Received 12/08/14

On the correctness of the main theorem for absolutely monotonic functions

Sitnik S.M.

Abstract

We prove that the main theorem for absolutely monotonic functions on $(0, \infty)$ from the book Mitrinović D.S., Pečarić J.E., Fink A.M. "Classical And New Inequalities In Analysis", Kluwer Academic Publishers, 1993, is not valid.

In the classical book [1], chapter XIII, page 365, there is a definition of absolutely monotonic on $(0, \infty)$ functions.

Definition. A function $f(x)$ is said to be *absolutely monotonic on* $(0, \infty)$ if it has derivatives of all orders and

$$f^{(k)}(x) \geqslant 0, x \in (0, \infty), k = 0, 1, 2, \dots \ . \tag{1}$$

For absolutely monotonic functions the next integral representation is essential:

$$f(x) = \int_0^\infty e^{xt} \ d\sigma(t), \tag{2}$$

where $\sigma(t)$ is bounded and nondecreasing and the integral converges for all $x \in (0, \infty)$.

Also the basic set of inequalities is considered.

Let $f(x)$ be an absolutely monotonic function on $(0, \infty)$. Then

$$f^{(k)}(x) f^{(k+2)}(x) \geqslant \left(f^{(k+1)}(x)\right)^2, k = 0, 1, 2, \dots \ . \tag{3}$$

After this definition in the book [1] the result which we classify as the main theorem for absolutely monotonic functions on $(0, \infty)$ is formulated (theorem 1, page 366).

The main theorem for absolutely monotonic functions.

The above definition (1), integral representation (2) and basic set of inequalities (3) are equivalent.

It means:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3). \tag{4}$$

In the book [1] for (1) $\Leftrightarrow$ (2) the reference is given to [2], and an equivalence (2) $\Leftrightarrow$ (3) is proved, it is in fact a consequence of Chebyschev inequality.

In this note we consider a counterexample to the equivalence (1) $\Leftrightarrow$ (2) of Widder. So unfortunately it seems that the main theorem for absolutely monotonic functions in the book [1] **is not valid !!!**.

This counterexample is very simple so it is strange enough it was not found before (cf. also [8]).

Really, consider a function $f(x) = x^2 + 1$. Obviously for all $x \in [0, \infty)$

$$f(x) \geqslant 0, f'(x) = 2x \geqslant 0, f''(x) = 2 \geqslant 0, f^{(k)}(x) = 0 \geqslant 0, k > 2. \quad (5)$$

So this function $f(x)$ is in the class of absolutely monotonic functions on $(0, \infty)$ due to the definition (1). If (1) $\Rightarrow$ (3) is valid then the next inequality must be true as a special case of (3) for all $x \in (0, \infty)$

$$f(x)f''(x) \geqslant (f'(x))^2 \Leftrightarrow 2(x^2 + 1) \geqslant 4x^2 \Leftrightarrow 1 \geqslant x^2$$

but this is not valid for all $x \in (0, \infty)$.

As a conclusion we see that implication (1) $\Rightarrow$ (3) in [1] is not valid. It also means that implication (1) $\Rightarrow$ (2) is also not valid. The implication (2) $\Rightarrow$ (3) is obviously valid due to the Chebyschev inequality.

And consequently also the theorem 2 in [1], pages 366–367 on determinant inequalities is not valid too if based only on definition (1).

In some papers the above implications are used to derive new results for absolutely monotonic functions. It seems not to be a correct way of reasoning. One way is to change the main theorem on absolutely monotonic functions to a proper one, otherwise for all special cases an integral representation must be proved independently.

Comment 1. On the other hand everything is OK with theorems on completely monotonic functions. An integral representation for them in [3] include the additional condition

$$\lim_{x \to \infty} f(x) = 0.$$

This condition is omitted in [2] but mysteriously mentioned in [3] with the reference again to [2]. May be something like it is needed also for absolutely monotonic functions.

Different aspects of completely monotonic functions are considered in ([2]–[3]), and also for example in the classical expository articles ([4]–[6]).

Comment 2. There are many ways to generalize notions of absolutely and completely monotonic functions. It seems that a first step was done by Sergei Bernstein [14] and very important generalizations were investigated by Bulgarian mathematicians Nikola Obreshkov [15]–[17] (also known for two celebrated named formulas: Obreshkov generalized Taylor expansion formula and the Obreshkov integral transform) and Jaroslav Tagamlitskii [18].

Comment 3. With absolute and complete monotonicity different functional classes are deeply connected: Stieltjes, Pick, Bernstein, Schoenberg, Schur and others, cf. [7]. Just mention recent papers [9]–[13].

So the next problems seem to be rather interesting and important.

Problem 1. Give a correct proof for the theorem under consideration from [1] and so give justification for equivalences (4).

Problem 2. Generalize the theorem under consideration from [1] *for fractional derivatives* and give justification for equivalences (4) for this case.

There are applications of considered inequalities in the theory of transmutation operators for estimating transmutation kernels and norms ([19]–[21]) and for problems of function expansions by systems of integer shifts of Gaussians ([22]–[24]).

The author is thankful to Prof. Ivan Dimovski for useful information on results of N. Obreshkov and J. Tagamlitskii.

References

[1] Mitrinović D.S., Pečarić J.E., Fink A.M. Classical And New Inequalities In Analysis, Kluwer Academic Publishers, 1993.

[2] Widder D.V. The Laplace Transform. Princeton Mathematical Series, 1946.

[3] Hirschman I.I., Widder D.V. The Convolution Transform. Princeton University Press, 1955.

[4] Miller K.S., Samko S.G. (2001): Completely monotonic functions. Integral Transforms and Special Functions, 12:4, 389-402.

[5] Alzer H., Berg Ch. Some classes of completely monotonic functions. Annales Academiæ Scientiarum Fennicæ, Mathematica, Volumen 27, 2002, 445–460.

[6] Alzer H., Berg Ch. Some classes of completely monotonic functions, II. Ramanujan J (2006) 11: 225–248.

[7] Berg Ch. Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity. In: Positive definite functions. From Schoenberg to Space-Time Challenges. Ed. J. Mateu and E. Porcu. Dept. of Mathematics, University Jaume I, Castellon, Spain, 2008.

[8] Sitnik S.M. A note on the main theorem for absolutely monotonic functions. arXiv:1202.1210v1, 2012.

[9] Kalugin G.A., Jeffrey D.J., Corless R.M. Stieltjes, Poisson and other integral representations for functions of Lambert W. arXiv:1103.5640v1 [math.CV] 27 Mar 2011.

[10] Karp D., Prilepkina E. Generalized Stieltjes transforms: basic aspects. arXiv:1111.4271v1, 2011, 22 P.

[11] Karp D., Prilepkina E. Hypergeometric functions as generalized Stieltjes transforms. arXiv:1112.5769v1, 2011.

[12] Karp D., Prilepkina E. Hypergeometric functions as generalized Stieltjes transforms Journal of Mathematical Analysis and Applications. 2012, 393,2, P. 348–359.

[13] Karp D., Prilepkina E. Generalized Stieltjes functions and their exact order Journal of Classical Analysis. V. 1, No. 1 (2012), P. 53–74.

[14] S.N. Bernstein. a) On the definition and the properties of analytic functions of a real variable. Collected Papers, 1, Moscow, 1952, P. 231–250. b) On the regularly–monotone functions. ibid., P. 487–496. (in Russian).

[15] N. Obreshkov. Sur quelques inégalités pour les dérivées des fonctions. C.R. Acad. Bulgare Sci., 2,1 (1949), P. 1–4. (also in: Nikola Obrecshkoff. Selected Papers. Sofia, 2006. V.1, P. 133–136.)

[16] N. Obreshkov. On certain classes of functions. Annuaire Univ. Sofia, Phys.–Math. Fac., 47, 1, 1951. P. 237–258. (also in: Nikola Obrecshkoff. Selected Papers. Sofia, 2006. V.1, P. 141–158.)

[17] N. Obreshkov. On the regularly–monotone functions. Annuaire Univ. Sofia, Phys.–Math. Fac., 56, 1, 1963, P. 27–34. (also in: Nikola Obrecshkoff. Selected Papers. Sofia, 2006. V.1, P. 159–164.)

[18] Tagamlitski Ya.A. On some interpolation expansions of regular–monotone functions. Izvestia Math. Institute. 3, 2, 1959, P. 187–200. (also in: East J. on Approximations, V. 16, No. 2, 2010, P. 93–107.)

[19] Sitnik S.M. Factorization and Estimates of Norms in Weighted Lebesgue Spaces of Buschman-Erdelyi Operators. Soviet Math. Dokl, 1991.

[20] Katrakhov V.V., Sitnik S.M. Boundary value problem for the stationary Schroedinger equation with a singular potential. Akademiia Nauk SSSR, Doklady, 1984.

[21] Sitnik S.M. Unitary and bounded Buschman–Erdelyi operators of zero order smothness. Preprint, Vladivostok, 1990. (In Russian).

[22] Zhuravlev M.V., Kiselev E.A., Minin L.A., Sitnik S.M. Jacobi theta-functions and systems of integral shifts of Gaussian functions. Journal of Mathematical Sciences, 2011, V. 173, No. 2, P. 231–241.

[23] Zhuravlev M.V., Minin L.A., Sitnik S.M. On numerical aspects of interpolation by integral shifts of Gaussian functions. Belgorod State University Scientific Bulletin. Mathematics, Physics. 2009, V. 13(68), P. 89–99. (In Russian).

[24] Kiselev E.A., Minin L.A., Novikov I.Ya., Sitnik S.M. On the Riesz Constants for Systems of Integer Translates. Mathematical Notes, 2014, Vol. 96, No. 2, pp. 228–238.