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Applications of the Cauchy-Bouniakowsky inequality in the theory of means

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Abstract

We offer various inequalities for means, as applications of the classical integral inequality

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right).$$

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1 Introduction

Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions. The classical inequality of Cauchy-Bouniakowsky states that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right). \tag{1}$$

One has equality in (1) iff there exists a real constant $k \in \mathbb{R}$ such that $f(x) = kg(x)$ almost everywhere in $x \in [a, b]$. When f and g are continuous, equality occurs when the above equality holds true for all $x \in [a, b]$ (see e.g. [2]).

Let $x, y > 0$ be positive real numbers. Let us denote by

$$A := A(x, y) = \frac{x + y}{2} \text{ and } G := G(x, y) = \sqrt{xy}$$

the classical arithmetic resp. geometric means of x and y.

The logarithmic and identric means L and I are defined by

$$L := L(x, y) = \frac{x - y}{\ln x - \ln y} \ (x \neq y), \ L(x, x) = x \tag{2}$$

and

$$I := I(x, y) = \frac{1}{e}(y^y/x^x)^{1/(y-x)} \ (x \neq y), \ I(x, x) = x, \tag{3}$$

respectively (see e.g. [4], [5], [11]).

One of the most important inequalities satisfied by the mean L is:

$$G < L < A \text{ for } x \neq y \tag{4}$$

Though the left side inequality of (4) is attributed to B.C. Carlson, while the right side to B. Ostle and H.L. Terwilliger (see [5] for references), the author has discovered recently ([13]) that (4) was proved in fact by Bouniakowsky in his paper [1] from 1859. In the proof, inequality (1) was used for certain particular continuous functions. The author has obtained more direct and simplified proofs of (4).

The aim of this paper is to obtain other applications of inequality (1) in the theory of means. Other means, besides L and I will be defined, when necessary.

Though there exist many integral inequalities with applications in the theory of means (some of them may be found e.g. in [5]) we will restrict here our interest **only** to the inequality (1) (in honour of V. Bouniakowsky).

2 Applications

1) Let $g(x) = 1$, $x \in [a, b]$ in (1). Then one obtains

$$\left(\int_a^b f(x)dx \right)^2 \leq (b-a) \int_a^b f^2(x)dx, \text{ with } a < b, \tag{5}$$

where equality occurs in case of continuous f, when f is constant.

a) For a new proof of (4), apply (5) for $f(x) = \frac{1}{x}$. One obtains

$$(\ln b - \ln a)^2 < (b-a) \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{(b-a)^2}{ab},$$

where the inequality is strict, as the function is not constant. The left side of (4) follows:

$$G(a, b) < L(a, b).$$

b) Apply now (1) for $f(x) = e^x$, implying:

$$(e^b - e^a)^2 < \frac{b-a}{2}(e^b - e^a)(e^b + e^a),$$

so

$$\frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}. \tag{6}$$

Replace $a = \ln x$, $b = \ln y$ in (6), obtaining

$$L(x, y) < A(x, y),$$

i.e. the right side of (4).

c) Apply now (5) for $f(x) = \frac{1}{\sqrt{x}}$. One obtains

$$4(\sqrt{b} - \sqrt{a})^2 < (b-a)(\ln b - \ln a),$$

or

$$\frac{b-a}{\ln b - \ln a} < \left[\frac{b-a}{2(\sqrt{b} - \sqrt{a})} \right]^2 = \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 = A_{1/2},$$

where $A_r = \left(\frac{a^r + b^r}{2}\right)^{1/r}$ is the Hölder mean of a and b.

As A_r is a strictly increasing function of r, we have obtained the following refinement of right side of (4):

$$L < A_{1/2} < A \tag{7}$$

In fact $A_{1/2} = \frac{A+G}{2}$, and inequality (7), with another method, has been deduced in [4], too.

d) Let now $f(x) = x^r$, where $r \neq -1$ and $-1/2$ (these cases have been applied in a), resp. c)). Then one obtains the inequality:

$$\left[\frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right]^2 < \frac{b^{2r+1} - a^{2r+1}}{(b-a)(2r+1)} \tag{8}$$

By denoting by $L_r = L_r(a,b)$ the usual r**-th logarithmic mean**

$$L_r(a,b) = \left[\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r} \quad \text{for } r \neq -1,\ r \neq 0$$

(and $L_{-1} = \lim_{r \to -1} L_r(a,b) = L$, $L_0 = \lim_{r \to 0} L_r(a,b) = I$), relation (8) can be rewritten as $L_r^{2r} < L_{2r}^{2r}$, or

$$L_r^r < L_{2r}^r \tag{9}$$

When $r > 0$, particularly (9) contains the inequality $L_r < L_{2r}$.

e) Let $f(x) = \ln x$ in (5). It is well-known that (see e.g. [5])

$$\frac{1}{b-a} \int_a^b \ln x dx = \ln I(a,b) \tag{10}$$

On the other hand, by partial integration we can deduce

$$\frac{1}{b-a} \int_a^b \ln^2 x dx = \frac{b \ln^2 b - a \ln^2 a}{b-a} - 2 \ln I(a,b), \tag{11}$$

where we have used (10). Therefore, by (5) we get:

$$\ln^2 I < \frac{b \ln^2 b - a \ln^2 a}{b-a} - 2 \ln I, \tag{12}$$

which seems to be new. Remark that (12) may be rewritten as

$$(\ln I + 1)^2 < \frac{b\ln^2 b - a\ln^2 a}{b-a} + 1 \tag{13}$$

Now we shall prove that the expression $K(a, b)$ given by

$$\ln K(a,b) + 1 = \sqrt{\frac{b\ln^2 b - a\ln^2 a}{b-a} + 1} \tag{14}$$

defines a mean. Indeed, by the mean value theorem of Lagrange one has for the function $g(x) = x\ln^2 x$:

$$\frac{g(b)-g(a)}{b-a} = \ln^2 \xi + 2\ln \xi, \text{ with } \xi \in (a,b).$$

Therefore,

$$\sqrt{\frac{g(b)-g(a)}{b-a} + 1} = \ln \xi + 1$$

which lies between $\ln a + 1$ and $\ln b + 1$.

Thus $\ln a + 1 < \ln K + 1 < \ln b + 1$, implying

$$a < K < b \text{ for } a < b. \tag{15}$$

Since $K(a, a) = a$, this means that K is indeed a mean. By (13) and (14) we get the inequality

$$I < K, \tag{16}$$

where $K := K(a,b) = \frac{1}{e} \cdot \exp\left(\sqrt{\frac{b\ln^2 b - a\ln^2 a}{b-a} + 1}\right)$.

f) Let $f(x) = \frac{1}{\sqrt{x(a+b-x)}}$ in (5). Then we get

$$\left[\frac{1}{b-a}\int_a^b \frac{1}{\sqrt{x(a+b-x)}}dx\right]^2 < \frac{1}{b-a}\int_a^b \left(\frac{1}{x} + \frac{1}{a+b-x}\right)\frac{1}{a+b}dx, \tag{17}$$

where we have used the remark that

$$\frac{1}{x(a+b-x)} = \left(\frac{1}{x} + \frac{1}{a+b-x}\right)\frac{1}{a+b}.$$

Remark also that

$$\frac{1}{b-a}\int_a^b \frac{1}{x}dx = \frac{1}{b-a}\int_a^b \frac{1}{a+b-x}dx = \frac{1}{L(a,b)}. \tag{18}$$

On the other hand one has

$$\frac{1}{b-a}\int_a^b \frac{1}{\sqrt{x(a+b-x)}} = \frac{1}{P(a,b)}, \tag{19}$$

where $P = P(a,b)$ is the Seiffert mean, defined by (see e.g. [10], [14], [15], [16])

$$P(a,b) = \frac{a-b}{2\arcsin\dfrac{a-b}{a+b}} \text{ for } a \neq b,\ P(a,a) = a \tag{20}$$

For the integral representation (19) of the mean P defined by (20), see e.g. [14]. Now, by (17), (18) and (19) we get the inequality

$$P^2 > L \cdot A, \tag{21}$$

discovered by more complicated arguments in [3].

Particularly, by the right side of (4), from (21) we get

$$P > L \tag{22}$$

Clearly, by $\sqrt{a(a+b-x)} < \dfrac{x+(a+b-x}{2} = A$ from (19) we get

$$A > P, \tag{23}$$

therefore (22) and (23) improve the right side of (4).

Remark 1. For improvements of (21) with stronger arguments, see [12].

As (21) is equivalent, with the following inequality (see [3]):

$$L^2 > P \cdot G \tag{21'}$$

inequality (21′) here follows by the proved inequality (21).

By (21) and (21′) one can deduce also

$$P^2 \cdot L^2 > (LA) \cdot (PG),$$

which implies the inequality

$$P \cdot L > A \cdot G, \tag{21''}$$

one of the main results in [16] (and proved by more difficult means).

2) Applying (1) for $f(x) = \dfrac{1}{\sqrt{x}}$ and $g(x) = \dfrac{1}{\sqrt{a+b-x}}$, and using (19) we can deduce again relation (22). We note that by the left side of (4) and (22) we get

$$P > G, \tag{24}$$

but this follows also from the observation that for any $t \in [a,b]$ one has $t(a+b-t) \geq ab$, or equivalently $(t-a)(b-t) \geq 0$. Now, using this fact, and the integral representation (19), we get (24).

b) Let $f(x) = \sqrt{\dfrac{\ln x}{x}}$, $g(x) = \sqrt{x \ln x}$ in (1), where $x > 1$. As

$$\frac{1}{b-a}\int_a^b \ln x dx = \ln I(a,b), \quad \frac{1}{b-a}\int_a^b \frac{\ln x}{x} dx = \frac{1}{2(b-a)}(\ln^2 b - \ln^2 a)$$

and

$$\frac{1}{b-a}\int_a^b x \ln x = \frac{A}{2} \ln I(a^2, b^2)$$

(see e.g. [7]), we get:

$$\ln^2 I < \frac{\ln b - lna}{b-a} \cdot \ln G \cdot \frac{A}{2} \cdot lnI(a^2, b^2) = \frac{A}{2L} \cdot \ln G \cdot \ln I(a^2, b^2).$$

Let $S = S(a,b)$ be the mean defined by

$$S = (a^a \cdot b^b)^{1/(a+b)} \tag{25}$$

Then it is known (see [5], [7]) that

$$S(a,b) = \frac{I(a^2,b^2)}{I(a,b)} \tag{26}$$

By using (26), from the above relations we get the inequality

$$\ln^2 I < \frac{A}{2L} \cdot \ln G \cdot \ln(S \cdot I), \tag{27}$$

which seems to be new.

Remark 2. As in the definitions of f and g we must suppose $x > 1$, clearly (27) holds true for $b > a > 1$, where $I = I(a,b)$, etc.

c) An exponential mean $E = E(a,b)$ is defined and studied e.g. in [6], [9] by

$$E = E(a,b) = \frac{be^b - ae^a}{e^b - e^a} - 1 \tag{28}$$

Apply now inequality (1) for $f(x) = \sqrt{e^x}$ and $g(x) = x\sqrt{e^x}$.

Remark that

$$\int_a^b xe^x dx = be^b - ae^a - (e^b - e^a),$$

$$\int_a^b x^2 e^x dx = b^2e^b - a^2e^a - 2\int_a^b xe^x dx$$

and that these imply

$$\int_a^b xe^x dx = (e^b - e^a)E, \quad \int_a^b x^2e^x dx = b^2e^b - a^2e^a - 2(e^b - e^a)E,$$

so we get:

$$(e^b - e^a)^2 E^2 < (e^b - e^a)[b^2e^b - a^2e^a - 2(e^b - e^a)E] \tag{29}$$

or

$$(e^b - e^a)(E^2 + 2E) < b^2e^b - a^2e^a,$$

so

$$(E+1)^2 < \frac{b^2e^b - a^2e^a}{e^b - e^a} + 1 \tag{30}$$

Define a new exponential mean F by

$$F = F(a,b) = \sqrt{\frac{b^2e^b - a^2e^a}{e^b - e^a} + 1} - 1 \tag{31}$$

By (30) we get

$$E < F \tag{32}$$

d) Let $f(x) = \dfrac{1}{\sqrt{x \ln x}}$ and $g(x) = \sqrt{\dfrac{\ln x}{x}}$ in (1). As

$$\int_a^b \frac{1}{x \ln x} dx = \ln(\ln b) - \ln(\ln a)$$

and

$$\int_a^b \frac{\ln x}{x} dx = \frac{1}{2}(\ln^2 b - \ln^2 a),$$

we get

$$(\ln b - \ln a)^2 < [\ln(\ln b) - \ln(\ln a)] \cdot \frac{1}{2}(\ln^2 b - \ln^2 a),$$

or

$$\ln b - \ln a < [\ln(\ln b) - \ln(\ln a)] \cdot \frac{1}{2}(\ln b + \ln a).$$

By letting $\ln b = y$, $\ln a = x$, this gives a new proof of right side of (4).

Applying (1) for $f(x) = \dfrac{1}{\sqrt{x} \cdot \ln x}$, $g(x) = \dfrac{1}{\sqrt{x}}$, as

$$\int_a^b \frac{1}{x \ln^2 x} dx = \frac{1}{\ln x} - \frac{1}{\ln b},$$

we get

$$[\ln(\ln b) - \ln(\ln a)]^2 < (\ln b - \ln a) \cdot \frac{\ln b - \ln a}{\ln b \cdot \ln a},$$

which by notation $\ln b = y$, $\ln a = x$, gives a new proof of left side of (4).

e) Let $f(x) = \sqrt{x}$, $g(x) = \sqrt{x} \cdot \ln x$ in (1). As

$$\int x \ln^2 x dx = \frac{x^2 \ln^2 x}{2} - \int x \ln x dx,$$

by the formula used in b) we get:

$$(b-a)^2\cdot\frac{A^2}{4}\cdot\ln^2(S\cdot I) < \frac{b^2-a^2}{2}\left[\frac{b^2\ln^2 b - a^2\ln^2 a}{2} - (b-a)\cdot\frac{A}{2}\cdot\ln(I\cdot S)\right] \tag{33}$$

As $\ln I = \dfrac{b\ln b - a\ln a}{b-a} - 1$ and $\ln S = a\ln a + b\ln b2A$, after certain transformations, we get from (33):

$$\ln^2(S\cdot I) + 2\ln(S\cdot I) < 4(1+\ln I)\cdot\ln S \tag{34}$$

Put $u = \ln I$, $v = \ln S$ in (34). It is easy to see that

$$v = \ln S > \ln I = u \tag{35}$$

becomes equivalent, after elementary transformations to

$$ab(\ln b^2 - \ln a^2) < b^2 - a^2, \text{ or } L(a^2, b^2) > G(a^2, b^2),$$

which is the left side of (4).

Now, (34) can be written as

$$(v+u)^2 + 2(v+u) < 4v(1+u), \text{ or } v^2 + u^2 + 2u < 2v + 2vu,$$

or

$$(v-u)^2 < 2(v-u) \tag{36}$$

as, by (35), $v - u > 0$, we get from (36) that $v - u < 2$, i.e.

$$S < e^2\cdot I \tag{37}$$

Therefore, inequality (37) is a consequence of the Cauchy-Bouniakowsky inequality.

3) We have shown by more applications of the inequality (1) that holds true relation (4). Now, this implies the logarithmic inequality

$$\ln x \leq x - 1, \tag{38}$$

with equality only for $x = 1$. Indeed, let $x > 1$. Then by $L(x, 1) > G(x, 1)$ one has $\frac{x-1}{\ln x} > \sqrt{x}$, so $\ln x < \frac{x-1}{\sqrt{x}} < x - 1$. If $0 < x < 1$, then apply $L(1, x) < A(1, x)$, i.e. $\frac{1-x}{-\ln x} < \frac{x+1}{2}$, where $\frac{x+1}{2} < 1$. Thus $\frac{1-x}{-\ln x} < 1$, so $1-x < -\ln x$ or $\ln x < x-1$. There is equality in (38) only for $x = 1$.

Let $A_p(x) = p_1 x_1 + \ldots + p_r x_r$, $G_p(x) = x_1^{p_1} \ldots x_r^{p_r}$ and

$$H_p(x) = \frac{1}{\dfrac{p_1}{x_1} + \ldots + \dfrac{p_r}{x_r}}$$

denote the weighted arithmetic, geometric resp. harmonic means of the positive real numbers $x_1, \ldots, x_r > 0$, where the positive weights p_i ($i = 1, 2, \ldots, r$) satisfy $p_1 + \ldots + p_r = 1$.

Apply now inequality (38) for $x = \frac{x_i}{A_p(x)}$, and multiply both sides with p_i:

$$p_i \ln \frac{x_i}{A_p(x)} \leq \frac{p_i x_i}{A_p(x)} - p_i \tag{39}$$

After summation in (39) we get

$$\ln \frac{x_1^{p_1} \ldots x_r^{p_r}}{A_p(x)^{p_1+\ldots+p_r}} \leq \frac{p_1 x_1 + \ldots + p_r x_r}{A_p(x)} - (p_1 + \ldots + p_r).$$

As $p_1 + \ldots + p_r = 1$, we get the weighted arithmetic-geometric inequality. This in turn gives also the weighted harmonic-geometric inequality:

$$H_p(x) \leq G_p(x) \leq A_p(x) \tag{40}$$

The left side of (40) follows by applying

$$G_p\left(\frac{1}{x}\right) \leq A_p\left(\frac{1}{x}\right),$$

where $\frac{1}{x} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_r}\right)$.

There is equality in both sides only if $\frac{x_i}{A_p(x)} = 1$ for all $i = 1, \ldots, r$, which means that $x_1 = \ldots = x_r$.

The continuous analogue of inequality (40) can be proved in the same manner. Let $f, p : [a, b] \to \mathbb{R}$ be two positive Riemann-integrable functions.

Suppose that $\int_a^b p(x)dx = 1$ and define

$$A_{p,f} = \int_a^b p(x)f(x)dx, \; G_{p,f} = e^{\int_a^b p(x)\ln f(x)dx}, \; H_{p,f} = \frac{1}{\displaystyle\int_a^b \frac{p(x)}{f(x)}dx} \tag{41}$$

Then one has

$$H_{p,f} \le G_{p,f} \le A_{p,f}. \tag{42}$$

Particularly, when $p(x) = \dfrac{1}{b-a}$, we get

$$A_f = \frac{1}{b-a}\int_a^b f(x)dx, \quad G_f = e^{\frac{1}{b-a}\int_a^b \ln f(x)}, \quad H_f = \frac{b-a}{\displaystyle\int_a^b dx/f(x)} \tag{43}$$

so

$$H_f \le G_f \le A_f \tag{44}$$

There is equality in both sides of (41) (or (44)) only if f is a constant almost everywhere. If f is continuous, the equality occurs only when f is a constant function.

For the proof of (42) apply the same method, as in the proof of (40), but in place of summation, use integration.

Therefore, let $x = \dfrac{x_i}{A_{p,f}}$ in (38), and multiply both sides with $p(x) > 0$:

$$\ln \frac{f(x)^{p(x)}}{A_{p,f}^{p(x)}} \le \frac{p(x)f(x)}{A_{p,f}} - p(x). \tag{45}$$

By integration in (45) we get the left side of (42). Then apply this inequality to $\dfrac{1}{f}$ in place of f in order to deduce the left side of (42).

There are many applications to the discrete form (40), or continuous form (42) of the arithmetic-geometric-harmonic inequalities.

We will be mainly interest in the means studied before.

a) For the means A, G, L, I and S, the following identities are easy to prove (see also [7], [8]):

$$\ln \frac{I}{G} = \frac{A}{L} - 1 \tag{46}$$

$$\ln \frac{S}{I} = 1 - \frac{G^2}{AL} \tag{47}$$

As $A \cdot L = A(a,b) \cdot L(a,b) = L(a^2,b^2)$ and $G^2(a,b) = G(a^2,b^2)$, by replacing a with $\sqrt{a}$ and b with $\sqrt{b}$ in (47), one obtains

$$\ln \frac{S}{I}(\sqrt{a},\sqrt{b}) = 1 - \frac{G}{L} \tag{47$'$}$$

In base of identities (46) and (47′) one can state the following:

$$L < A \Leftrightarrow I > G \tag{48}$$

$$G < L \Leftrightarrow S > I \tag{49}$$

Therefore, inequalities (4) are equivalent to the following:

$$G < I < S \tag{4$'$}$$

Applying (44) to $f(x) = x$ we get

$$L < I < A \tag{50}$$

On the other hand, applying the left side of (40) for $r = 2$ and $p_1 = \dfrac{a}{a+b}$, $p_2 = \dfrac{b}{a+b}$, $x_1 = a$, $x_2 = b$, one has

$$\frac{1}{\dfrac{a}{a+b} \cdot \dfrac{1}{a} + \dfrac{b}{a+b} \cdot \dfrac{1}{b}} < a^{a/(a+b)} \cdot b^{b/(a+b)},$$

which gives

$$A < S, \tag{51}$$

see e.g. [8].

In fact, relations (4), (4′), (50) and (51) may be rewritten as

$$G < L < I < A < S \tag{52}$$

b) By (22) and (23) P lies between L and A, but we can strengthen this fact by applying the right side of (44) to

$$f(x) = \frac{1}{\sqrt{x(a+b-x)}}.$$

As $\int_a^b \ln(a+b-x)dx = \int_a^b \ln x dx = \ln I$, by (19) we get

$$P < I \tag{53}$$

Therefore, (52) may by completed as

$$G < L < P < I < A < S \tag{54}$$

Remark. Inequalities $L < P < I$ have been obtained for the first time by H.-J. Seiffert [15], by using more complicated arguments.

Let us apply now the left side of (44) to the same function f as above. Let us introduce the new mean

$$J = J(a,b) = \frac{1}{b-a}\int_a^b \sqrt{x(a+b-x)}dx \tag{55}$$

As $G < \sqrt{x(a+b-x)} < A$ (see 1f) and 2a)), we get also

$$G < J < A \tag{56}$$

By the left side of (44) however, the left side of (56) may be improved to

$$I < J \tag{57}$$

Therefore the chain (54) may by rewritten as

$$G < L < P < I < J < A < S \tag{54'}$$

Remark 3. By inequalities (21) and (21′) one can strengthen the first two inequalities:

$$G < \sqrt{P \cdot G} < L < \sqrt{L \cdot A} < P$$

c) Let us introduce another new mean R by

$$R = R(a,b) = 1/\left(\frac{1}{b-a} \cdot \int_a^b \frac{1}{\sqrt[4]{x(a+b-x)}}dx\right)^2 \tag{58}$$

As $\sqrt{G} < \sqrt[4]{x(a+b-x)} < \sqrt{A}$, clearly

$$G < R < A, \tag{59}$$

too. By inequality (5) applied to $f(x) = \dfrac{1}{\sqrt[4]{x(a+b-x)}}$ we get, using (19), that

$$P < R \tag{60}$$

Applying, as in b) the right side of (44) to this function, we get

$$R < I \tag{61}$$

Therefore, a completion of (54′) is valid:

$$G < L < P < R < I < J < A < S \tag{54''}$$

with two new means J, resp. R defined by (55), resp. (58).

d) Apply now (41) for $p(x) = \dfrac{2x}{b^2 - a^2}$ and $f(x) = \dfrac{1}{x}$. As

$$A_{p,f} = \int_a^b p(x)f(x)dx = \frac{2}{b^2 - a^2}(b-a) = \frac{1}{A},$$

$$G_{p,f} = e^{\int_a^b p(x)\ln f(x)dx} = e^{-\frac{2}{b^2 a^2}\int_a^b x\ln x dx} = \frac{1}{\sqrt{I \cdot S}},$$

as

$$\int_a^b x\ln x dx = (b-a) \cdot \frac{A}{2}\ln(I \cdot S).$$

On the other hand,

$$H_{p,f} = \frac{1}{\displaystyle\int_a^b \frac{p(x)}{f(x)}dx} = \frac{b^2-a^2}{2} \cdot \frac{3}{b^2+ab+a^2} = \frac{A}{He(a^2,b^2)},$$

where $He(x,y) = \dfrac{x+\sqrt{xy}+y}{3}$ denotes the **Heronian mean** of x and y. One obtains the double inequality:

$$A^2 < I \cdot S < \left(\frac{He(a^2,b^2)}{A}\right)^2 \tag{59}$$

The left side of (59) has been proved also in [8], while the right side seems to be new.

For an extension of (59) repeat all above computations with

$$p(x) = \frac{x^{n-1}n}{b^n - a^n}.$$

Since by partial integration we get

$$\int_a^b x^{n-1} \ln x dx = \frac{(b^n - a^n)\ln I(a^n, b^n)}{n^2} \tag{60}$$

from (42) we get

$$\frac{b^n - a^n}{n(b^{n-1} - a^{n-1})} < \sqrt[n]{I(a^n,b^n)} < \frac{n}{n+1} \cdot \frac{b^{n+1} - a^{n+1}}{b^n - a^n} \tag{61}$$

This new inequality extends (59), as for $n = 2$, by $I(a^2, b^2) = I \cdot S$, one reobtains (59). Here n is a positive integer, but as the proof shows, it holds true by replacing n with any $r > 1$, i.e.

$$\frac{b^r - a^r}{r(b^{r-1} - a^{r-1})} < (I(a^r, b^r))^{1/r} < \frac{r}{r+1} \cdot \frac{b^{r+1} - a^{r+1}}{b^r - a^r},\ r > 1. \tag{62}$$

By putting $a^n = x$, $b^n = y$ in (61), this inequality appears as

$$\frac{y-x}{n[y^{(n-1)/n} - x^{(n-1)/n}]} < \sqrt[n]{I(x,y)} < \frac{n}{n+1} \cdot \frac{y^{(n+1)/n} - x^{(n+1)/n}}{y-x}. \tag{61$'$}$$

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