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A DOUBLE INEQUALITY FOR RATIOS OF BERNOULLI NUMBERS

FENG QI

ABSTRACT. In the paper, the author find a double inequality for ratios of two consecutive Bernoulli numbers with even indexes.

1. MAIN RESULT

Recall from [1, p. 804, 23.1.1] and [21, p. 3, (1.1)] that Bernoulli numbers B_n may be defined by the power series expansion

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!},\quad |z|<2\pi. \tag{1.1}$$

Because the function $\frac{x}{e^x-1}-1+\frac{x}{2}$ is odd in $x\in\mathbb{R}$, all the Bernoulli numbers B_{2k+1} for $k\in\mathbb{N}$ equal 0. The first few Bernoulli numbers B_{2k} are

$$B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},$$
$$B_{12}=-\frac{691}{2730},\quad B_{14}=\frac{7}{6},\quad B_{16}=-\frac{3617}{510},\quad B_{18}=\frac{43867}{798},\quad B_{20}=-\frac{174611}{330}.$$

It has been being a classical topic to find explicit formulas, recurrent formulas, closed form expressions, and integral representations of Bernoulli numbers B_{2k}. For detailed information and recently published results, please refer to [5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20] and closely related references therein.

Another topic is to bound Bernoulli numbers B_{2k}. In [1, p. 805, 23.1.15], [3, 14, 15], and [21, p. 14, (1.23) and p. 23, Exercise 1.2], some double inequalities for bounding Bernoulli numbers B_{2k} were established and listed. These inequalities were refined and sharpened in [2] by

$$\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\alpha-2k}}\le|B_{2k}|\le\frac{2(2k)!}{(2\pi)^{2k}}\frac{1}{1-2^{\beta-2k}},\quad k\in\mathbb{N}, \tag{1.2}$$

where $\alpha=0$ and

$$\beta=2+\frac{\ln(1-6/\pi^2)}{\ln 2}=0.649\dots$$

are the best possible. In [4, Theorem 1.1], an alternative upper bound for Bernoulli numbers B_{2k} was given by

$$|B_{2k}|\le\frac{2(2^{2m}-1)}{2^{2m}}\zeta(2m)\frac{(2k)!}{\pi^{2k}(2^{2k}-1)}, \tag{1.3}$$

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where $k \ge m$ and $m \in \mathbb{N}$, the equality in (1.3) is valid if and only if $k = m$, and the constant $\frac{2(2^{2m}-1)}{2^{2m}}\zeta(2m)$ is the best possible.

An interesting topic is to consider and investigate differences $|B_{2k+2}| - |B_{2k}|$ and ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$ for $k \in \mathbb{N}$.

In this paper, we are interested in considering and investigating ratios $\frac{|B_{2k+2}|}{|B_{2k}|}$ for $k \in \mathbb{N}$. Our main result is the following double inequality.

Theorem 1.1. *For $k \in \mathbb{N}$, we have*

$$\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{2(k+1)(2k+1)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{2^{2k}-1}{2^{2k+2}-1}\frac{2(k+1)(2k+1)}{\pi^2}. \tag{1.4}$$

2. Proof of main result

Recall from [1, p. 807, 23.2.1] and [21, p. 57] that Riemann zeta function ζ may be defined by the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{2.1}$$

under the condition $\Re(z) > 1$, and by analytic continuation elsewhere.

In [21, p. 5, (1.14)], it was listed that

$$B_{2k} = \frac{(-1)^{k+1}2(2k)!}{(2\pi)^{2k}}\zeta(2k), \quad k \in \mathbb{N}. \tag{2.2}$$

Then

$$\frac{|B_{2k+2}|}{|B_{2k}|} = \frac{2(k+1)(2k+1)}{\pi^2}\frac{1}{4}\frac{\zeta(2k+2)}{\zeta(2k)}, \quad k \in \mathbb{N}. \tag{2.3}$$

Hence, to prove the right-hand side of the inequality (1.4), it is sufficient to verify

$$\frac{1}{4}\frac{\zeta(2k+2)}{\zeta(2k)} < \frac{2^{2k}-1}{2^{2k+2}-1}$$

which may be rearranged as

$$\left(1 - \frac{1}{2^{2k+2}}\right)\zeta(2k+2) < \left(1 - \frac{1}{2^{2k}}\right)\zeta(2k), \quad k \in \mathbb{N}.$$

This inequality is a special case of [4, Lemma 2.1] and [23, Lemma 2.1] which may be slightly modified as follows: the sequence

$$\left(1 - \frac{1}{2^k}\right)\zeta(k) = \zeta(k) - \sum_{m=1}^{\infty}\frac{1}{(2m)^k} = \sum_{m=1}^{\infty}\frac{1}{(2m-1)^k}, \quad k \ge 2$$

is decreasing in k. The right-hand side of the inequality (1.4) is thus proved.

In [22, p. 4, Corollary 1], it was derived that Dirichlet eta function

$$\eta(x) = \left(1 - \frac{1}{2^{x-1}}\right)\zeta(x) \tag{2.4}$$

is strictly logarithmically concave on $(0, \infty)$. This implies that the logarithmic derivative

$$[\ln \eta(x)]' = \frac{\eta'(x)}{\eta(x)} = \frac{\zeta'(x)}{\zeta(x)} + \frac{\ln 2}{2^{x-1}-1}$$

is strictly decreasing on $(0,\infty)$. It is clear that

$$\lim_{x\to\infty}\frac{\eta'(x)}{\eta(x)}=\lim_{x\to\infty}\frac{\zeta'(x)}{\zeta(x)}=0. \tag{2.5}$$

Consequently, it follows that $\frac{\eta'(x)}{\eta(x)}>0$, which implies that $\eta'(x)>0$ and $\eta(x)$ is strictly increasing, on $(0,\infty)$. As a result, we have

$$\biggl(1-\frac{1}{2^{2k+1}}\biggr)\zeta(2k+2)>\biggl(1-\frac{1}{2^{2k-1}}\biggr)\zeta(2k)$$

which may be rearranged as

$$\frac{1}{4}\frac{\zeta(2k+2)}{\zeta(2k)}>\frac{2^{2k-1}-1}{2^{2k+1}-1}$$

for $k\in\mathbb{N}$. Combining this with (2.3) brings about the right-hand side of the inequality (1.4). The proof of Theorem 1.1 is complete.

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Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: http://qifeng618.wordpress.com