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SOME INEQUALITIES FOR TRACE CLASS OPERATORS VIA A KATO'S RESULT

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Abstract. By the use of the celebrated Kato's inequality we obtain in this paper some new inequalities for trace class operators on a complex Hilbert space H. Natural applications for functions defined by power series of normal operators are given as well.

1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px,x\rangle\geq 0$ for any $x\in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle, \tag{1.1}$$

for any $x,y\in H$.

The following inequality is of interest as well, see [18, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2\leq\|P\|\langle Px,x\rangle \tag{1.2}$$

for any $x\in H$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [18, p. 240]: *If the operator $A\in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B:=\sqrt{A}\in B(H)$ such that $B^2=A$. If A is invertible, then so is B.*

If $A\in\mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A|:=\sqrt{A^*A}$.

In 1952, Kato [19] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx,y\rangle|^2\leq\langle(T^*T)^\alpha x,x\rangle\left\langle(TT^*)^{1-\alpha}y,y\right\rangle, \tag{1.3}$$

for any $x,y\in H$, $\alpha\in[0,1]$. Utilizing the modulus notation introduced before, we can write (1.3) as follows

$$|\langle Tx,y\rangle|^2\leq\left\langle|T|^{2\alpha}x,x\right\rangle\left\langle|T^*|^{2(1-\alpha)}y,y\right\rangle \tag{1.4}$$

for any $x,y\in H$, $\alpha\in[0,1]$.

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It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (1.4) can be written as

$$|\langle Nx, y\rangle|^2 \le \left\langle |N|^{2\alpha} x, x\right\rangle \left\langle |N|^{2(1-\alpha)} y, y\right\rangle, \tag{1.5}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \le \||A|^{\alpha} x\| \left\||A|^{1-\alpha} y\right\| \tag{1.6}$$

for any $x, y \in H$, $\alpha \in [0,1]$.

If $T = U$, a unitary operator, i.e., we recall that $UU^* = U^*U = 1_H$, then the inequality (1.4) becomes

$$|\langle Ux, y\rangle| \le \|x\| \|y\|$$

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in H.

The symmetric powers in the inequalities above are natural to be considered, so if we choose in (1.4), (1.5) and in (1.6) $\alpha = 1/2$ then we get for any $x, y \in H$

$$|\langle Tx, y\rangle|^2 \le \langle |T| x, x\rangle \langle |T^*| y, y\rangle, \tag{1.7}$$

$$|\langle Nx, y\rangle|^2 \le \langle |N| x, x\rangle \langle |N| y, y\rangle, \tag{1.8}$$

and

$$|\langle Ax, y\rangle| \le \left\||A|^{1/2} x\right\| \left\||A|^{1/2} y\right\| \tag{1.9}$$

respectively.

It is also worthwhile to observe that, if we take the supremum over $y \in H$, $\|y\| = 1$ in (1.4) then we get

$$\|Tx\|^2 \le \|T\|^{2(1-\alpha)} \left\langle |T|^{2\alpha} x, x\right\rangle \tag{1.10}$$

for any $x \in H$, or in an equivalent form

$$\|Tx\| \le \||T|^{\alpha} x\| \|T\|^{1-\alpha} \tag{1.11}$$

for any $x \in H$.

If we take $\alpha = 1/2$ in (1.10), then we get

$$\|Tx\|^2 \le \|T\| \langle |T| x, x\rangle \tag{1.12}$$

for any $x \in H$, which in the particular case of $T = P$, a positive operator, provides the result from (1.2).

For various interesting generalizations, extension and Kato related results, see the papers [7]-[17], [23]-[29] and [34].

In order to state our results concerning new trace inequalities for operators in Hilbert spaces we need some preliminary facts as follows.

2. Trace of Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$,one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If $A \in \mathcal{B}(H)$,then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;
(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leq \|A\|_1 \tag{2.8}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\left\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \|B\| \leq 1\right\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have*
(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [33].

For some classical trace inequalities see [4], [6], [30] and [38], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [14], [22], [27], [28], [31] and [35].

3. Trace Inequalities Via Kato's Result

We start with the following result:

Theorem 4. *Let $T\in\mathcal{B}(H)$.*

(i) If for some $\alpha\in(0,1)$ we have $|T|^{2\alpha}$, $|T^*|^{2(1-\alpha)}\in\mathcal{B}_1(H)$, then $T\in\mathcal{B}_1(H)$ and we have the inequality

$$|\operatorname{tr}(T)|^2\le\operatorname{tr}\left(|T|^{2\alpha}\right)\operatorname{tr}\left(|T^*|^{2(1-\alpha)}\right);\tag{3.1}$$

(ii) If for some $\alpha\in[0,1]$ and an orthonormal basis $\{e_i\}_{i\in I}$ the sum

$$\sum_{i\in I}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}$$

is finite, then $T\in\mathcal{B}_1(H)$ and we have the inequality

$$|\operatorname{tr}(T)|\le\sum_{i\in I}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}.\tag{3.2}$$

Moreover, if the sums $\sum_{i\in I}\|Te_i\|$ and $\sum_{i\in I}\|T^*e_i\|$ are finite for an orthonormal basis $\{e_i\}_{i\in I}$, then $T\in\mathcal{B}_1(H)$ and we have

$$|\operatorname{tr}(T)|\le\inf_{\alpha\in[0,1]}\left\{\sum_{i\in I}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}\right\}\le\min\left\{\sum_{i\in F}\|Te_i\|,\sum_{i\in F}\|T^*e_i\|\right\}.\tag{3.3}$$

Proof. (i) Assume that $\alpha\in(0,1)$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (1.4) we have

$$\left|\sum_{i\in F}\langle Te_i,e_i\rangle\right|\le\sum_{i\in F}|\langle Te_i,e_i\rangle|\le\sum_{i\in F}\left\langle|T|^{2\alpha}e_i,e_i\right\rangle^{1/2}\left\langle|T^*|^{2(1-\alpha)}e_i,e_i\right\rangle^{1/2}.\tag{3.4}$$

By Cauchy-Buniakovski-Schwarz inequality for finite sums we have

$$\begin{aligned}&\sum_{i\in F}\left\langle|T|^{2\alpha}e_i,e_i\right\rangle^{1/2}\left\langle|T^*|^{2(1-\alpha)}e_i,e_i\right\rangle^{1/2}\\&\le\left(\sum_{i\in F}\left[\left\langle|T|^{2\alpha}e_i,e_i\right\rangle^{1/2}\right]^2\right)^{1/2}\left(\sum_{i\in F}\left[\left\langle|T^*|^{2(1-\alpha)}e_i,e_i\right\rangle^{1/2}\right]^2\right)^{1/2}\\&=\left(\sum_{i\in F}\left\langle|T|^{2\alpha}e_i,e_i\right\rangle\right)^{1/2}\left(\sum_{i\in F}\left\langle|T^*|^{2(1-\alpha)}e_i,e_i\right\rangle\right)^{1/2}.\end{aligned}\tag{3.5}$$

Therefore, by (3.4) and (3.5) we have

$$\left|\sum_{i\in F}\langle Te_i,e_i\rangle\right|\le\left(\sum_{i\in F}\left\langle|T|^{2\alpha}e_i,e_i\right\rangle\right)^{1/2}\left(\sum_{i\in F}\left\langle|T^*|^{2(1-\alpha)}e_i,e_i\right\rangle\right)^{1/2}\tag{3.6}$$

for any finite part F of I.

If for some $\alpha \in (0,1)$ we have $|T|^{2\alpha}, |T^*|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then the sums $\sum_{i\in I}\left\langle |T|^{2\alpha} e_i, e_i\right\rangle$ and $\sum_{i\in I}\left\langle |T^*|^{2(1-\alpha)} e_i, e_i\right\rangle$ are finite and by (3.6) we have that $\sum_{i\in I}\langle Te_i, e_i\rangle$ is also finite and we have the inequality (3.1).

(ii) Assume that $\alpha \in [0,1]$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Utilising McCarthy's inequality for the positive operator P, namely

$$\left\langle P^{\beta} x, x\right\rangle \le \langle Px, x\rangle^{\beta},$$

that holds for $\beta \in [0,1]$ and $x \in H$, $\|x\| = 1$, we have

$$\left\langle |T|^{2\alpha} e_i, e_i\right\rangle \le \left\langle |T|^{2} e_i, e_i\right\rangle^{\alpha}$$

and

$$\left\langle |T^*|^{2(1-\alpha)} e_i, e_i\right\rangle \le \left\langle |T^*|^{2} e_i, e_i\right\rangle^{1-\alpha}$$

for any $i \in I$.

Making use of (3.4) we have

$$\begin{aligned} (3.7)\quad \left|\sum_{i\in F}\langle Te_i, e_i\rangle\right| &\le \sum_{i\in F}|\langle Te_i, e_i\rangle| \le \sum_{i\in F}\left\langle |T|^{2\alpha} e_i, e_i\right\rangle^{1/2}\left\langle |T^*|^{2(1-\alpha)} e_i, e_i\right\rangle^{1/2} \\ &\le \sum_{i\in F}\left\langle |T|^{2} e_i, e_i\right\rangle^{\alpha/2}\left\langle |T^*|^{2} e_i, e_i\right\rangle^{(1-\alpha)/2} \\ &= \sum_{i\in F}\langle T^*Te_i, e_i\rangle^{\alpha/2}\langle TT^*e_i, e_i\rangle^{(1-\alpha)/2} \\ &= \sum_{i\in F}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}. \end{aligned}$$

Utilizing Hölder's inequality for finite sums and $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ we also have

$$\begin{aligned} (3.8)\quad & \sum_{i\in F}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha} \\ &\le \left[\sum_{i\in F}\left(\|Te_i\|^{\alpha}\right)^{1/\alpha}\right]^{\alpha}\left[\sum_{i\in F}\left(\|T^*e_i\|^{1-\alpha}\right)^{1/(1-\alpha)}\right]^{1-\alpha} \\ &= \left[\sum_{i\in F}\|Te_i\|\right]^{\alpha}\left[\sum_{i\in F}\|T^*e_i\|\right]^{1-\alpha}. \end{aligned}$$

Since all the series involved in (3.7) and (3.8) are convergent, then we get

$$\begin{aligned} (3.9)\quad \left|\sum_{i\in I}\langle Te_i, e_i\rangle\right| &\le \sum_{i\in I}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha} \\ &\le \left[\sum_{i\in I}\|Te_i\|\right]^{\alpha}\left[\sum_{i\in I}\|T^*e_i\|\right]^{1-\alpha} \end{aligned}$$

for any $\alpha \in [0,1]$.

Taking the infimum over $\alpha \in [0,1]$ in (3.9) produces

$$\left|\sum_{i\in I}\langle Te_i, e_i\rangle\right| \le \inf_{\alpha\in[0,1]}\left\{\sum_{i\in F}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}\right\} \le \inf_{\alpha\in[0,1]}\left[\sum_{i\in F}\|Te_i\|\right]^{\alpha}\left[\sum_{i\in F}\|T^*e_i\|\right]^{1-\alpha} = \min\left\{\sum_{i\in F}\|Te_i\|, \sum_{i\in F}\|T^*e_i\|\right\}. \tag{3.10}$$

□

Corollary 1. *Let $T \in \mathcal{B}(H)$.*

(i) If we have $|T|, |T^*| \in \mathcal{B}_1(H)$, then $T \in \mathcal{B}_1(H)$ and we have the inequality

$$|\mathrm{tr}(T)|^2 \le \mathrm{tr}(|T|)\,\mathrm{tr}(|T^*|); \tag{3.11}$$

(ii) If for an orthonormal basis $\{e_i\}_{i\in I}$ the sum $\sum_{i\in I}\sqrt{\|Te_i\|\,\|T^*e_i\|}$ is finite, then $T \in \mathcal{B}_1(H)$ and we have the inequality

$$|\mathrm{tr}(T)| \le \sum_{i\in I}\sqrt{\|Te_i\|\,\|T^*e_i\|}. \tag{3.12}$$

Corollary 2. *Let $N \in \mathcal{B}(H)$ be a normal operator. If for some $\alpha \in (0,1)$ we have $|N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and we have the inequality*

$$|\mathrm{tr}(N)|^2 \le \mathrm{tr}\left(|N|^{2\alpha}\right)\mathrm{tr}\left(|N|^{2(1-\alpha)}\right). \tag{3.13}$$

In particular, if $|N| \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(N)| \le \mathrm{tr}(|N|). \tag{3.14}$$

The following result also holds.

Theorem 5. *Let $T \in \mathcal{B}(H)$ and $A, B \in \mathcal{B}_2(H)$.*

(i) For any $\alpha \in [0,1]$ we have $|A^*|^2|T|^{2\alpha}$, $|B^*|^2|T^*|^{2(1-\alpha)}$ and $B^*TA \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(AB^*T)|^2 \le \mathrm{tr}\left(|A^*|^2|T|^{2\alpha}\right)\mathrm{tr}\left(|B^*|^2|T^*|^{2(1-\alpha)}\right); \tag{3.15}$$

(ii) We also have

$$|\mathrm{tr}(AB^*T)|^2 \le \min\left\{\mathrm{tr}\left(|B|^2\right)\mathrm{tr}\left(|A^*|^2|T|^2\right), \mathrm{tr}\left(|A|^2\right)\mathrm{tr}\left(|B^*|^2|T^*|^2\right)\right\}. \tag{3.16}$$

Proof. (i) Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (1.4) we have

$$|\langle TAe_i, Be_i\rangle|^2 \le \left\langle |T|^{2\alpha}Ae_i, Ae_i\right\rangle\left\langle |T^*|^{2(1-\alpha)}Be_i, Be_i\right\rangle \tag{3.17}$$

for any $i \in I$. This is equivalent to

$$|\langle B^*TAe_i, e_i\rangle| \le \left\langle A^*|T|^{2\alpha}Ae_i, e_i\right\rangle^{1/2}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i, e_i\right\rangle^{1/2} \tag{3.18}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Cauchy-Bunyakowsky-Schwarz inequality for finite sums we have from (3.18) that

$$
\begin{aligned}
(3.19)\qquad & \left|\sum_{i\in F}\langle B^*TAe_i,e_i\rangle\right| \\
&\le \sum_{i\in F}\left|\langle B^*TAe_i,e_i\rangle\right| \\
&\le \sum_{i\in F}\left\langle A^*\left|T\right|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\left\langle B^*\left|T^*\right|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2} \\
&\le \left[\sum_{i\in F}\left(\left\langle A^*\left|T\right|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2} \\
&\times \left[\sum_{i\in F}\left(\left\langle B^*\left|T^*\right|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2} \\
&= \left[\sum_{i\in F}\left\langle A^*\left|T\right|^{2\alpha}Ae_i,e_i\right\rangle\right]^{1/2}\left[\sum_{i\in F}\left\langle B^*\left|T^*\right|^{2(1-\alpha)}Be_i,e_i\right\rangle\right]^{1/2}
\end{aligned}
$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,B\in\mathcal{B}_2(H)$, then $A^*\left|T\right|^{2\alpha}A$, $B^*\left|T^*\right|^{2(1-\alpha)}B$ and $B^*TA\in\mathcal{B}_1(H)$ and by (3.19) we have

$$
(3.20)\qquad \left|\operatorname{tr}(B^*TA)\right|\le\left[\operatorname{tr}\left(A^*\left|T\right|^{2\alpha}A\right)\right]^{1/2}\left[\operatorname{tr}\left(B^*\left|T^*\right|^{2(1-\alpha)}B\right)\right]^{1/2}.
$$

Since, by the properties of trace we have

$$
\operatorname{tr}(B^*TA)=\operatorname{tr}(AB^*T),
$$

$$
\operatorname{tr}\left(A^*\left|T\right|^{2\alpha}A\right)=\operatorname{tr}\left(AA^*\left|T\right|^{2\alpha}\right)=\operatorname{tr}\left(\left|A^*\right|^2\left|T\right|^{2\alpha}\right)
$$

and

$$
\operatorname{tr}\left(B^*\left|T^*\right|^{2(1-\alpha)}B\right)=\operatorname{tr}\left(\left|B^*\right|^2\left|T^*\right|^{2(1-\alpha)}\right),
$$

then by (3.20) we get (3.15).

(ii) Utilising McCarthy's inequality [29] for the positive operator P

$$
\left\langle P^\beta x,x\right\rangle\le\langle Px,x\rangle^\beta
$$

that holds for $\beta\in(0,1)$ and $x\in H$, $\|x\|=1$, we have

$$
(3.21)\qquad \left\langle P^\beta y,y\right\rangle\le\|y\|^{2(1-\beta)}\langle Py,y\rangle^\beta
$$

for any $y\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. From (3.21) we have

$$
\left\langle\left|T\right|^{2\alpha}Ae_i,Ae_i\right\rangle\le\|Ae_i\|^{2(1-\alpha)}\left\langle\left|T\right|^2Ae_i,Ae_i\right\rangle^\alpha
$$

and

$$
\left\langle\left|T^*\right|^{2(1-\alpha)}Be_i,Be_i\right\rangle\le\|Be_i\|^{2\alpha}\left\langle\left|T^*\right|^2Be_i,Be_i\right\rangle^{1-\alpha}
$$

for any $i\in I$.

Making use of the inequality (3.17) we get

$$\begin{aligned}|\langle TAe_i, Be_i\rangle|^2 &\le \|Ae_i\|^{2(1-\alpha)}\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Be_i\|^{2\alpha}\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha} \\ &= \|Be_i\|^{2\alpha}\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Ae_i\|^{2(1-\alpha)}\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha}\end{aligned}$$

and taking the square root we get

$$|\langle TAe_i, Be_i\rangle| \le \|Be_i\|^{\alpha}\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha}\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \tag{3.22}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Hölder's inequality for finite sums and $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ we get from (3.22) that

$$\begin{aligned}&\left|\sum_{i\in F}\langle B^*TAe_i, e_i\rangle\right| \\ &\le \sum_{i\in F}|\langle B^*TAe_i, e_i\rangle| \\ &\le \sum_{i\in F}\|Be_i\|^{\alpha}\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha}\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \\ &\le \left(\sum_{i\in F}\left[\|Be_i\|^{\alpha}\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}}\right]^{1/\alpha}\right)^{\alpha} \\ &\times \left(\sum_{i\in F}\left[\|Ae_i\|^{1-\alpha}\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}}\right]^{1/(1-\alpha)}\right)^{1-\alpha} \\ &= \left(\sum_{i\in F}\|Be_i\|\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}}\right)^{\alpha}\left(\sum_{i\in F}\|Ae_i\|\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}}\right)^{1-\alpha}.\end{aligned} \tag{3.23}$$

By Cauchy-Bunyakowsky-Schwarz inequality for finite sums we also have

$$\begin{aligned}\sum_{i\in F}\|Be_i\|\left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F}\|Be_i\|^2\right)^{1/2}\left(\sum_{i\in F}\left\langle |T|^2 Ae_i, Ae_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F}\left\langle |B|^2 e_i, e_i\right\rangle\right)^{1/2}\left(\sum_{i\in F}\left\langle A^*|T|^2 Ae_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in F}\|Ae_i\|\left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F}\|Ae_i\|^2\right)^{1/2}\left(\sum_{i\in F}\left\langle |T^*|^2 Be_i, Be_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F}\left\langle |A|^2 e_i, e_i\right\rangle\right)^{1/2}\left(\sum_{i\in F}\left\langle B^*|T^*|^2 Be_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and by (3.23) we obtain

$$\left|\sum_{i\in F}\langle B^{*}TAe_i,e_i\rangle\right| \le \left(\sum_{i\in F}\left\langle |B|^2 e_i,e_i\right\rangle\right)^{\alpha/2}\left(\sum_{i\in F}\left\langle A^{*}|T|^2 Ae_i,e_i\right\rangle\right)^{\alpha/2} \times\left(\sum_{i\in F}\left\langle |A|^2 e_i,e_i\right\rangle\right)^{(1-\alpha)/2}\left(\sum_{i\in F}\left\langle B^{*}|T^{*}|^2 Be_i,e_i\right\rangle\right)^{(1-\alpha)/2} \tag{3.24}$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,B\in\mathcal{B}_2(H)$, then $A^{*}|T|^2 A$ and $B^{*}|T^{*}|^2 B\in\mathcal{B}_1(H)$ and by (3.24) we get

$$\begin{aligned}|\mathrm{tr}(AB^{*}T)|^2 &\le \left[\mathrm{tr}\left(|B|^2\right)\mathrm{tr}\left(A^{*}|T|^2 A\right)\right]^{\alpha}\left[\mathrm{tr}\left(|A|^2\right)\mathrm{tr}\left(B^{*}|T^{*}|^2 B\right)\right]^{1-\alpha}\\ &=\left[\mathrm{tr}\left(|B|^2\right)\mathrm{tr}\left(|A^{*}|^2|T|^2\right)\right]^{\alpha}\left[\mathrm{tr}\left(|A|^2\right)\mathrm{tr}\left(|B^{*}|^2|T^{*}|^2\right)\right]^{1-\alpha}.\end{aligned} \tag{3.25}$$

Taking the infimum over $\alpha\in[0,1]$ we get (3.16). □

Corollary 3. *Let $T\in\mathcal{B}(H)$ and $A,B\in\mathcal{B}_2(H)$. We have $|A^{*}|^2|T|$, $|B^{*}|^2|T^{*}|$ and $B^{*}TA\in\mathcal{B}_1(H)$ and*

$$|\mathrm{tr}(AB^{*}T)|^2\le\mathrm{tr}\left(|A^{*}|^2|T|\right)\mathrm{tr}\left(|B^{*}|^2|T^{*}|\right). \tag{3.26}$$

Corollary 4. *Let $N\in\mathcal{B}(H)$ be a normal operator and $A,B\in\mathcal{B}_2(H)$.*

(i) For any $\alpha\in[0,1]$ we have $|A^{*}|^2|N|^{2\alpha}$, $|B^{*}|^2|N|^{2(1-\alpha)}$ and $B^{*}NA\in\mathcal{B}_1(H)$ and

$$|\mathrm{tr}(AB^{*}N)|^2\le\mathrm{tr}\left(|A^{*}|^2|N|^{2\alpha}\right)\mathrm{tr}\left(|B^{*}|^2|N|^{2(1-\alpha)}\right). \tag{3.27}$$

In particular, we have $|A^{*}|^2|N|$, $|B^{*}|^2|N|$ and $B^{*}NA\in\mathcal{B}_1(H)$ and

$$|\mathrm{tr}(AB^{*}N)|^2\le\mathrm{tr}\left(|A^{*}|^2|N|\right)\mathrm{tr}\left(|B^{*}|^2|N|\right). \tag{3.28}$$

(ii) We also have

$$|\mathrm{tr}(AB^{*}N)|^2 \le\min\left\{\mathrm{tr}\left(|B|^2\right)\mathrm{tr}\left(|A^{*}|^2|N|^2\right),\mathrm{tr}\left(|A|^2\right)\mathrm{tr}\left(|B^{*}|^2|N|^2\right)\right\}. \tag{3.29}$$

Remark 1. *Let $\alpha\in[0,1]$. By replacing A with A^{*} and B with B^{*} in (3.15) we get*

$$|\mathrm{tr}(A^{*}BT)|^2\le\mathrm{tr}\left(|A|^2|T|^{2\alpha}\right)\mathrm{tr}\left(|B|^2|T^{*}|^{2(1-\alpha)}\right) \tag{3.30}$$

for any $T\in\mathcal{B}(H)$ and $A,B\in\mathcal{B}_2(H)$.

If in this inequality we take $A=B$, then we get

$$\left|\mathrm{tr}\left(|B|^2 T\right)\right|^2\le\mathrm{tr}\left(|B|^2|T|^{2\alpha}\right)\mathrm{tr}\left(|B|^2|T^{*}|^{2(1-\alpha)}\right) \tag{3.31}$$

for any $T\in\mathcal{B}(H)$ and $B\in\mathcal{B}_2(H)$.

If in (3.30) we take $A = B^*$ then we get

$$\left|\operatorname{tr}\left(B^2 T\right)\right|^2 \leq \operatorname{tr}\left(\left|B^*\right|^2 |T|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 \left|T^*\right|^{2(1-\alpha)}\right) \tag{3.32}$$

for any $T \in \mathcal{B}(H)$ and $B \in \mathcal{B}_2(H)$.

Also, if $T = N$, a normal operator, then (3.31) and (3.32) become

$$\left|\operatorname{tr}\left(|B|^2 N\right)\right|^2 \leq \operatorname{tr}\left(|B|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right) \tag{3.33}$$

and

$$\left|\operatorname{tr}\left(B^2 N\right)\right|^2 \leq \operatorname{tr}\left(\left|B^*\right|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right), \tag{3.34}$$

for any $B \in \mathcal{B}_2(H)$.

4. Some Functional Properties

Let $A \in \mathcal{B}_2(H)$ and $P \in \mathcal{B}(H)$ with $P \geq 0$. Then $Q := A^*PA \in \mathcal{B}_1(H)$ with $Q \geq 0$ and writing the inequality (3.31) for $B = (A^*PA)^{1/2} \in \mathcal{B}_2(H)$ we get

$$\left|\operatorname{tr}(A^*PAT)\right|^2 \leq \operatorname{tr}\left(A^*PA|T|^{2\alpha}\right) \operatorname{tr}\left(A^*PA\left|T^*\right|^{2(1-\alpha)}\right),$$

which, by the properties of trace, is equivalent to

$$\left|\operatorname{tr}(PATA^*)\right|^2 \leq \operatorname{tr}\left(PA|T|^{2\alpha}A^*\right) \operatorname{tr}\left(PA\left|T^*\right|^{2(1-\alpha)}A^*\right), \tag{4.1}$$

where $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$.

For a given $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$, we consider the functional $\sigma_{A,T,\alpha}$ defined on the cone $\mathcal{B}_+(H)$ of nonnegative operators on $\mathcal{B}(H)$ by

$$\begin{aligned}\sigma_{A,T,\alpha}(P) := &\left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA\left|T^*\right|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\ &-\left|\operatorname{tr}(PATA^*)\right|.\end{aligned}$$

The following theorem collects some fundamental properties of this functional.

Theorem 6. *Let $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$.*

(i) For any $P, Q \in \mathcal{B}_+(H)$ we have

$$\sigma_{A,T,\alpha}(P+Q) \geq \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q) \, (\geq 0), \tag{4.2}$$

namely, $\sigma_{A,T,\alpha}$ is a superadditive functional on $\mathcal{B}_+(H)$;

(ii) For any $P, Q \in \mathcal{B}_+(H)$ with $P \geq Q$ we have

$$\sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(Q) \, (\geq 0), \tag{4.3}$$

namely, $\sigma_{A,T,\alpha}$ is a monotonic nondecreasing functional on $\mathcal{B}_+(H)$;

(iii) If $P, Q \in \mathcal{B}_+(H)$ and there exist the constants $M > m > 0$ such that $MQ \geq P \geq mQ$ then

$$M\sigma_{A,T,\alpha}(Q) \geq \sigma_{A,T,\alpha}(P) \geq m\sigma_{A,T,\alpha}(Q) \, (\geq 0). \tag{4.4}$$

Proof. (i) Let $P, Q \in \mathcal{B}_+(H)$. On utilizing the elementary inequality

$$\left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2} \geq ac+bd, \; a,b,c,d \geq 0$$

and the triangle inequality for the modulus, we have

$$\begin{aligned}
&\sigma_{A,T,\alpha}(P+Q) \\
&= \left[\operatorname{tr}\left((P+Q)A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left((P+Q)A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}((P+Q)ATA^*)| \\
&= \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^* + QA|T|^{2\alpha}A^*\right)\right]^{1/2} \\
&\times\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^* + QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^* + QATA^*)| \\
&= \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right) + \operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2} \\
&\times\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right) + \operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^*) + \operatorname{tr}(QATA^*)| \\
&\geq \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&+ \left[\operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(PATA^*)| - |\operatorname{tr}(QATA^*)| \\
&= \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (4.2) is proved.

(ii) Let $P, Q \in \mathcal{B}_+(H)$ with $P \geq Q$. Utilising the superadditivity property we have

$$\begin{aligned}
\sigma_{A,T,\alpha}(P) &= \sigma_{A,T,\alpha}((P-Q)+Q) \geq \sigma_{A,T,\alpha}(P-Q) + \sigma_{A,T,\alpha}(Q) \\
&\geq \sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (4.3) is obtained.

(iii) From the monotonicity property we have

$$\sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(mQ) = m\sigma_{A,T,\alpha}(Q)$$

and a similar inequality for M, which prove the desired result (4.4). □

Corollary 5. *Let $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$. If $P \in \mathcal{B}(H)$ is such that there exist the constants $M > m > 0$ with $M1_H \geq P \geq m1_H$, then we have*

$$\begin{aligned}
&M\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(ATA^*)|\right) \\
&\geq \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(PATA^*)| \\
&\geq m\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(ATA^*)|\right).
\end{aligned} \tag{4.5}$$

For a given $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$, if we take $P = |V|^2$ with $V \in \mathcal{B}(H)$, we have

$$\begin{aligned}
\sigma_{A,T,\alpha}\left(|V|^2\right) &= \left[\operatorname{tr}\left(|V|^2 A|T|^{2\alpha} A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(|V|^2 A|T^*|^{2(1-\alpha)} A^*\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(|V|^2 ATA^*\right)\right| \\
&= \left[\operatorname{tr}\left(V^*VA|T|^{2\alpha} A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(V^*VA|T^*|^{2(1-\alpha)} A^*\right)\right]^{1/2} \\
&- |\operatorname{tr}(V^*VATA^*)| \\
&= \left[\operatorname{tr}\left(A^*V^*VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(A^*V^*VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&- |\operatorname{tr}(A^*V^*VAT)| \\
&= \left[\operatorname{tr}\left((VA)^* VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left((VA)^* VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left((VA)^* VAT\right)\right| \\
&= \left[\operatorname{tr}\left(|VA|^2 |T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2 T\right)\right|.
\end{aligned}$$

Assume that $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$.

If we use the superadditivity property of the functional $\sigma_{A,T,\alpha}$ we have for any $V, U \in \mathcal{B}(H)$ that

$$\begin{aligned}
&\left[\operatorname{tr}\left(\left(|VA|^2 + |UA|^2\right)|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(|VA|^2 + |UA|^2\right)|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(\left(|VA|^2 + |UA|^2\right)T\right)\right| \\
&\geq \left[\operatorname{tr}\left(|VA|^2 |T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2 T\right)\right| \\
&+ \left[\operatorname{tr}\left(|UA|^2 |T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|UA|^2 T\right)\right| (\geq 0).
\end{aligned} \tag{4.6}$$

Also, if $|V|^2 \geq |U|^2$ with $V, U \in \mathcal{B}(H)$, then

$$\begin{aligned}
&\left[\operatorname{tr}\left(|VA|^2 |T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|VA|^2 T\right)\right| \\
&\geq \left[\operatorname{tr}\left(|UA|^2 |T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2 |T^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(|UA|^2 T\right)\right| (\geq 0).
\end{aligned} \tag{4.7}$$

If $U \in \mathcal{B}(H)$ is invertible then

$$\frac{1}{\|U^{-1}\|}\|x\| \leq \|Ux\| \leq \|U\|\|x\| \text{ for any } x \in H,$$

which implies that

$$\frac{1}{\|U^{-1}\|^2} 1_H \leq |U|^2 \leq \|U\|^2 1_H.$$

Utilising (4.5) we get

$$\|U\|^2\left(\left[\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|A|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|A|^2T\right)\right|\right) \tag{4.8}$$
$$\geq\left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|UA|^2T\right)\right|$$
$$\geq\frac{1}{\|U^{-1}\|^2}\left(\left[\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|A|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|A|^2T\right)\right|\right).$$

5. Inequalities for Sequences of Operators

For $n\geq 2$, define the Cartesian products $\mathcal{B}^{(n)}(H):=\mathcal{B}(H)\times...\times\mathcal{B}(H)$, $\mathcal{B}_2^{(n)}(H):=\mathcal{B}_2(H)\times...\times\mathcal{B}_2(H)$ and $\mathcal{B}_+^{(n)}(H):=\mathcal{B}_+(H)\times...\times\mathcal{B}_+(H)$ where $\mathcal{B}_+(H)$ denotes the convex cone of nonnegative selfadjoint operators on H, i.e. $P\in\mathcal{B}_+(H)$ if $\langle Px,x\rangle\geq 0$ for any $x\in H$.

Proposition 2. *Let* $\mathbf{P}=(P_1,...,P_n)\in\mathcal{B}_+^{(n)}(H)$, $\mathbf{T}=(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)$, $\mathbf{A}=(A_1,...,A_n)\in\mathcal{B}_2^{(n)}(H)$ *and* $\mathbf{z}=(z_1,...,z_n)\in\mathbb{C}^n$ *with* $n\geq 2$. *Then*

$$\left|\operatorname{tr}\left(\sum_{k=1}^n z_kP_kA_kT_kA_k^*\right)\right|^2 \tag{5.1}$$
$$\leq\operatorname{tr}\left(\sum_{k=1}^n|z_k|P_kA_k|T_k|^{2\alpha}A_k^*\right)\operatorname{tr}\left(\sum_{k=1}^n|z_k|P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)$$

for any $\alpha\in[0,1]$.

Proof. Using the properties of modulus and the inequality (4.1) we have

$$\left|\operatorname{tr}\left(\sum_{k=1}^n z_kP_kA_kT_kA_k^*\right)\right|$$
$$=\left|\sum_{k=1}^n z_k\operatorname{tr}(P_kA_kT_kA_k^*)\right|\leq\sum_{k=1}^n|z_k|\left|\operatorname{tr}(P_kA_kT_kA_k^*)\right|$$
$$\leq\sum_{k=1}^n|z_k|\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}.$$

Utilizing the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\sum_{k=1}^n|z_k|\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}$$
$$\leq\left(\sum_{k=1}^n|z_k|\left(\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2}$$
$$\times\left(\sum_{k=1}^n|z_k|\left(\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2}$$
$$=\left(\sum_{k=1}^n|z_k|\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right)^{1/2}\left(\sum_{k=1}^n|z_k|\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right)^{1/2},$$

which imply the desired result (5.1). □

Remark 2. *If we take $P_k = 1_H$ for any $k \in \{1, ..., n\}$ in (5.1), then we have the simpler inequality*

$$\left| \operatorname{tr}\left(\sum_{k=1}^{n} z_k |A_k|^2 T_k \right) \right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| |A_k|^2 |T_k|^{2\alpha} \right) \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| |A_k|^2 |T_k^*|^{2(1-\alpha)} \right) \tag{5.2}$$

provided that $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$, $\alpha \in [0, 1]$ and $\mathbf{z} = (z_1, ..., z_n) \in \mathbb{C}^n$.

We consider the functional for n-tuples of nonnegative operators $\mathbf{P} = (P_1, ..., P_n) \in \mathcal{B}_+^{(n)}(H)$ as follows:

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k |T_k|^{2\alpha} A_k^* \right) \right]^{1/2} \times \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k |T_k^*|^{2(1-\alpha)} A_k^* \right) \right]^{1/2} - \left| \operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k T_k A_k^* \right) \right|, \tag{5.3}$$

where $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$ and $\alpha \in [0, 1]$.

Utilising a similar argument to the one in Theorem 6 we can state:

Proposition 3. *Let $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)$, $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{B}_2^{(n)}(H)$ and $\alpha \in [0, 1]$.*

(i) For any $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ we have

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P} + \mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) + \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \, (\geq 0), \tag{5.4}$$

namely, $\sigma_{\mathbf{A},\mathbf{T},\alpha}$ is a superadditive functional on $\mathcal{B}_+^{(n)}(H)$;

(ii) For any $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ with $\mathbf{P} \geq \mathbf{Q}$, namely $P_k \geq Q_k$ for all $k \in \{1, ..., n\}$ we have

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \, (\geq 0), \tag{5.5}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ is a monotonic nondecreasing functional on $\mathcal{B}_+^{(n)}(H)$;

(iii) If $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ and there exist the constants $M > m > 0$ such that $M\mathbf{Q} \geq \mathbf{P} \geq m\mathbf{Q}$ then

$$M\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq m\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \, (\geq 0). \tag{5.6}$$

If $\mathbf{P} = (p_1 1_H, ..., p_n 1_H)$ with $p_k \geq 0$, $k \in \{1, ..., n\}$ then the functional of real nonnegative weights $\mathbf{p} = (p_1, ..., p_n)$ defined by

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{p}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha} \right) \right]^{1/2} \times \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left| \operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k \right) \right| \tag{5.7}$$

has the same properties as in Theorem 6.

Moreover, we have the simple bounds

$$\begin{aligned} (5.8)\quad & \max_{k\in\{1,\dots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\right. \\ & \times\left.\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right|\right) \\ & \geq\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} \\ & -\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right| \\ & \geq\min_{k\in\{1,\dots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k|^{2\alpha}\right)\right]^{1/2}\right. \\ & \times\left.\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2|T_k^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2T_k\right)\right|\right). \end{aligned}$$

6. Inequalities for Power Series of Operators

Denote by:

$$D(0,R)=\begin{cases}\{z\in\mathbb{C}:|z|<R\}, & \text{if } R<\infty\\ \mathbb{C}, & \text{if } R=\infty,\end{cases}$$

and consider the functions:

$$\lambda\mapsto f(\lambda):D(0,R)\to\mathbb{C},\ f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$$

and

$$\lambda\mapsto f_a(\lambda):D(0,R)\to\mathbb{C},\ f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} (6.1)\quad f(\lambda) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1); \end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1);\tag{6.2}$$

$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$

$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$

$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{6.3}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$

$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$

$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function.*

Theorem 7. *Let $f(\lambda):=\sum_{n=1}^{\infty}\alpha_n\lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. Let $N\in\mathcal{B}(H)$ be a normal operator. If for some $\alpha\in(0,1)$ we have $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(|N|^{2\alpha}\right)$, $\operatorname{tr}\left(|N|^{2(1-\alpha)}\right)<R$, then we have the inequality*

$$|\operatorname{tr}(f(N))|^2\le\operatorname{tr}\left(f_a\left(|N|^{2\alpha}\right)\right)\operatorname{tr}\left(f_a\left(|N|^{2(1-\alpha)}\right)\right).\tag{6.4}$$

Proof. Since N is a normal operator, then for any natural number $k\ge1$ we have $\left|N^k\right|^{2\alpha}=|N|^{2\alpha k}$ and $\left|N^k\right|^{2(1-\alpha)}=|N|^{2(1-\alpha)k}$.

By the generalized triangle inequality for the modulus we have for $n\ge2$

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n}\alpha_kN^k\right)\right|=\left|\sum_{k=1}^{n}\alpha_k\operatorname{tr}\left(N^k\right)\right|\le\sum_{k=1}^{n}|\alpha_k|\left|\operatorname{tr}\left(N^k\right)\right|.\tag{6.5}$$

If for some $\alpha\in(0,1)$ we have $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$, then by Corollary 2 we have $N\in\mathcal{B}_1(H)$. Now, since N, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$ then any natural power of these operators belong to $\mathcal{B}_1(H)$ and by (3.13) we have

$$\left|\operatorname{tr}\left(N^k\right)\right|^2 \leq \operatorname{tr}\left(|N|^{2\alpha k}\right)\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right), \tag{6.6}$$

for any natural number $k \geq 1$.

Making use of (6.6) we have

$$\sum_{k=1}^{n}|\alpha_k|\left|\operatorname{tr}\left(N^k\right)\right| \leq \sum_{k=1}^{n}|\alpha_k|\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}. \tag{6.7}$$

Utilising the weighted Cauchy-Bunyakovsky-Schwarz inequality for sums we also have

$$\begin{aligned}
&\sum_{k=1}^{n}|\alpha_k|\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2} \\
&\leq \left[\sum_{k=1}^{n}|\alpha_k|\left(\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\right)^2\right]^{1/2} \\
&\times \left[\sum_{k=1}^{n}|\alpha_k|\left(\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}\right)^2\right]^{1/2} \\
&= \left[\sum_{k=1}^{n}|\alpha_k|\operatorname{tr}\left(|N|^{2\alpha k}\right)\right]^{1/2}\left[\sum_{k=1}^{n}|\alpha_k|\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right]^{1/2}.
\end{aligned} \tag{6.8}$$

Making use of (6.5), (6.7) and (6.8) we get the inequality

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n}\alpha_k N^k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n}|\alpha_k|\,|N|^{2\alpha k}\right)\operatorname{tr}\left(\sum_{k=1}^{n}|\alpha_k|\,|N|^{2(1-\alpha)k}\right) \tag{6.9}$$

for any $n \geq 2$.

Due to the fact that $\operatorname{tr}\left(|N|^{2\alpha}\right)$, $\operatorname{tr}\left(|N|^{2(1-\alpha)}\right) < R$ it follows by (3.13) that $\operatorname{tr}(|N|) < R$ and the operator series

$$\sum_{k=1}^{\infty}\alpha_k N^k,\ \sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2\alpha k} \text{ and } \sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n \to \infty$ in (6.9) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (6.4). □

Example 1. *a) If we take in $f(\lambda) = (1 \pm \lambda)^{-1} - 1 = \mp\lambda\left((1 \pm \lambda)^{-1}\right)$, $|\lambda| < 1$ then we get from (6.4) the inequality*

$$\begin{aligned}
&\left|\operatorname{tr}\left(N\left((1 \pm N)^{-1}\right)\right)\right|^2 \\
&\leq \operatorname{tr}\left(|N|^{2\alpha}\left(1 - |N|^{2\alpha}\right)^{-1}\right)\operatorname{tr}\left(|N|^{2(1-\alpha)}\left(1 - |N|^{2(1-\alpha)}\right)^{-1}\right),
\end{aligned} \tag{6.10}$$

provided that $N \in \mathcal{B}(H)$ is a normal operator and for $\alpha \in (0,1)$ we have $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(|N|^{2\alpha}\right)$, $\operatorname{tr}\left(|N|^{2(1-\alpha)}\right) < 1$.

b) If we take in (6.4) $f(\lambda)=\exp(\lambda)-1$, $\lambda\in\mathbb{C}$ *then we get the inequality*

$$|\operatorname{tr}(\exp(N)-1_H)|^2\le\operatorname{tr}\left(\exp\left(|N|^{2\alpha}\right)-1_H\right)\operatorname{tr}\left(\exp\left(|N|^{2(1-\alpha)}\right)-1_H\right), \tag{6.11}$$

provided that $N\in\mathcal{B}(H)$ *is a normal operator and for* $\alpha\in(0,1)$ *we have* $|N|^{2\alpha}$, $|N|^{2(1-\alpha)}\in\mathcal{B}_1(H)$.

The following result also holds:

Theorem 8. *Let* $f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ *are normal operators that double commute, i.e.* $TA=AT$ *and* $TA^*=A^*T$ *and* $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<R$ *for some* $\alpha\in[0,1]$, *then*

$$\left|\operatorname{tr}\left(f\left(|A|^2T\right)\right)\right|^2\le\operatorname{tr}\left(f_a\left(|A|^2|T|^{2\alpha}\right)\right)\operatorname{tr}\left(f_a\left(|A|^2|T|^{2(1-\alpha)}\right)\right). \tag{6.12}$$

Proof. From the inequality (5.2) we have

$$\begin{aligned}&\left|\operatorname{tr}\left(\sum_{k=0}^{n}\alpha_k\left|A^k\right|^2T^k\right)\right|^2\\&\le\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left|A^k\right|^2\left|T^k\right|^{2\alpha}\right)\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left|A^k\right|^2\left|T^k\right|^{2(1-\alpha)}\right).\end{aligned} \tag{6.13}$$

Since A and T are normal operators, then $\left|A^k\right|^2=|A|^{2k}$, $\left|T^k\right|^{2\alpha}=|T|^{2\alpha k}$ and $\left|T^k\right|^{2(1-\alpha)}=|T|^{2(1-\alpha)k}$ for any natural number $k\ge0$ and $\alpha\in[0,1]$.

Since T and A double commute, then is easy to see that

$$|A|^{2k}T^k=\left(|A|^2T\right)^k,\ \ |A|^{2k}|T|^{2\alpha k}=\left(|A|^2|T|^{2\alpha}\right)^k$$

and

$$|A|^{2k}|T|^{2(1-\alpha)k}=\left(|A|^2|T|^{2(1-\alpha)}\right)^k$$

for any natural number $k\ge0$ and $\alpha\in[0,1]$.

Therefore (6.13) is equivalent to

$$\begin{aligned}&\left|\operatorname{tr}\left(\sum_{k=0}^{n}\alpha_k\left(|A|^2T\right)^k\right)\right|^2\\&\le\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2\alpha}\right)^k\right)\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2(1-\alpha)}\right)^k\right),\end{aligned} \tag{6.14}$$

for any natural number $n\ge 1$and $\alpha\in[0,1]$.

Due to the fact that $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<R$ it follows by (5.2) for $n=1$ that $\operatorname{tr}\left(|A|^2T\right)<R$ and the operator series

$$\sum_{k=1}^{\infty}\alpha_kN^k,\ \ \sum_{k=1}^{\infty}|\alpha_k||N|^{2\alpha k}\ \text{ and }\ \sum_{k=1}^{\infty}|\alpha_k||N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n\to\infty$ in (6.14) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (6.12). □

Example 2. *a) If we take* $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ *then we get from (6.12) the inequality*

$$\left|\operatorname{tr}\left(\left(1 \pm |A|^2 T\right)^{-1}\right)\right|^2 \leq \operatorname{tr}\left(\left(1 - |A|^2 |T|^{2\alpha}\right)^{-1}\right) \operatorname{tr}\left(\left(1 - |A|^2 |T|^{2(1-\alpha)}\right)^{-1}\right), \tag{6.15}$$

provided that $T \in \mathcal{B}(H)$, $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < 1$ *for* $\alpha \in [0,1]$.

b) If we take in (6.12) $f(\lambda) = \exp(\lambda)$, $\lambda \in \mathbb{C}$ *then we get the inequality*

$$\left|\operatorname{tr}\left(\exp\left(|A|^2 T\right)\right)\right|^2 \leq \operatorname{tr}\left(\exp\left(|A|^2 |T|^{2\alpha}\right)\right) \operatorname{tr}\left(\exp\left(|A|^2 |T|^{2(1-\alpha)}\right)\right), \tag{6.16}$$

provided that $T \in \mathcal{B}(H)$ *and* $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\alpha \in [0,1]$.

Theorem 9. *Let* $f(z) := \sum_{j=0}^{\infty} p_j z^j$ *and* $g(z) := \sum_{j=0}^{\infty} q_j z^j$ *be two power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$, $R > 0$. *If* $T \in \mathcal{B}(H)$, $A \in \mathcal{B}_2(H)$ *are normal operators that double commute and* $\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < R$ *for* $\alpha \in [0,1]$, *then*

$$\begin{aligned}
&\left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right) + g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \\
&\times \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right) + g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(f\left(|A|^2 T\right) + g\left(|A|^2 T\right)\right)\right| \\
&\geq \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\
&+ \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right| (\geq 0).
\end{aligned} \tag{6.17}$$

Moreover, if $p_j \geq q_j$ *for any* $j \in \mathbb{N}$, *then, with the above assumptions on* T *and* A, *we have*

$$\begin{aligned}
&\left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\
&\geq \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2\alpha}\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|A|^2 |T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\
&- \left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right| (\geq 0).
\end{aligned} \tag{6.18}$$

The proof follows in a similar way to the proof of Theorem 8 by making use of the superadditivity and monotonicity properties of the functional $\sigma_{\mathbf{A},\mathbf{T},\alpha}(\cdot)$. We omit the details.

Example 3. *Now, observe that if we take*

$$f(\lambda)=\sinh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}$$

and

$$g(\lambda)=\cosh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}$$

then

$$f(\lambda)+g(\lambda)=\exp\lambda=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n}$$

for any $\lambda\in\mathbb{C}$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ *are normal operators that double commute and* $\alpha\in[0,1]$, *then by (6.17) we have*

$$\begin{aligned}(6.19)\qquad &\left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\ &-\left|\operatorname{tr}\left(\exp\left(|A|^2T\right)\right)\right|\\ &\geq\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\ &-\left|\operatorname{tr}\left(\sinh\left(|A|^2T\right)\right)\right|\\ &+\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}\\ &-\left|\operatorname{tr}\left(\cosh\left(|A|^2T\right)\right)\right|(\geq 0).\end{aligned}$$

Now, consider the series $\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^n$, $\lambda\in D(0,1)$ *and* $\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n$, $\lambda\in D(0,1)$ *and define* $p_n=1$, $n\geq 0$, $q_0=0$, $q_n=\frac{1}{n}$, $n\geq 1$, *then we observe that for any* $n\geq 0$ *we have* $p_n\geq q_n$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ *are normal operators that double commute,* $\alpha\in[0,1]$ *and* $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<1$, *then by (6.18) we have*

$$\begin{aligned}(6.20)\qquad &\left[\operatorname{tr}\left(\left(1-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(1-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2}\\ &-\left|\operatorname{tr}\left(\left(1-|A|^2T\right)^{-1}\right)\right|\\ &\geq\left[\operatorname{tr}\left(\ln\left(1-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\ln\left(1-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2}\\ &-\left|\operatorname{tr}\left(\ln\left(1-|A|^2T\right)^{-1}\right)\right|(\geq 0).\end{aligned}$$

References

[1] T. Ando, Matrix Young inequalities, *Oper. Theory Adv. Appl.* **75** (1995), 33–38.

[2] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[3] E. V. Belmega, M. Jungers and S. Lasaulce, A generalization of a trace inequality for positive definite matrices. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 26, 5 pp.

[4] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[5] L. Chen and C. Wong, Inequalities for singular values and traces, *Linear Algebra Appl.* **171** (1992), 109–120.

[6] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[7] M. Fujii, C.-S. Lin and R. Nakamoto, Alternative extensions of Heinz-Kato-Furuta inequality. *Sci. Math.* **2** (1999), no. 2, 215–221.

[8] M. Fujii and T. Furuta, Löwner-Heinz, Cordes and Heinz-Kato inequalities. *Math. Japon.* **38** (1993), no. 1, 73–78.

[9] M. Fujii, E. Kamei, C. Kotari and H. Yamada, Furuta's determinant type generalizations of Heinz-Kato inequality. *Math. Japon.* **40** (1994), no. 2, 259–267.

[10] M. Fujii, Y.O. Kim, and Y. Seo, Further extensions of Wielandt type Heinz-Kato-Furuta inequalities via Furuta inequality. *Arch. Inequal. Appl.* **1** (2003), no. 2, 275–283

[11] M. Fujii, Y.O. Kim and M. Tominaga, Extensions of the Heinz-Kato-Furuta inequality by using operator monotone functions. *Far East J. Math. Sci.* (FJMS) **6** (2002), no. 3, 225–238

[12] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **128** (2000), no. 1, 223–228.

[13] M. Fujii and R. Nakamoto, Extensions of Heinz-Kato-Furuta inequality. II. *J. Inequal. Appl.* **3** (1999), no. 3, 293–302,

[14] S. Furuichi and M. Lin, Refinements of the trace inequality of Belmega, Lasaulce and Debbah. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 23, 4 pp.

[15] T. Furuta, Equivalence relations among Reid, Löwner-Heinz and Heinz-Kato inequalities, and extensions of these inequalities. *Integral Equations Operator Theory* **29** (1997), no. 1, 1–9.

[16] T. Furuta, Determinant type generalizations of Heinz-Kato theorem via Furuta inequality. *Proc. Amer. Math. Soc.* **120** (1994), no. 1, 223–231.

[17] T. Furuta, An extension of the Heinz-Kato theorem. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 785–787.

[18] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[19] T. Kato, Notes on some inequalities for linear operators, *Math. Ann.* **125**(1952), 208-212.

[20] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[21] F. Kittaneh, Norm inequalities for fractional powers of positive operators. *Lett. Math. Phys.* **27** (1993), no. 4, 279–285.

[22] H. D. Lee, On some matrix inequalities, *Korean J. Math.* **16** (2008), No. 4, pp. 565-571.

[23] C.-S. Lin, On Heinz-Kato-Furuta inequality with best bounds. J. *Korea Soc. Math. Educ. Ser. B Pure Appl. Math.* **15** (2008), no. 1, 93–101.

[24] C.-S. Lin, On chaotic order and generalized Heinz-Kato-Furuta-type inequality. *Int. Math. Forum* **2** (2007), no. 37-40, 1849–1858,

[25] C.-S. Lin, On inequalities of Heinz and Kato, and Furuta for linear operators. *Math. Japon.* **50** (1999), no. 3, 463–468.

[26] C.-S. Lin, On Heinz-Kato type characterizations of the Furuta inequality. II. *Math. Inequal. Appl.* **2** (1999), no. 2, 283–287.

[27] L. Liu, A trace class operator inequality, *J. Math. Anal. Appl.* 328 (2007) 1484–1486.

[28] S. Manjegani, Hölder and Young inequalities for the trace of operators, *Positivity* **11** (2007), 239–250.

[29] C.A. McCarthy, c_p. *Israel J. Math.*, **5**(1967), 249-271.

[30] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.

[31] K. Shebrawi and H. Albadawi, Operator norm inequalities of Minkowski type, *J. Inequal. Pure Appl. Math.* **9**(1) (2008), 1–10, article 26.

[32] K. Shebrawi and H. Albadawi, Trace inequalities for matrices, *Bull. Aust. Math. Soc.* **87** (2013), 139–148.

[33] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.

[34] M. Uchiyama, Further extension of Heinz-Kato-Furuta inequality. *Proc. Amer. Math. Soc.* **127** (1999), no. 10, 2899–2904.

[35] Z. Ulukök and R. Türkmen, On some matrix trace inequalities. *J. Inequal. Appl.* **2010**, Art. ID 201486, 8 pp.

[36] X. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **250** (2000) 372–374.

[37] X. M. Yang, X. Q. Yang and K. L. Teo, A matrix trace inequality, *J. Math. Anal. Appl.* **263** (2001), 327–331.

[38] Y. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988) 573–574.

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