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Conjectures on monotonicity of ratios of Kummer and Gauss hypergeometric functions

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Abstract

In the preprint [1] the author formulated some conjectures on monotonicity of ratios for exponential series remainders. They are equivalent to conjectures on monotonicity of a ratio of Kummer hypergeometric functions and presumably not proved still. In this short note the most interesting conjecture from [1] is reproduced with its generalizations.

Let us consider exponential series remainder in the form

$$R_n(x) = \exp(x) - \sum_{k=0}^{k=n} \frac{x^k}{k!} = \sum_{k=n+1}^{k=\infty} \frac{x^k}{k!},\ x \geq 0. \tag{1}$$

Different inequalities for $R_n(x)$ and its combinations were proved by H. Hardy, W. Gautschi, H. Alzer and many others [1].

In the preprint [1] the author studied inequalities of the form

$$m(n) \leq f_n(x) = \frac{R_{n-1}(x)R_{n+1}(x)}{[R_n(x)]^2} \leq M(n),\ x \geq 0. \tag{2}$$

The search for best constants $m(n) = m_{best}(n)$, $M(n) = M_{best}(n)$ has some history. The left–hand side of (2) was first proved by Kesava Menon in 1943 with $m(n) = \frac{1}{2}$ (not best) and by Horst Alzer in 1990 with $m_{best}(n) = \frac{n+1}{n+2} = f_n(0)$ [1]. In fact this result is a special case of an inequality proved earlier by Walter Gautschi in 1982 in [2] (cf. [1]).

It seems that the right–hand side of (2) was first proved by the author in [1] with $M_{best} = 1 = f_n(\infty)$. In [1] dozens of generalizations of inequality (2) and related results were proved. May be in fact it was the first example of so called Turan–type inequality for special case of the Kummer hypergeometric functions, recently this class of inequalities became thoroughly studied (cf. just for examples [12] and [17]–[20]).

Obviously the above inequalities are consequences of the next conjecture formulated in [1].

Conjecture 1. *The function $f_n(x)$ in (2) is monotone increasing for $x \in [0;\infty), n \in \mathbb{N}$.*

In 1990's we tried to prove this conjecture by considering $(f_n(x))' \geq 0$ and expanding triple products of hypergeometric functions but failed ([3]–[4]).

Consider a representation via Kummer hypergeometric functions

$$f_n(x) = \frac{n+1}{n+2}\, g_n(x), g_n(x) = \frac{{}_1F_1(1; n+1; x)\,{}_1F_1(1; n+3; x)}{[{}_1F_1(1; n+2; x)]^2}. \tag{3}$$

So the conjecture 1 is equivalent to the next conjecture 2.

Conjecture 2. *The function $g_n(x)$ in (3) is monotone increasing for $x \in [0;\infty), n \in \mathbb{N}$.*

This leads us to the next more general

Problem 1. *Find monotonicity in x conditions for $x \in [0;\infty)$ for all parameters a,b,c for the function*

$$h(a,b,c,x) = \frac{{}_1F_1(a;b-c;x)\,{}_1F_1(a;b+c;x)}{\left[{}_1F_1(a;b;x)\right]^2}. \tag{4}$$

We may also call (4) *The abc–problem* for Kummer hypergeometric functions, why not?

Another generalization is to change Kummer hypergeometric functions to higher ones.

Problem 2. *Find monotonicity in x conditions for $x \in [0;\infty)$ for all vector–valued parameters a,b,c for the function*

$$h_{p,q}(a,b,c,x) = \frac{{}_pF_q(a;b-c;x)\,{}_pF_q(a;b+c;x)}{\left[{}_pF_q(a;b;x)\right]^2}, \tag{5}$$

$$a = (a_1,\ldots,a_p), b = (b_1,\ldots,b_q), c = (c_1,\ldots,c_q).$$

This is *The abc–problem* for Gauss hypergeometric functions.

Similar problems were recently carefully studied by D. Karp and his coauthors, cf. [5]–[16]. These problems are also connected with famous Turán–type inequalities [17]–[20].

There are applications of considered inequalities in the theory of transmutation operators for estimating transmutation kernels and norms ([21]–[23]) and for problems of function expansions by systems of integer shifts of Gaussians ([24]–[26]).

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