{N}_0$. The proof of Theorem 1 is complete. \square

Under the inequality $(-1)^n F_m^{(n)}(x) > 0$ for $x > 0$ and $m, n \in \mathbb{N}_0$, we obtain the following

Corollary 1. (i) Let $m \in \mathbb{N}_0$. Then for $x > 0$,

$$\begin{aligned} \sqrt{x} \exp \left(\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right) &< \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \\ &< \sqrt{x} \exp \left(\sum_{j=1}^{2m+1} \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \right). \end{aligned} \quad (3.7)$$

(ii) Let $m, n \in \mathbb{N}$. Then for $x > 0$,

$$\begin{aligned} \sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}} \\ &< (-1)^n \left(\psi^{(n-1)}(x+1) - \psi^{(n-1)} \left(x + \frac{1}{2} \right) \right) + \frac{(n-1)!}{2x^n} \\ &< \sum_{j=1}^{2m-1} \left(1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{(2j)!} \frac{(2j+n-2)!}{x^{2j+n-1}}. \end{aligned} \quad (3.8)$$

By using the obtained results above, we here present inequalities and integral representations for the constant π .

The problem of finding new and sharp inequalities for the gamma function Γ and in particular about the Wallis ratio

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} = \frac{1}{\sqrt{\pi}} \cdot \frac{(2n)!!}{(2n-1)!!}, \quad n \in \mathbb{N} \quad (3.9)$$

has attracted the attention of many researchers (see [10, 18, 19, 23] and references therein). Some inequalities for π can be found (see, for example, [14, 20, 24, 25]). Here, we employ the special double factorial notation as follows:

$$\begin{aligned} (2n)!! &= 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \\ (2n-1)!! &= 1 \cdot 3 \cdot 5 \cdots (2n-1) = \pi^{-1/2} 2^n \Gamma(n+\frac{1}{2}), \\ 0!! &= 1, \quad (-1)!! = 1 \end{aligned}$$

(see [1, p. 258]). Very recently, Lin [20, Theorem 2.4] proved that for all $n \in \mathbb{N}$,

$$\begin{aligned} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \frac{1}{n} \exp \left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} - \frac{31}{9216n^9} \right) \\ &< \pi < \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \frac{1}{n} \exp \left(-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} \right). \end{aligned} \quad (3.10)$$

Setting $x = n$ in (3.7), we obtain estimate for the constant π :

$$\begin{aligned} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \frac{1}{n} \exp \left(-\sum_{j=1}^{2m+1} \left(1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{j(2j-1)n^{2j-1}} \right) \\ &< \pi < \left(\frac{(2n)!!}{(2n-1)!!} \right)^2 \frac{1}{n} \exp \left(-\sum_{j=1}^{2m} \left(1 - \frac{1}{2^{2j}} \right) \frac{2B_{2j}}{j(2j-1)n^{2j-1}} \right). \end{aligned} \quad (3.11)$$

Obviously, (3.11) is a generalization of (3.10).

Formula (3.6) gives the following integral representation

$$\begin{aligned} \ln \left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right) - \frac{1}{2} \ln x - \sum_{j=1}^m \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j-1)x^{2j-1}} \\ = -\frac{1}{2} \int_0^\infty \left(\frac{2}{e^{t/2} + 1} - 1 - \sum_{j=1}^m \frac{(1-2^{2j})B_{2j}}{j(2j-1)!} \left(\frac{t}{2} \right)^{2j-1} \right) \frac{e^{-xt}}{t} dt, \end{aligned} \quad (3.12)$$

which implies

$$\begin{aligned} \psi(x+1) - \psi \left(x + \frac{1}{2} \right) - \frac{1}{2x} + \sum_{j=1}^m \left(1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{jx^{2j}} \\ = \frac{1}{2} \int_0^\infty \left(\frac{2}{e^{t/2} + 1} - 1 - \sum_{j=1}^m \frac{(1-2^{2j})B_{2j}}{j(2j-1)!} \left(\frac{t}{2} \right)^{2j-1} \right) e^{-xt} dt \end{aligned} \quad (3.13)$$

for $x > 0$ and $m \in \mathbb{N}_0$. Formulas (3.12) and (3.13) can provide integral representations for the constant π . For example, setting $(x, m) = (1/2, 0)$ in (3.12) yields

$$\int_0^\infty \left(1 - \frac{2}{e^u + 1} \right) \frac{e^{-u}}{u} du = \ln \left(\frac{\pi}{2} \right). \quad (3.14)$$

Setting $(x, m) = (1/4, 1)$ in (3.13) yields

$$\int_0^\infty \left(\frac{2}{e^u + 1} - 1 + \frac{u}{2} \right) e^{-u/2} du = 4 - \pi. \quad (3.15)$$

Many formulas exist for the representation of π , and a collection of these formulas is listed [30, 31]. For more history of π see [6, 7, 13].

4. ASYMPTOTIC EXPANSIONS

Stirling's formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n, \quad n \in \mathbb{N} := \{1, 2, \dots\} \quad (4.1)$$

has many applications in statistical physics, probability theory and number theory. Actually it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667-1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n} (n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692-1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Equation (6.1.40)]) by

$$\begin{aligned} \Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \exp \left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}} \right) \\ = \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \exp \left(\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \dots \right) \end{aligned} \quad (4.2)$$

as $x \rightarrow \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers. The following asymptotic formula is due to Laplace

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \dots\right) \quad (4.3)$$

as $x \rightarrow \infty$ (see [1, p. 257, Equation (6.1.37)]). The expression (4.3) is sometimes incorrectly called Stirling's series (see [11, pp. 2-3]). Stirling's formula is in fact the first approximation to the asymptotic formula (4.3). Stirling's formula has attracted much interest of many mathematicians and have motivated a large number of research papers concerning various generalizations and improvements (see [8] and the references cited therein). See also an overview at [22].

Windschitl (see [5, p. 128] and [37]) had noted that for $x > 8$, the approximation

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} \quad (4.4)$$

gives at least eight decimal places of the gamma function. The formula (4.4) derives

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \quad x \rightarrow \infty. \quad (4.5)$$

Inspired by (4.4) and (4.5), Alzer [3] proved that for all $x > 0$,

$$\sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{\alpha}{x^5}\right) < \Gamma(x+1) < \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{\beta}{x^5}\right) \quad (4.6)$$

with the best possible constants $\alpha = 0$ and $\beta = 1/1620$.

Smith [29, Eq. (43)] presented the following analogous result to (4.4):

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \left(1 + O\left(\frac{1}{x^5}\right)\right), \quad x \rightarrow \infty. \quad (4.7)$$

Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. We determine the coefficients $a_j = a_j(\ell, r)$ and $d_j = d_j(\ell, r)$ (for $j \in \mathbb{N}$) such that

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}$$

and

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x} + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell/r}$$

as $x \rightarrow \infty$. From which we derive the extensions for (4.5) and (4.7) as follows:

$$\begin{aligned} \Gamma(x+1) &\sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \\ &\quad \times \left(1 + \frac{1}{1620x^5} - \frac{11}{18900x^7} + \frac{143}{170100x^9} + \frac{1}{5248800x^{10}} \right. \\ &\quad \left. - \frac{2260261}{1178793000x^{11}} - \frac{11}{30618000x^{12}} + \dots\right) \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} \Gamma\left(x + \frac{1}{2}\right) &\sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \\ &\times \left(1 - \frac{31}{51840x^5} + \frac{1397}{2419200x^7} - \frac{10439}{12441600x^9} + \frac{961}{5374771200x^{10}} \right. \\ &\left. + \frac{4626754267}{2414168064000x^{11}} - \frac{43307}{125411328000x^{12}} - \frac{26820782411}{4184557977600x^{13}} + \dots\right) \end{aligned} \quad (4.9)$$

as $x \rightarrow \infty$.

Theorem 2. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \sum_{j=1}^{\infty} \frac{a_j}{x^j}\right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (4.10)$$

with the coefficients $a_j = a_j(\ell, r)$ ($j \in \mathbb{N}$) given by

$$a_{2j-1} = b_{2j-1}(\ell, r), \quad a_{2j} = b_{2j}(\ell, r) - \frac{1}{(2j+1)!}, \quad (4.11)$$

where $b_j(\ell, r)$ ($j \in \mathbb{N}$) can be calculated by (2.9).

Proof. The Maclaurin expansion of $\sinh t$ with $t = 1/x$ gives

$$x \sinh \frac{1}{x} = 1 + \sum_{j=1}^{\infty} \frac{1}{(2j+1)!x^{2j}}, \quad x \neq 0. \quad (4.12)$$

In view of (2.8) and (4.12), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^{r/x^\ell} - x \sinh \frac{1}{x} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \quad x \rightarrow \infty, \quad (4.13)$$

where a_j ($j \in \mathbb{N}$) are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{b_j}{x^j} - \sum_{j=1}^{\infty} \frac{1}{(2j+1)!x^{2j}} \sim \sum_{j=1}^{\infty} \frac{a_j}{x^j}, \quad x \rightarrow \infty, \quad (4.14)$$

where $b_j = b_j(\ell, r)$ ($j \in \mathbb{N}$) are given in (2.9). Equating coefficients of equal powers of x in (4.14) yields (4.11). The proof of Theorem 2 is complete. \square

Remark 1. *Setting $(r, \ell) = (2, 1)$ in (4.10) yields full asymptotic expansion of Windschitl's formula (4.4):*

$$\begin{aligned} \Gamma(x+1) &\sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6} - \frac{163}{170100x^8} + \frac{1019}{680400x^{10}} \right. \\ &\left. - \frac{50405273}{14145516000x^{12}} + \frac{698912219}{57316896000x^{14}} - \dots\right)^{x/2}, \quad x \rightarrow \infty. \end{aligned} \quad (4.15)$$

Write (4.15) as

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{\lambda(x)}{x \sinh \frac{1}{x}}\right)^{x/2}, \quad (4.16)$$

where

$$\lambda(x) = \frac{1}{810x^6} - \frac{163}{170100x^8} + \frac{1019}{680400x^{10}} - \frac{50405273}{14145516000x^{12}} + \frac{698912219}{57316896000x^{14}} - \cdots.$$

We find that

$$\begin{aligned} \left(1 + \frac{\lambda(x)}{x \sinh \frac{1}{x}}\right)^{x/2} &\sim 1 + \frac{1}{1620x^5} - \frac{11}{18900x^7} + \frac{143}{170100x^9} + \frac{1}{5248800x^{10}} \\ &\quad - \frac{2260261}{1178793000x^{11}} - \frac{11}{30618000x^{12}} + \cdots, \quad x \rightarrow \infty. \end{aligned} \quad (4.17)$$

Combining (4.16) with (4.17) yields (4.8).

Theorem 3. Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The gamma function has the following asymptotic expansion:

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x} + \sum_{j=1}^{\infty} \frac{d_j}{x^j}\right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (4.18)$$

with the coefficients $d_j = d_j(\ell, r)$ ($j \in \mathbb{N}$) given by

$$d_{2j-1} = c_{2j-1}(\ell, r), \quad d_{2j} = c_{2j}(\ell, r) - \frac{4(2^{2j+2} - 1)B_{2j+2}}{(2j+2)!}, \quad (4.19)$$

where $c_j(\ell, r)$ ($j \in \mathbb{N}$) can be calculated by (2.11).

Proof. The power series expansion of $\tanh t$ with $t = 1/(2x)$ gives

$$2x \tanh \frac{1}{2x} = 1 + \sum_{j=2}^{\infty} \frac{4(2^{2j} - 1)B_{2j}}{(2j)!} \frac{1}{x^{2j-2}}, \quad |x| > \frac{1}{\pi}. \quad (4.20)$$

In view of (2.10) and (4.20), we can let

$$\left(\frac{\Gamma(x + \frac{1}{2})}{\sqrt{2\pi}(x/e)^x}\right)^{r/x^\ell} - 2x \tanh \frac{1}{2x} \sim \sum_{j=1}^{\infty} \frac{d_j}{x^j}, \quad x \rightarrow \infty, \quad (4.21)$$

where d_j ($j \in \mathbb{N}$) are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{c_j}{x^j} - \sum_{j=1}^{\infty} \frac{4(2^{2j+2} - 1)B_{2j+2}}{(2j+2)!} \frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty} \frac{d_j}{x^j}, \quad x \rightarrow \infty, \quad (4.22)$$

where $c_j = c_j(\ell, r)$ ($j \in \mathbb{N}$) are given in (2.11). Equating coefficients of equal powers of x in (4.22) yields (4.27). The proof of Theorem 3 is complete. \square

Remark 2. Setting $(r, \ell) = (2, 1)$ in (4.18) yields the following full asymptotic expansion:

$$\begin{aligned} \Gamma\left(x + \frac{1}{2}\right) &\sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x} - \frac{31}{25920x^6} + \frac{6829}{5443200x^8} - \frac{38849}{21772800x^{10}} \right. \\ &\quad \left. + \frac{3606938263}{905313024000x^{12}} - \frac{530812593173}{40351094784000x^{14}} + \cdots\right)^{x/2}, \quad x \rightarrow \infty. \end{aligned} \quad (4.23)$$

Write (4.23) as

$$\Gamma\left(x + \frac{1}{2}\right) \sim \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \left(1 + \frac{\mu(x)}{2x \tanh \frac{1}{2x}}\right)^{x/2}, \quad x \rightarrow \infty, \quad (4.24)$$

where

$$\begin{aligned} \mu(x) = & -\frac{31}{25920x^6} + \frac{6829}{5443200x^8} - \frac{38849}{21772800x^{10}} \\ & + \frac{3606938263}{905313024000x^{12}} - \frac{530812593173}{40351094784000x^{14}} + \cdots. \end{aligned}$$

We find that

$$\begin{aligned} \left(1 + \frac{\mu(x)}{2x \tanh \frac{1}{2x}}\right)^{x/2} \sim & 1 - \frac{31}{51840x^5} + \frac{1397}{2419200x^7} - \frac{10439}{12441600x^9} + \frac{961}{5374771200x^{10}} \\ & + \frac{4626754267}{2414168064000x^{11}} - \frac{43307}{125411328000x^{12}} \\ & - \frac{26820782411}{4184557977600x^{13}} + \cdots, \quad x \rightarrow \infty. \end{aligned} \quad (4.25)$$

Combining (4.24) with (4.25) yields (4.9).

From (4.5) and (4.7), we derive

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \left(1 + O\left(\frac{1}{x^5}\right) \right), \quad x \rightarrow \infty.$$

This fact motivated us to observe the following

Theorem 4. *Let $r \neq 0$ be a given real number and $\ell \geq 0$ be a given integer. The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \left(\cosh \frac{1}{2x} + \sum_{j=1}^{\infty} \frac{q_j}{x^j} \right)^{x^\ell/r}, \quad x \rightarrow \infty, \quad (4.26)$$

with the coefficients $q_j = q_j(\ell, r)$ ($j \in \mathbb{N}$) given by

$$2q_{j-1} = p_{2j-1}(\ell, r), \quad q_{2j} = p_{2j}(\ell, r) - \frac{1}{2^{2j}(2j)!}, \quad (4.27)$$

where $p_j(\ell, r)$ ($j \in \mathbb{N}$) can be calculated by (2.17).

Proof. The Maclaurin expansion of $\cosh t$ with $t = 1/(2x)$ gives

$$\cosh \frac{1}{2x} = 1 + \sum_{j=1}^{\infty} \frac{1}{2^{2j}(2j)!} \frac{1}{x^{2j}}, \quad |x| \neq 0. \quad (4.28)$$

In view of (2.16) and (4.28), we can let

$$\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})} \right)^{r/x^\ell} - \cosh \frac{1}{2x} \sim \sum_{j=1}^{\infty} \frac{q_j}{x^j}, \quad x \rightarrow \infty, \quad (4.29)$$

where q_j ($j \in \mathbb{N}$) are real numbers to be determined. It follows that

$$\sum_{j=1}^{\infty} \frac{p_j}{x^j} - \sum_{j=1}^{\infty} \frac{1}{2^{2j}(2j)!} \frac{1}{x^{2j}} \sim \sum_{j=1}^{\infty} \frac{q_j}{x^j}, \quad x \rightarrow \infty, \quad (4.30)$$

where $p_j = p_j(\ell, r)$ ($j \in \mathbb{N}$) are given in (2.17). Equating coefficients of equal powers of x in (4.30) yields (4.27). The proof of Theorem 3 is complete. \square

Remark 3. Setting $(r, \ell) = (1, 1)$ in (4.26) yields the following full asymptotic expansion:

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \left(\cosh \frac{1}{2x} + \frac{7}{5760x^6} - \frac{65}{64512x^8} + \frac{357179}{232243200x^{10}} - \frac{31761707}{8758886400x^{12}} + \cdots \right)^x, \quad x \rightarrow \infty. \quad (4.31)$$

Write (4.31) as

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \sim \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \left(1 + \frac{\nu(x)}{\cosh \frac{1}{2x}} \right)^x, \quad x \rightarrow \infty, \quad (4.32)$$

where

$$\nu(x) = \frac{7}{5760x^6} - \frac{65}{64512x^8} + \frac{357179}{232243200x^{10}} - \frac{31761707}{8758886400x^{12}} + \cdots.$$

We find that

$$\left(1 + \frac{\nu(x)}{\cosh \frac{1}{2x}} \right)^x \sim 1 + \frac{7}{5760x^5} - \frac{187}{161280x^7} + \frac{48763}{29030400x^9} + \frac{49}{66355200x^{10}} - \frac{29383393}{7664025600x^{11}} - \frac{187}{132710400x^{12}} + \cdots \quad (4.33)$$

as $x \rightarrow \infty$. Combining (4.32) with (4.33) yields

$$\begin{aligned} \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} &\sim \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \\ &\times \left(1 + \frac{7}{5760x^5} - \frac{187}{161280x^7} + \frac{48763}{29030400x^9} + \frac{49}{66355200x^{10}} - \frac{29383393}{7664025600x^{11}} - \frac{187}{132710400x^{12}} + \cdots \right) \quad \text{as } x \rightarrow \infty. \end{aligned} \quad (4.34)$$

5. INEQUALITIES

Theorem 5. For $x > 0$,

$$\sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \left(1 + \frac{\theta_1}{x^5} \right) < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \left(1 + \frac{\theta_2}{x^5} \right) \quad (5.1)$$

with the best possible constants

$$\theta_1 = 0 \quad \text{and} \quad \theta_2 = \frac{7}{5760}. \quad (5.2)$$

Proof. We first prove the inequality (5.1) with $\theta_1 = 0$ and $\theta_2 = \frac{7}{5760}$. That is

$$\sqrt{x} \left(\cosh \frac{1}{2x} \right)^x < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \left(1 + \frac{7}{5760x^5} \right), \quad x > 0. \quad (5.3)$$

Lambert's continued fraction [35, p. 349]

$$\tanh(z) = \frac{z}{1 + \frac{z^2}{3 + \frac{z^2}{5 + \frac{z^2}{7 + \cdots}}}}$$

is valid for all values of z . Hence for $x > 0$,

$$\frac{6x}{12x^2 + 1} = \frac{1/(2x)}{1 + \frac{(1/(2x))^2}{3}} < \tanh \frac{1}{2x} < \frac{1/(2x)}{1 + \frac{(1/(2x))^2}{3 + \frac{(1/(2x))^2}{5}}} = \frac{60x^2 + 1}{12x(10x^2 + 1)}. \quad (5.4)$$

The proof of the inequality (5.3) make use of the inequality (5.4).

The lower bound in (5.3) is obtained by considering the function $f(x)$ defined for $x > 0$ by

$$f(x) = \ln \Gamma(x+1) - \ln \Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2} \ln x - x \ln \left(\cosh \frac{1}{2x} \right).$$

Differentiation yields

$$f'(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2x} - \ln \left(\cosh \frac{1}{2x} \right) + \frac{1}{2x} \tanh \frac{1}{2x} \quad (5.5)$$

and

$$\begin{aligned} f''(x) &= \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3} \left(\tanh \frac{1}{2x} \right)^2 \\ &> \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3} \left(\frac{6x}{12x^2 + 1} \right)^2 =: g(x), \end{aligned}$$

by applying the left-hand inequality of (5.4). By using the recurrence formula

$$\psi'(x+1) = \psi'(x) - \frac{1}{x^2}, \quad (5.6)$$

we find that

$$g(x) - g(x+1) = \frac{h(x)}{4x^3(12x^2 + 1)^2(x+1)^3(2x+1)^2(12x^2 + 24x + 13)^2}$$

with

$$\begin{aligned} h(x) &= 1540909 + 8983929(x-1) + 22585735(x-1)^2 + 32006952(x-1)^3 \\ &\quad + 27981444(x-1)^4 + 15459984(x-1)^5 + 5274000(x-1)^6 \\ &\quad + 1016064(x-1)^7 + 84672(x-1)^8. \end{aligned}$$

Hence, for $x \geq 1$,

$$g(x) > g(x+1) \quad \text{and} \quad g(x) > g(x+n).$$

Therefore, for $x \geq 1$,

$$g(x) > \lim_{n \rightarrow \infty} g(x+n) = 0 \quad \text{and} \quad f''(x) > 0.$$

We then obtain that

$$f'(x) < \lim_{x \rightarrow \infty} f'(x) = 0 \quad \text{for} \quad x \geq 1. \quad (5.7)$$

We now show that (5.7) is also valid for $0 < x \leq 1$. It follows from (5.5) that

$$-f'(x) = y_1(x) + y_2(x)$$

with

$$y_1(x) = \ln \left(\cosh \frac{1}{2x} \right) - \frac{1}{2x} \tanh \frac{1}{2x}$$

and

$$y_2(x) = -\psi(x+1) + \psi\left(x + \frac{1}{2}\right) + \frac{1}{2x}.$$

Differentiation yields

$$y_1'(x) = \frac{1}{4x^3 \left(\cosh \frac{1}{2x}\right)^2} > 0.$$

Using the following representations:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt$$

in [1, p. 259, 6.3.21] and

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt,$$

we conclude that

$$y_2(x) = \int_0^\infty \left(\frac{1}{2} - \frac{1}{e^{t/2} + 1} \right) e^{-xt} dt$$

and

$$y_2'(x) = - \int_0^\infty \left(\frac{1}{2} - \frac{1}{e^{t/2} + 1} \right) t e^{-xt} dt < 0.$$

Let $0 \leq r \leq x \leq s \leq 1$. Since $y_1(x)$ is increasing and $y_2(x)$ is decreasing, we obtain

$$-f'(x) \geq y_1(r) + y_2(s) =: \sigma_1(r, s).$$

We have

$$\sigma_1\left(\frac{k}{100}, \frac{k+1}{100}\right) > 0 \quad \text{for } k = 0, 1, 2, \dots, 99.$$

This implies that $f'(x)$ is negative on $(0, 1]$.

We then obtain that for all $x > 0$,

$$f(x) > \lim_{x \rightarrow \infty} f(x) = 0.$$

This means that the first inequality in (5.3) holds for $x > 0$.

The upper bound in (5.3) is obtained by considering the function $u(x)$ defined for $x > 0$ by

$$u(x) = \ln \Gamma(x+1) - \ln \Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2} \ln x - x \ln \left(\cosh \frac{1}{2x} \right) - \ln \left(1 + \frac{7}{5760x^5} \right).$$

Differentiation yields

$$u'(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2x} - \ln \left(\cosh \frac{1}{2x} \right) + \frac{1}{2x} \tanh \frac{1}{2x} + \frac{35}{x(5760x^5 + 7)} \quad (5.8)$$

and

$$\begin{aligned} u''(x) &= \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} + \frac{1}{4x^3} \left(\tanh \frac{1}{2x}\right)^2 - \frac{35(34560x^5 + 7)}{x^2(5760x^5 + 7)^2} \\ &< \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2x^2} - \frac{1}{4x^3} \\ &\quad + \frac{1}{4x^3} \left(\frac{60x^2 + 1}{12x(10x^2 + 1)}\right)^2 - \frac{35(34560x^5 + 7)}{x^2(5760x^5 + 7)^2} =: v(x), \end{aligned}$$

by applying the right-hand inequality of (5.4). By using the recurrence formula (5.6), we find that

$$v(x) - v(x+1) = -\frac{w_1(x)}{576w_2(x)}$$

with

$$\begin{aligned} w_1(x) &= 868270269104794924464036897 + 18765499992969382192107571641(x-1) \\ &\quad + \cdots + 14820540300656640000000(x-1)^{30} \end{aligned}$$

has all coefficients positive, and

$$\begin{aligned} w_2(x) &= x^5(10x^2 + 1)^2(5760x^5 + 7)^2(x+1)^5(2x+1)^2(10x^2 + 20x + 11)^2 \\ &\quad \times (5760x^5 + 28800x^4 + 57600x^3 + 57600x^2 + 28800x + 5767)^2. \end{aligned}$$

Hence, for $x \geq 1$,

$$v(x) < v(x+1) \quad \text{and} \quad v(x) < v(x+n).$$

Therefore, for $x \geq 1$,

$$v(x) < \lim_{n \rightarrow \infty} v(x+n) = 0 \quad \text{and} \quad u''(x) < 0.$$

We then obtain that

$$u'(x) > \lim_{x \rightarrow \infty} u'(x) = 0 \quad \text{for} \quad x \geq 1. \tag{5.9}$$

We now show that (5.9) is also valid for $0 < x \leq 1$. It follows from (5.8) that

$$u'(x) = y_3(x) + y_4(x),$$

where $y_3(x) = -y_2(x)$ and

$$y_4(x) = -\ln\left(\cosh \frac{1}{2x}\right) + \frac{1}{2x} \tanh \frac{1}{2x} + \frac{35}{x(5760x^5 + 7)}.$$

Differentiation yields

$$y_4'(x) = -\frac{1}{4x^3(\cosh \frac{1}{2x})^2} - \frac{35(34560x^5 + 7)}{x^2(5760x^5 + 7)^2} < 0.$$

Let $0 \leq r \leq x \leq s \leq 1$. Since $y_3(x)$ is increasing and $y_4(x)$ is decreasing, we obtain

$$u'(x) \geq y_3(r) + y_4(s) =: \sigma_2(r, s).$$

We have

$$\sigma_2\left(\frac{k}{100}, \frac{k+1}{100}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \dots, 99.$$

This implies that $u'(x)$ is positive on $(0, 1]$.

We then obtain that for all $x > 0$,

$$u(x) < \lim_{x \rightarrow \infty} u(x) = 0.$$

This means that the second inequality in (5.3) holds for $x > 0$.

The inequality (5.1) can be written as

$$\theta_1 < x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] < \theta_2, \quad x > 0.$$

We find that

$$\theta_1 \leq \lim_{x \rightarrow 0^+} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = 0$$

and

$$\lim_{x \rightarrow \infty} x^5 \left[\frac{\Gamma(x+1)}{\sqrt{x}\Gamma\left(x+\frac{1}{2}\right)\left(\cosh\frac{1}{2x}\right)^x} - 1 \right] = \frac{7}{5760} \leq \theta_2.$$

Hence, inequality (5.1) holds with the best possible constants given in equation (5.2). The proof of Theorem 5 is complete. \square

Remark 4. From (5.3), we derive new inequality for the constant π :

$$\left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n\left(1+\frac{7}{5760n^5}\right)} \frac{(2n)!!}{(2n-1)!!} \right)^2 < \pi < \left(\frac{1}{\sqrt{n}\left(\cosh\frac{1}{2n}\right)^n} \frac{(2n)!!}{(2n-1)!!} \right)^2 \quad (5.10)$$

for $n \in \mathbb{N}$.

Theorem 6. For $x \geq 1$,

$$\begin{aligned} \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x}\right)^{x/2} \left(1 - \frac{31}{51840x^5}\right) \\ < \Gamma\left(x + \frac{1}{2}\right) < \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh\frac{1}{2x}\right)^{x/2}. \end{aligned} \quad (5.11)$$

Proof. From the well-known continued fraction for ψ' (see [35, p. 373])

$$\psi'\left(x + \frac{1}{2}\right) = \frac{1}{x + \frac{a_1}{x + \frac{a_2}{x + \dots}}}, \quad x > 0,$$

where

$$a_p = \frac{p^4}{4(2p-1)(2p+1)}, \quad p = 1, 2, \dots,$$

we find that $x > 0$,

$$\frac{20x(84x^2 + 71)}{3(560x^4 + 520x^2 + 27)} = \frac{1}{x + \frac{\frac{1}{12}}{x + \frac{\frac{4}{15}}{x + \frac{\frac{81}{140}}{x + \frac{1}{x}}}}} < \psi'\left(x + \frac{1}{2}\right) < \frac{1}{x + \frac{\frac{1}{12}}{x + \frac{4}{15}}} = \frac{4(15x^2 + 4)}{3x(20x^2 + 7)}. \quad (5.12)$$

The proof of the inequality (5.11) make use of the inequalities (2.22) and (5.12).

The upper bound in (5.11) is obtained by considering the function $U(x)$ defined for $x \geq 1$ by

$$U(x) = \ln \Gamma \left(x + \frac{1}{2} \right) - \ln(\sqrt{2\pi}) - x \ln x + x - \frac{x}{2} \ln(2x) - \frac{x}{2} \ln \left(\tanh \frac{1}{2x} \right).$$

Differentiation yields

$$U'(x) = \psi \left(x + \frac{1}{2} \right) - \frac{3}{2} \ln x - \frac{1}{2} - \frac{1}{2} \ln 2 - \frac{1}{2} \ln \left(\tanh \frac{1}{2x} \right) + \frac{1}{2x \sinh \frac{1}{x}}.$$

Differentiating $U'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$\begin{aligned} U''(x) &= \psi' \left(x + \frac{1}{2} \right) + \frac{\cosh \frac{1}{x}}{x^3 (\cosh \frac{2}{x} - 1)} - \frac{3}{2x} \\ &< \frac{4(15x^2 + 4)}{3x(20x^2 + 7)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{3}{2x} \\ &= -\frac{3074x^2 - 1085}{30240x^7(20x^2 + 7)} < 0 \quad \text{for } x \geq 1. \end{aligned}$$

We then obtain that for $x \geq 1$,

$$U'(x) > \lim_{x \rightarrow \infty} U'(x) = 0 \implies U(x) < \lim_{x \rightarrow \infty} U(x) = 0.$$

This means that the second inequality in (5.11) holds for $x \geq 1$.

The lower bound in (5.11) is obtained by considering the function $F(x)$ defined for $x \geq 1$ by

$$\begin{aligned} F(x) &= \ln \Gamma \left(x + \frac{1}{2} \right) - \ln(\sqrt{2\pi}) - x \ln x + x - \frac{x}{2} \ln(2x) \\ &\quad - \frac{x}{2} \ln \left(\tanh \frac{1}{2x} \right) - \ln \left(1 - \frac{31}{51840x^5} \right). \end{aligned}$$

Differentiation yields

$$\begin{aligned} F'(x) &= \psi \left(x + \frac{1}{2} \right) - \frac{3}{2} \ln x - \frac{1}{2} - \frac{1}{2} \ln 2 - \frac{1}{2} \ln \left(\tanh \frac{1}{2x} \right) \\ &\quad + \frac{1}{2x \sinh \frac{1}{x}} - \frac{155}{x(51840x^5 - 31)}. \end{aligned}$$

Differentiating $F'(x)$ and applying the inequalities (2.22) and (5.12) yield

$$\begin{aligned} F''(x) &= \psi' \left(x + \frac{1}{2} \right) + \frac{\cosh \frac{1}{x}}{x^3 (\cosh \frac{2}{x} - 1)} \\ &\quad - \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5 - 31)^2} \\ &> \frac{20x(84x^2 + 71)}{3(560x^4 + 520x^2 + 27)} + \frac{1}{2x} + \frac{1}{12x^3} - \frac{7}{240x^5} + \frac{31}{6048x^7} - \frac{127}{172800x^9} \\ &\quad - \frac{8062156800x^{11} - 96422400x^5 - 9642240x^6 + 2883x + 9610}{2x^2(51840x^5 - 31)^2} \\ &= \frac{G(x)}{1209600x^9(560x^4 + 520x^2 + 27)(51840x^5 - 31)^2}, \end{aligned}$$

where

$$\begin{aligned}
G(x) = & 35836321468874877 + 500332411677183040(x-1) \\
& + 3243650443203650720(x-1)^2 + 12942574029209436800(x-1)^3 \\
& + 35510225676993688800(x-1)^4 + 70870406789351495040(x-1)^5 \\
& + 106102603246810676800(x-1)^6 + 121054817642726534400(x-1)^7 \\
& + 105767748316645632000(x-1)^8 + 70425167657151744000(x-1)^9 \\
& + 35177843816429875200(x-1)^{10} + 12782474258840064000(x-1)^{11} \\
& + 3194045448118272000(x-1)^{12} + 491271286947840000(x-1)^{13} \\
& + 35090806210560000(x-1)^{14}.
\end{aligned}$$

Hence, $F''(x) > 0$ for $x \geq 1$. We then obtain that for $x \geq 1$,

$$F'(x) < \lim_{x \rightarrow \infty} F'(x) = 0 \implies F(x) > \lim_{x \rightarrow \infty} F(x) = 0.$$

This means that the first inequality in (5.11) holds for $x \geq 1$. The proof of Theorem 6 is complete. \square

Remark 5. *Some computer experiments indicate that for $x > 0$,*

$$\begin{aligned}
\sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \left(1 - \frac{\vartheta_1}{x^5}\right) \\
< \Gamma\left(x + \frac{1}{2}\right) < \sqrt{2\pi} \left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \left(1 - \frac{\vartheta_2}{x^5}\right)
\end{aligned} \tag{5.13}$$

with the best possible constants $\vartheta_1 = \frac{31}{51840}$ and $\vartheta_2 = 0$.

Although the double inequality (5.11) is given only for $x \geq 1$, it attains the main scope to evaluate the function $\Gamma\left(x + \frac{1}{2}\right)$ for large values of the argument.

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