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# NEW BOUNDS FOR ČEBYŠEV FUNCTIONAL FOR POWER SERIES IN BANACH ALGEBRAS VIA A GRÜSS-LUPAŞ TYPE INEQUALITY

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Abstract. Some Grüss-Lupaş type inequalities for sequences in Banach algebras are obtained. Moreover, if $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ is a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$, a Banach algebra, with $xy = yx$, then we also establish some new upper bounds for the norm of the *Čebyšev type difference*

$$f(\lambda) f(\lambda xy) - f(\lambda x) f(\lambda y), \lambda \in D(0, R).$$

These results complement the earlier resuls obtained by the authors. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

## 1. Introduction

In 1935, G. Grüss [45] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integral means integrals as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) \tag{1.1}$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfying the assumption

$$\varphi \leq f(x) \leq \Phi, \gamma \leq g(x) \leq \Gamma$$

for each $x \in [a, b]$ where $\varphi, \Phi, \gamma, \Gamma$ are given real constants.

Moreover the constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss' type see the Chapter X of the recent book [51] by Mitrinović, Pečarić and Fink.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardzewski [4] established the following discrete version of Grüss' inequality, see also [51, Ch. X]:

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**Theorem 1.** *Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n)$ be two $n$-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, ..., n$. Then one has the inequality:*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \frac{1}{n} \left[ \frac{n}{2} \right] \left( 1 - \frac{1}{n} \left[ \frac{n}{2} \right] \right) (R - r)(S - s) \tag{1.2}$$

*when $[x]$ is the integer part of $x, x \in \mathbb{R}$.*

A weighted version of Grüss' discrete inequality was proved by J.E. Pečarić in 1979, see for instance [51, Ch. X]:

**Theorem 2.** *Let $a, b$ be two monotonic $n$-tuples and $p$ a positive one. Then*

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right| \leq |a_n - a_1| |b_n - b_1| \max_{1 \leq k \leq n-1} \left( \frac{P_k \bar{P}_{k+1}}{P_n^2} \right) \tag{1.3}$$

*where $P_n := \sum_{i=1}^{n} p_i$ , $\bar{P}_{k+1} = P_n - P_{k+1}$ .*

In 1981 , A. Lupaş [51, Ch. X] proved some similar results for the first difference of $a$ as follows :

**Theorem 3.** *Let $a, b$ two monotonic $n$-tuples in the same sense and $p$ a positive $n$-tuple. Then*

$$\begin{aligned} &\min_{1 \leq i \leq n-1} |a_{i+1} - a_i| \min_{1 \leq i \leq n-1} |b_{i+1} - b_i| \left[ \frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left( \frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \\ &\leq \max_{1 \leq i \leq n-1} |a_{i+1} - a_i| \max_{1 \leq i \leq n-1} |b_{i+1} - b_i| \left[ \frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left( \frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right]. \end{aligned} \tag{1.4}$$

*If there exists the numbers $\bar{a}, \bar{a}_1, r, r_1, (rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then in (1.4) the equality holds.*

For some generalizations of Grüss' inequality for isotonic linear functionals defined on certain spaces of mappings see Chapter X of the book [51] where further references are given .

For related results, see [1]-[33], [37]-[43] and [46]-[62].

In order to extend the above results for Banach algebras, we need some preliminary facts as follows:

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \leq \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element $b$ is unique; it is called the *inverse* of $a$ and written $a^{-1}$ or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;
(ii) $\{a \in \mathcal{B}: \ \|1 - b\| < 1\} \subset$Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B} \ni a \longmapsto a^{-1} \in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by $\lambda$. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : \ \lambda - a \in \text{Inv}\mathcal{B}\};$$

the *spectrum* of $a$ is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of $a$ is $R_a : \rho(a) \to$Inv$\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} : \ |\lambda| \le \|a\|\}$. The *spectral radius* of $a$ is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If $a, b$ are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \le \nu(a)\nu(b) \ \text{ and } \nu(a+b) \le \nu(a) + \nu(b).$$

Let $f$ be an analytic functions on the open disk $D(0, R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j \ (|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for $a$ and $b$ from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if $a$ is an element in $B$ such that $\|1 - a\| < 1$, then $a$ is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is known that if $x$ and $y$ are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if $x$ is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}\left[\exp(bx) - \exp(ax)\right].$$

Moreover, if $x$ and $y$ are commuting and $y - x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)x + sy)\,ds &= \int_0^1 \exp(s(y-x))\exp(x)\,ds \\ &= \left(\int_0^1 \exp(s(y-x))\,ds\right)\exp(x) \\ &= (y-x)^{-1}\left[\exp(y-x) - I\right]\exp(x) \\ &= (y-x)^{-1}\left[\exp(y) - \exp(x)\right].\end{aligned}$$

Inequalities for functions of operators in Hilbert spaces may be found in the papers [15], [14] and in the recent monographs [34], [35], [44] and the references therein.

## 2. Some Grüss-Lupaş' Type Inequalities in Banach Algebras

The following inequality of Grüss'-Lupaş type in Banach algebras holds:

**Theorem 4.** *Let $\mathcal{B}$ be a Banach algebra over $\mathbb{K}$ $(=\mathbb{R},\mathbb{C})$, $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ $(i = 1, ..., n)$. Then we have the inequality :*

$$\begin{aligned}&\left\|\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i a_i b_i - \sum_{i=1}^n \alpha_i a_i \sum_{i=1}^n \alpha_i b_i\right\| \\ &\le \max_{1\le j\le n-1}\|a_{j+1} - a_j\| \max_{1\le j\le n-1}\|b_{j+1} - b_j\| \\ &\times \left[\sum_{i=1}^n |\alpha_i| \sum_{i=1}^n i^2|\alpha_i| - \left(\sum_{i=1}^n i|\alpha_i|\right)^2\right]\end{aligned} \tag{2.1}$$

*The inequality (2.1) is sharp in the sense that the multiplicative constant $C = 1$ in the right membership can not be replaced by a smaller one.*

*Proof.* Let us start with the following identity which can be proved by direct computation:

$$\begin{aligned}&\sum_{i=1}^n \alpha_i \sum_{i=1}^n \alpha_i a_i b_i - \sum_{i=1}^n \alpha_i a_i \sum_{i=1}^n \alpha_i b_i \\ &= \frac{1}{2}\sum_{i,j=1}^n \alpha_i\alpha_j (a_j - a_i)(b_j - b_i) = \sum_{1\le i<j\le n} \alpha_i\alpha_j (a_j - a_i)(b_j - b_i).\end{aligned} \tag{2.2}$$

As $i < j$ we can write that

$$a_j - a_i = \sum_{k=i}^{j-1}(a_{k+1} - a_k)$$

and

$$b_j - b_i = \sum_{k=i}^{j-1} (b_{k+1} - b_k) .$$

Using the generalized triangle inequality we have successively from (2.2) :

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\
&= \left\| \sum_{1 \le i < j \le n}^{n} \alpha_i \alpha_j \sum_{k=i}^{j-1} (a_{k+1} - a_k) \sum_{k=i}^{j-1} (b_{k+1} - b_k) \right\| \\
&\le \sum_{1 \le i < j \le n}^{n} |\alpha_i \alpha_j| \left\| \sum_{k=i}^{j-1} (a_{k+1} - a_k) \right\| \left\| \sum_{k=i}^{j-1} (b_{k+1} - b_k) \right\| \\
&\le \sum_{1 \le i < j \le n}^{n} |\alpha_i \alpha_j| \sum_{k=i}^{j-1} \|a_{k+1} - a_k\| \sum_{k=i}^{j-1} \|b_{k+1} - b_k\| =: A.
\end{aligned} \tag{2.3}$$

Now, we have

$$\|a_{k+1} - a_k\| \le \max_{1 \le s \le n-1} \|a_{s+1} - a_s\| \text{ and } \|b_{k+1} - b_k\| \le \max_{1 \le s \le n-1} \|b_{s+1} - b_s\|$$

for all $k = i, ..., j-1$ and then by summation,

$$\sum_{k=i}^{j-1} \|a_{k+1} - a_k\| \le (j - i) \max_{1 \le s \le n-1} \|a_{s+1} - a_s\|$$

and

$$\sum_{k=i}^{j-1} \|b_{k+1} - b_k\| \le (j - i) \max_{1 \le s \le n-1} \|b_{s+1} - b_s\| .$$

Taking into account the above estimations, we can write

$$A \le \left[ \sum_{1 \le i < j \le n}^{n} |\alpha_i \alpha_j| (j - i)^2 \right] \max_{1 \le s \le n-1} \|a_{s+1} - a_s\| \max_{1 \le s \le n-1} \|b_{s+1} - b_s\| .$$

As a simple calculation shows that

$$\sum_{1 \le i < j \le n}^{n} |\alpha_i| |\alpha_j| (j - i)^2 = \sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left( \sum_{i=1}^{n} i |\alpha_i| \right)^2$$

the inequality (2.1) is proved.

Assume that the inequality (2.1) holds with a constant $c > 0$, *i.e.*,

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} \alpha_i a_i b_i - \sum_{i=1}^{n} \alpha_i a_i \sum_{i=1}^{n} \alpha_i b_i \right\| \\
&\le c \max_{1 \le j \le n-1} \|a_{j+1} - a_j\| \max_{1 \le j \le n-1} \|b_{j+1} - b_j\| \\
&\times \left[ \sum_{i=1}^{n} |\alpha_i| \sum_{i=1}^{n} i^2 |\alpha_i| - \left( \sum_{i=1}^{n} i |\alpha_i| \right)^2 \right].
\end{aligned} \tag{2.4}$$

Now, choose the sequences $a_k = a + kz\,(z \neq 0)\,,\ b_k = b + ky\,(y \neq 0)\ (k = 1, ..., n)\,.$ We get

$$\begin{aligned}&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i b_i-\sum_{i=1}^{n}\alpha_i a_i\sum_{i=1}^{n}\alpha_i b_i\right\|\\&=\frac{1}{2}\left\|\sum_{i,j=1}^{n}\alpha_i\alpha_j\,(i-j)^2\,zy\right\|=\|z\|\,\|y\|\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2\,|\alpha_i|-\left(\sum_{i=1}^{n}i\,|\alpha_i|\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}&\max_{1\le j\le n-1}\|a_{j+1}-a_j\|\,\|b_{j+1}-b_j\|\left[\sum_{i=1}^{n}i^2\alpha_i-\left(\sum_{i=1}^{n}i\alpha_i\right)^2\right]\\&=\|z\|\,\|y\|\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2\,|\alpha_i|-\left(\sum_{i=1}^{n}i\,|\alpha_i|\right)^2\right]\end{aligned}$$

and then by (2.4) we get $c \geq 1$ , which proves the statement. □

**Remark 1.** *In particular, we have*

$$\begin{aligned}(2.5)\qquad&\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i^2-\left(\sum_{i=1}^{n}\alpha_i a_i\right)^2\right\|\\&\le\max_{1\le j\le n-1}\|a_{j+1}-a_j\|^2\left[\sum_{i=1}^{n}|\alpha_i|\sum_{i=1}^{n}i^2\,|\alpha_i|-\left(\sum_{i=1}^{n}i\,|\alpha_i|\right)^2\right].\end{aligned}$$

The following corollary holds:

**Corollary 1.** *Under the above assumptions for $a_i, b_i\,(i = 1, ..., n)$ we have the inequality:*

$$\begin{aligned}(2.6)\qquad&\left\|\frac{1}{n}\sum_{i=1}^{n}a_i b_i-\frac{1}{n}\sum_{i=1}^{n}a_i\frac{1}{n}\sum_{i=1}^{n}b_i\right\|\\&\le\frac{n^2-1}{12}\max_{1\le j\le n-1}\|a_{j+1}-a_j\|\max_{1\le j\le n-1}\|b_{j+1}-b_j\|\,.\end{aligned}$$

*The constant $\frac{1}{12}$ is sharp in the sense that it can not be replaced by a smaller one.*
*In particular,*

$$(2.7)\qquad\left\|\frac{1}{n}\sum_{i=1}^{n}a_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)^2\right\|\le\frac{n^2-1}{12}\max_{1\le j\le n-1}\|a_{j+1}-a_j\|^2\,.$$

The proof follows by the above theorem putting $\alpha_i = \frac{1}{n}$ and taking into account that

$$\begin{aligned}
&\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}i^2\alpha_i-\left(\sum_{i=1}^{n}i\alpha_i\right)^2\\
&=\frac{1}{n^2}\left[n\sum_{i=1}^{n}i^2-\left(\sum_{i=1}^{n}i\right)^2\right]\\
&=\frac{1}{n^2}\left[\frac{n^2\left(n+1\right)\left(2n+1\right)}{6}-\frac{n^2\left(n+1\right)^2}{4}\right]=\frac{n^2-1}{12}.
\end{aligned}$$

**Remark 2.** *If $a_i, b_i \in \mathcal{B}$ and $\alpha_i \in \mathbb{K}$ with $|\alpha_i| = 1$, $(i = 1, ..., n)$, then we have the inequality :*

$$\begin{aligned}
(2.8)\qquad &\left\|\sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}\alpha_i a_i b_i-\sum_{i=1}^{n}\alpha_i a_i\cdot\sum_{i=1}^{n}\alpha_i b_i\right\|\\
&\leq\frac{n^2-1}{12}\max_{1\leq j\leq n-1}\|a_{j+1}-a_j\|\max_{1\leq j\leq n-1}\|b_{j+1}-b_j\|
\end{aligned}$$

## 3. Inequalities for Power Series

Let $\alpha_n$ be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.

Denote by:

$$D(0,R)=\begin{cases}\{\lambda\in\mathbb{C}:|\lambda|<R\}, & \text{if } R<\infty\\ \mathbb{C}, & \text{if } R=\infty,\end{cases}$$

consider the functions:

$$\lambda\mapsto f(\lambda):D(0,R)\to\mathbb{C}, f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$$

and

$$\lambda\mapsto f_A(\lambda):D(0,R)\to\mathbb{C}, f_A(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R)=\begin{cases}\{x\in\mathcal{B}:\|x\|<R\}, & \text{if } R<\infty\\ \mathcal{B}, & \text{if } R=\infty.\end{cases}$$

We associate to $f$ the map:

$$x\mapsto\widetilde{f}(x):B(0,R)\to\mathcal{B},\ \widetilde{f}(x):=\sum_{n=0}^{\infty}\alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty}\alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty}\|\alpha_n x^n\| \leq \sum_{n=0}^{\infty}|\alpha_n|\,\|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have that [36]:

**Theorem 5.** *Let $\lambda \in \mathbb{C}$ such that $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ and let $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le 1$ and $xy = yx$. Then:*

(i) *We have*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \le \sqrt{2}\psi \min\{\|x-1\|, \|y-1\|\} f_A\left(|\lambda|^2\right) \tag{3.1}$$

*where:*

$$\psi^2 := s_0 s_2 - s_1^2. \tag{3.2}$$

(ii) *We also have*

$$\begin{aligned} &\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \\ &\le \sqrt{2} \min\{\|x-1\|, \|y-1\|\} f_A(|\lambda|) \\ &\times \left\{ f_A(|\lambda|) \left[ |\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[ |\lambda| f_A'(|\lambda|) \right]^2 \right\}^{1/2}. \end{aligned} \tag{3.3}$$

For other similar results, see [36].

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \end{aligned} \tag{3.4}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \\ g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C}; \\ h_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C}; \\ l_A(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1). \end{aligned} \tag{3.5}$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)\,n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where $\Gamma$ is *Gamma function*.

The following new result holds:

**Theorem 6.** *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0,R)$, with $R > 0$. If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|x\|, \|y\| \le 1$, then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ the inequality:*

$$\left\|\widetilde{f}(\lambda\cdot 1)\,\widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x)\,\widetilde{f}(\lambda y)\right\| \tag{3.7}$$
$$\le \|x-1\|\,\|y-1\| \left\{ f_A(|\lambda|) \left[ |\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[ |\lambda| f_A'(|\lambda|) \right]^2 \right\}.$$

*Proof.* From the inequality (2.1) we have

$$\left\| \sum_{i=0}^{n} \alpha_i \lambda^i \sum_{i=0}^{n} \alpha_i \lambda^i x^i y^i - \sum_{i=0}^{n} \alpha_i \lambda^i x^i \sum_{i=0}^{n} \alpha_i \lambda^i y^i \right\| \tag{3.8}$$
$$\le \max_{0\le j\le n-1} \left\|x^{j+1} - x^j\right\| \max_{0\le j\le n-1} \left\|y^{j+1} - y^j\right\|$$
$$\times \left[ \sum_{i=0}^{n} |\alpha_i|\,|\lambda|^i \sum_{i=0}^{n} i^2 |\alpha_i|\,|\lambda|^i - \left( \sum_{i=0}^{n} i\,|\alpha_i|\,|\lambda|^i \right)^2 \right]$$

for all $n \ge 1$.

Observe that, since $\|x\| \le 1$, then

$$\max_{0\le j\le n-1} \left\|x^{j+1} - x^j\right\| = \max_{0\le j\le n-1} \left\|x^j(x-1)\right\| \le \max_{0\le j\le n-1} \left\{ \left\|x^j\right\| \|x-1\| \right\}$$
$$\le \max_{0\le j\le n-1} \left\{ \|x\|^j \|x-1\| \right\} \le \|x-1\|.$$

Also,

$$\max_{0\le j\le n-1} \left\|y^{j+1} - y^j\right\| \le \|y-1\|$$

for $\|y\| \le 1$.

Therefore, since $xy = yx$ from (3.8) then we have

$$\left\| \sum_{j=0}^{n} \alpha_j \lambda^j \sum_{j=0}^{n} \alpha_j \lambda^j (xy)^j - \sum_{j=0}^{n} \alpha_j \lambda^j x^j \sum_{j=0}^{n} \alpha_j \lambda^j y^j \right\| \tag{3.9}$$
$$\leq \|x-1\| \|y-1\| \left[ \sum_{j=0}^{n} |\alpha_j| |\lambda|^j \sum_{j=0}^{n} j^2 |\alpha_j| |\lambda|^j - \left( \sum_{j=0}^{n} j |\alpha_j| |\lambda|^j \right)^2 \right]$$

for all $n \geq 1$.

If we denote $f(u) := \sum_{j=0}^{\infty} \alpha_j u^j$, then for $|u| < R$ we have

$$\sum_{j=0}^{\infty} j \alpha_j u^j = u f'(u)$$

and

$$\sum_{j=0}^{\infty} j^2 \alpha_j u^j = u (u g'(u))'.$$

However

$$u (u g'(u))' = u g'(u) + u^2 g''(u)$$

and then

$$\sum_{j=0}^{\infty} j^2 \alpha_j u^j = u g'(u) + u^2 g''(u).$$

Therefore

$$\sum_{j=0}^{\infty} j^2 |\alpha_j| |\lambda|^j = |\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|)$$

and

$$\sum_{j=0}^{\infty} j |\alpha_j| |\lambda|^j = |\lambda| f'(|\lambda|)$$

for $|\lambda| < R$.

Since all the series involved in (3.9) are convergent, then by letting $n \to \infty$ in (3.9) we deduce the desired result (3.7). □

**Corollary 2.** *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $x \in \mathcal{B}$ with $\|x\| \leq 1$ then we have for $\lambda \in \mathbb{C}$ with $|\lambda| < R$ that*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda x^2) - \left[ \widetilde{f}(\lambda x) \right]^2 \right\| \tag{3.10}$$
$$\leq \|x-1\|^2 \left\{ f_A(|\lambda|) \left[ |\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[ |\lambda| f_A'(|\lambda|) \right]^2 \right\}.$$

**Example 1.** *If we apply the inequality (3.7) to the exponential function, then we have*

$$\|\exp[\lambda(1+xy)] - \exp[\lambda(x+y)]\| \leq \|x-1\| \|y-1\| |\lambda| \exp(2|\lambda|) \tag{3.11}$$

*for any $x, y \in \mathcal{B}$ with $xy = yx$, $\|x\|, \|y\| < 1$ and $\lambda \in \mathbb{C}$.*

*In particular, we have*

$$\left\| \exp\left[\lambda\left(1+x^2\right)\right] - \exp(2\lambda x) \right\| \leq \|x-1\|^2 |\lambda| \exp(2|\lambda|) \tag{3.12}$$

*for any* $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\lambda \in \mathbb{C}$.

*Also we have*

$$\left\|\exp\left[\lambda\left(1-x^2\right)\right]-1\right\| \leq \|x-1\|\,\|x+1\|\,|\lambda|\exp\left(2\,|\lambda|\right) \tag{3.13}$$

*for any* $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\lambda \in \mathbb{C}$.

**Example 2.** *Now, consider the function* $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$, $\lambda \in D(0,1)$. *If we apply the inequality (3.7) for this function, then we get the result:*

$$\left\|(1-\lambda)^{-1}(1-\lambda xy)^{-1} - (1-\lambda x)^{-1}(1-\lambda y)^{-1}\right\| \leq \frac{\|x-1\|\,\|y-1\|}{(1-|\lambda|)^4} \tag{3.14}$$

*for any* $x, y \in \mathcal{B}$ *with* $xy = yx$, $\|x\|, \|y\| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

*In particular, we have the inequality*

$$\left\|(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1} - (1-\lambda x)^{-2}\right\| \leq \frac{\|x-1\|^2}{(1-|\lambda|)^4} \tag{3.15}$$

*and the inequality*

$$\left\|(1-\lambda)^{-1}\left(1+\lambda x^2\right)^{-1} - \left(1-\lambda^2 x^2\right)^{-1}\right\| \leq \frac{\|x-1\|\,\|x+1\|}{(1-|\lambda|)^4} \tag{3.16}$$

*for any* $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\lambda \in \mathbb{C}$ *with* $|\lambda| < 1$.

*Now, if we take* $\lambda = \frac{1}{\gamma}$ *with* $|\gamma| > 1$ *then we get from (3.14) the inequality*

$$\left\|\gamma^2(\gamma-1)^{-1}(\gamma-xy)^{-1} - \gamma^2(\gamma-x)^{-1}(\gamma-y)^{-1}\right\| \leq \frac{\|x-1\|\,\|y-1\|\,|\gamma|^4}{(|\gamma|-1)^4} \tag{3.17}$$

*which is equivalent to*

$$\left\|(\gamma-1)^{-1}(\gamma-xy)^{-1} - (\gamma-x)^{-1}(\gamma-y)^{-1}\right\| \leq \frac{\|x-1\|\,\|y-1\|\,|\gamma|^2}{(|\gamma|-1)^4} \tag{3.18}$$

*for any* $x, y \in \mathcal{B}$ *with* $xy = yx$, $\|x\|, \|y\| < 1$ *and* $\gamma \in \mathbb{C}$ *with* $|\gamma| > 1$.

*If we use the resolvent function notation we then have the following inequality:*

$$\left\|(\gamma-1)^{-1}R_{xy}(\gamma) - R_x(\gamma)\,R_y(\gamma)\right\| \leq \frac{\|x-1\|\,\|y-1\|\,|\gamma|^2}{(|\gamma|-1)^4} \tag{3.19}$$

*for any* $x, y \in \mathcal{B}$ *with* $xy = yx$, $\|x\|, \|y\| < 1$ *and* $\gamma \in \mathbb{C}$ *with* $|\gamma| > 1$.

*In particular, we have*

$$\left\|(\gamma-1)^{-1}R_{x^2}(\gamma) - R_x^2(\gamma)\right\| \leq \frac{\|x-1\|^2\,|\gamma|^2}{(|\gamma|-1)^4} \tag{3.20}$$

*for any* $x \in \mathcal{B}$ *with* $\|x\| < 1$ *and* $\gamma \in \mathbb{C}$ *with* $|\gamma| > 1$.

**Remark 3.** *Similar inequalities may be stated for the other power series mentioned above. However, the details are not presented here.*

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