Received 17/08/14

OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES WITH COMPLEX EXPONENTIAL WEIGHT FOR APPROXIMATING THE LAPLACE AND FOURIER TRANSFORMS

PIETRO CERONE[1], SEVER S. DRAGOMIR[2,3], AND EDER KIKIANTY[4,*]

ABSTRACT. We present some inequalities of Ostrowski and trapezoid type, for complex-valued absolutely continuous functions. These inequalities are related to Pompeiu's mean value theorem. The applications of these inequalities to obtain some approximation results for the finite Fourier and Laplace transforms are also given.

1. INTRODUCTION

In 1938, Ostrowski [17] proved the following estimate of the integral mean:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)|\le M<\infty$ for all $t\in(a,b)$. Then, for any $x\in[a,b]$, we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a). \tag{1}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Inequality (1) is referred to, in the literature, as the Ostrowski inequality. For its generalisations and related results we refer the readers to Dragomir and Rassias [15].

Inequalities providing upper bounds for the quantity

$$\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|,\qquad x\in[a,b] \tag{2}$$

are known in the literature as the *(generalized) trapezoid inequalities*. Cerone and Dragomir [7] proved the following result:

Theorem 2. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}&\left|\frac{(x-a)f(a)+(b-x)f(b)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\ &\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]M(b-a),\end{aligned} \tag{3}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is best possible.

For its generalisations and related results, we refer the readers to Cerone and Dragomir [7].

1991 *Mathematics Subject Classification.* Primary: 26D10, 26D15.

Key words and phrases. Ostrowski inequality, trapezoid inequality, complex exponential weight, Laplace transform, Fourier transform.

It is important to note that the bounds in inequalities (1) and (3) are the same. Cerone [6, Remark 1] stated that there is a strong relationship between the Ostrowski and the trapezoidal functionals which is highlighted by the symmetric transformations amongst their kernels.

In 1946, Pompeiu [19] derived a variant of Lagrange's mean value theorem, known as *Pompeiu's mean value theorem* (cf. Sahoo and Riedel [21, p. 83]), as given below:

Theorem 3. *For every real-valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{4}$$

In the next theorem, Pompeiu's mean value theorem is utilised in order to provide another approximation of the integral mean.

Theorem 4 (Dragomir, 2005 [9])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f-\ell f'\|_\infty . \tag{5}$$

where $\ell(t)=t$, $t\in[a,b]$. The constant $\frac{1}{4}$ is best possible.

We refer the readers to Popa [20], Pečarić and Ungar [18], Acu and Sofonea [1] and Acu et al. [2] for the generalisations and extensions of Theorem 4.

Inequalities of Ostrowski type which are related to the Pompeiu's mean value theorem are given in the papers by Dragomir [10, 11]. Further inequalities of Ostrowski and trapezoid types which are related to the Pompeiu's mean value theorem can be found in Cerone, Dragomir and Kikianty [8].

Some exponential Pompeiu type inequalities for complex-valued absolutely continuous functions are given in Dragomir [12], with applications to obtain some new Ostrowksi type inequalities. We recall the results on the Ostrowski type inequalities, in the next two theorems.

Theorem 5. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on the interval $[a,b]$ and $\alpha=\beta+i\gamma\in\mathbb{C}$ with $\beta>0$. Then, for any $x\in[a,b]$ we have*

$$\left|f(x)\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha} - \exp(\alpha x)\int_a^b f(t)\,dt\right| \tag{6}$$
$$\le \begin{cases} |\beta| B_1(a,b,x,\alpha)\|f'-\alpha f\|_\infty, & \text{if } f'-\alpha f\in L_\infty[a,b], \\[2ex] q^{1/q}|\beta|^{1/q}(b-a)^{1/p} & \\ \times |B_q(a,b,x,\alpha)|^{1/q}\|f'-\alpha f\|_p, & \text{if } f'-\alpha f\in L_p[a,b],\ p>1 \\ & \frac{1}{p}+\frac{1}{q}=1, \\[2ex] B_\infty(a,b,x,\alpha)\|f'-\alpha f\|_1. & \end{cases}$$

where

$$\begin{aligned} B_q(a,b,x,\alpha) \quad := \quad & 2\Bigg[\exp(xq\beta)\left(x-\frac{a+b}{2}\right) \\ & +\frac{1}{q\beta}\left(\frac{\exp(bq\beta)+\exp(aq\beta)}{2}-\exp(xq\beta)\right)\Bigg] \end{aligned}$$

for $q \geq 1$ *and*

$$B_\infty(a,b,x,\alpha) := \exp(x\beta)(x-a) + \frac{\exp(b\beta)-\exp(x\beta)}{\beta}.$$

Theorem 6. *Let* $f : [a,b] \to \mathbb{C}$ *be an absolutely continuous function on the interval* $[a,b]$ *and* $\alpha = \beta + i\gamma \in \mathbb{C}$ *with* $\beta = 0$*. Then, for any* $x \in [a,b]$ *we have*

$$\left| f(x)\frac{\exp(i\gamma b)-\exp(i\gamma a)}{i\gamma} - \exp(i\gamma x)\int_a^b f(t)\,dt \right| \tag{7}$$

$$\leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2\|f' - i\gamma f\|_\infty, & \text{if } f' - i\gamma f \in L_\infty[a,b], \\ \left[\left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{\frac{q+1}{q}} & \\ \times \frac{q}{q+1}\|f' - i\gamma f\|_p, & \text{if } f' - i\gamma f \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (b-a)\|f' - i\gamma f\|_1. & \end{cases}$$

In this paper, we give refinements of the inequalities in Theorems 5 and 6 in Section 2. We also present similar results for trapezoid inequalities in Section 3. In Section 4, we apply these inequalities to obtain some approximation results for the finite Fourier and Laplace transforms.

2. Ostrowski type inequalities

The following Ostrowski type inequalities with complex weights hold:

Theorem 7. *Let* $f : [a,b] \to \mathbb{C}$ *an absolutely continuous function on* $[a,b]$ *and* $\alpha = \beta + i\gamma \in \mathbb{C}$.

If $\beta \neq 0$*, then we have*

$$\begin{aligned} &\left| \frac{f(x)(b-a)}{e^{\alpha x}} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \\ &\leq \frac{e^{-a\beta} - [(x-a)\beta+1]e^{-x\beta}}{\beta^2}\|f' - \alpha f\|_{[a,x],\infty} \\ &\quad + \frac{e^{-b\beta} + [(b-x)\beta-1]e^{-x\beta}}{\beta^2}\|f' - \alpha f\|_{[x,b],\infty} \\ &\leq \frac{1}{\beta^2}\left[e^{-a\beta} + e^{-b\beta}\right. \\ &\quad \left. +2\left(\left(\frac{a+b}{2} - x\right)\beta - 1\right)e^{-x\beta}\right]\|f' - \alpha f\|_{[a,b],\infty}, \end{aligned} \tag{8}$$

for any $x \in [a,b]$.

If $\beta = 0$*, then we have*

$$\begin{aligned} &\left| \frac{f(x)(b-a)}{e^{ix\gamma}} - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt \right| \\ &\leq \frac{1}{2}(x-a)^2\|f' - i\gamma f\|_{[a,x],\infty} + \frac{1}{2}(b-x)^2\|f' - i\gamma f\|_{[x,b],\infty} \\ &\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\|f' - i\gamma f\|_{[a,b],\infty}, \end{aligned} \tag{9}$$

for any $x \in [a,b]$.

The constants $\frac{1}{2}$ *and* $\frac{1}{4}$ *in* (9) *are sharp.*

Proof. We use the Montgomery identity for the absolutely continuous function $g : [a,b] \to \mathbb{C}$ (cf. Mitrinović, Pečarić and Fink [16, p. 585]):

$$g(x)(b-a) - \int_a^b g(t)\,dt = \int_a^x (t-a)g'(t)\,dt + \int_x^b (t-b)g'(t)\,dt, \tag{10}$$

where $x \in [a,b]$. If $g(t) = f(t)/e^{\alpha t}$, then $g'(t) = (f'(t) - \alpha f(t))/e^{\alpha t}$; and with this choice of g, (10) becomes:

$$\begin{aligned} &\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \\ &= \int_a^x (t-a)\frac{f'(t)-\alpha f(t)}{e^{\alpha t}}\,dt + \int_x^b (t-b)\frac{f'(t)-\alpha f(t)}{e^{\alpha t}}\,dt. \end{aligned} \tag{11}$$

Taking the modulus in (11) we get

$$\begin{aligned} &\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\le \int_a^x (t-a)|e^{-\alpha t}||f'(t)-\alpha f(t)|\,dt + \int_x^b (b-t)|e^{-\alpha t}||f'(t)-\alpha f(t)|\,dt \end{aligned} \tag{12}$$

Note that $\alpha = \beta + i\gamma$, and thus

$$|e^{-\alpha t}| = |e^{-t\beta}||e^{-it\gamma}| = e^{-t\beta}. \tag{13}$$

Thus, (12) becomes

$$\begin{aligned} &\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\le \int_a^x (t-a)e^{-t\beta}\,dt\ \|f'-\alpha f\|_{[a,x],\infty} + \int_x^b (b-t)e^{-\beta}\,dt\ \|f'-\alpha f\|_{[x,b],\infty}. \end{aligned}$$

We have,

$$\begin{aligned} \int_a^x (t-a)e^{-t\beta}\,dt &= -\frac{t-a}{\beta}e^{-t\beta}\Big|_a^x + \frac{1}{\beta}\int_a^x e^{-t\beta}\,dt \\ &= -\frac{x-a}{\beta}e^{-x\beta} - \frac{1}{\beta^2}\left(e^{-x\beta} - e^{-a\beta}\right) \\ &= \frac{e^{-a\beta} - [(x-a)\beta + 1]e^{-x\beta}}{\beta^2}; \end{aligned}$$

and

$$\begin{aligned} \int_x^b (b-t)e^{-t\beta}\,dt &= -\frac{b-t}{\beta}e^{-t\beta}\Big|_x^b - \frac{1}{\beta}\int_x^b e^{-t\beta}\,dt \\ &= \frac{b-x}{\beta}e^{-x\beta} + \frac{1}{\beta^2}\left(e^{-b\beta} - e^{-x\beta}\right) \\ &= \frac{e^{-b\beta} + [(b-x)\beta - 1]e^{-x\beta}}{\beta^2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{f(x)}{e^{\alpha x}}(b-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \int_a^x (t-a)e^{-t\beta}\,dt\;\|f'-\alpha f\|_{[a,x],\infty}+\int_x^b (b-t)e^{-\beta}\,dt\;\|f'-\alpha f\|_{[x,b],\infty} \\
&= \frac{e^{-a\beta}-[(x-a)\beta+1]e^{-x\beta}}{\beta^2}\|f'-\alpha f\|_{[a,x],\infty} \\
&\quad+\frac{e^{-b\beta}+[(b-x)\beta-1]e^{-x\beta}}{\beta^2}\|f'-\alpha f\|_{[x,b],\infty} \\
&\le \frac{1}{\beta^2}\left[e^{-a\beta}+e^{-b\beta}+2\left(\left(\frac{a+b}{2}-x\right)\beta-1\right)e^{-x\beta}\right]\|f'-\alpha f\|_{[a,b],\infty},
\end{aligned}$$

and this proves (8).

If $\beta = 0$, then

$$\begin{aligned}
&\left|\frac{f(x)}{e^{ix\gamma}}(b-a)-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\
&\le \int_a^x (t-a)\,dt\;\|f'-i\gamma f\|_{[a,x],\infty}+\int_x^b (b-t)\,dt\;\|f'-i\gamma f\|_{[x,b],\infty} \\
&= \frac{(x-a)^2}{2}\;\|f'-i\gamma f\|_{[a,x],\infty}+\frac{(b-x)^2}{2}\;\|f'-i\gamma f\|_{[x,b],\infty} \\
&\le \frac{(x-a)^2+(b-x)^2}{2}\;\|f'-i\gamma f\|_{[a,b],\infty} \\
&= \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\;\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned}$$

This completes the proof. The sharpness of the constants in (9) is given by Corollary 9. □

Remark 8. If $\alpha = 0$ in Theorem 7, then we have a refinement for the Ostrowski inequality:

$$\begin{aligned}
\left|f(x)(b-a)-\int_a^b f(t)\,dt\right| &\le \frac{1}{2}(x-a)^2\|f'\|_{[a,x],\infty}+\frac{1}{2}(b-x)^2\|f'\|_{[x,b],\infty} \\
&\le \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}, \qquad (14)
\end{aligned}$$

for any $x \in [a,b]$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are sharp (cf. Corollary 9).

Let $h(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h(t) = g(t)e^{\alpha t}$ in Theorem 7, then we have the Ostrowski inequalities:

$$\begin{aligned}
\left|g(x)(b-a)-\int_a^b g(t)\,dt\right| &\le \frac{e^{-a\beta}-[(x-a)\beta+1]e^{-x\beta}}{\beta^2}\|g'h\|_{[a,x],\infty} \\
&\quad+\frac{e^{-b\beta}+[(b-x)\beta-1]e^{-x\beta}}{\beta^2}\|g'h\|_{[x,b],\infty} \qquad (15) \\
&\le \frac{1}{\beta^2}\Big[e^{-a\beta}+e^{-b\beta} \\
&\quad+2\left(\left(\frac{a+b}{2}-x\right)\beta-1\right)e^{-x\beta}\Big]\|g'h\|_{[a,b],\infty},
\end{aligned}$$

for any $x \in [a,b]$ and $\beta \neq 0$. If $\beta = 0$, then we have

$$\begin{aligned}\left|g(x)(b-a) - \int_a^b g(t)\,dt\right| &\leq \frac{1}{2}(x-a)^2\|g'h\|_{[a,x],\infty} + \frac{1}{2}(b-x)^2\|g'h\|_{[x,b],\infty} \\ &\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\|g'h\|_{[a,b],\infty}, \qquad (16)\end{aligned}$$

for any $x \in [a,b]$.

Corollary 9. *Under the assumptions of Theorem 7, we have the special cases as follows:*

If $\beta \neq 0$, then we have

$$\begin{aligned}&\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{\frac{\alpha(a+b)}{2}}} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\leq \frac{e^{-a\beta} - [\frac{b-a}{2}\beta + 1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],\infty} \\ &\quad + \frac{e^{-b\beta} + [\frac{b-a}{2}\beta - 1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f' - \alpha f\|_{[\frac{a+b}{2},b],\infty} \qquad (17) \\ &\leq \frac{1}{\beta^2}\left[e^{-a\beta} + e^{-b\beta} - 2e^{-\frac{a+b}{2}\beta}\right]\|f' - \alpha f\|_{[a,b],\infty}.\end{aligned}$$

If $\beta = 0$, then we have

$$\begin{aligned}&\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{i\frac{a+b}{2}\gamma}} - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\leq \frac{1}{8}(b-a)^2\left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],\infty} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right] \qquad (18) \\ &\leq \frac{1}{4}(b-a)^2\|f' - i\gamma f\|_{[a,b],\infty}.\end{aligned}$$

The constants $\frac{1}{8}$ and $\frac{1}{4}$ in (18) *is sharp.*

Proof. We obtain (17) and (18) by taking $x = (a+b)/2$ in (8) and (9), respectively. To prove the sharpness of the constants in (18), we assume that the inequalities hold for $A, B > 0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$, respectively:

$$\begin{aligned}&\left|\frac{f\left(\frac{a+b}{2}\right)(b-a)}{e^{i\frac{a+b}{2}\gamma}} - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\leq A(b-a)^2\left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],\infty} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right] \\ &\leq B(b-a)^2\|f' - i\gamma f\|_{[a,b],\infty}.\end{aligned}$$

Let $\gamma = 0$ and choose $f(x) = |x - \frac{a+b}{2}|$ on $[a,b]$, thus we have

$$\frac{(b-a)^2}{4} \leq 2A(b-a)^2 \leq B(b-a)^2,$$

which yields $A \geq \frac{1}{8}$ and $B \geq \frac{1}{4}$. □

We recall the definition of the Gamma and incomplete Gamma functions:

$$\begin{aligned}\Gamma(t) &= \int_0^\infty x^{t-1}e^{-x}\,dx, \\ \Gamma(s,x) &= \int_x^\infty t^{s-1}e^{-t}\,dt.\end{aligned}$$

Throughout the text, for $\alpha = \beta + i\gamma \in \mathbb{C}$ and $1 < q < \infty$, we use the following notation:

$$\Psi^{+}_{q,\alpha}(s,t) = e^{-s\beta}(q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, q\beta(t-s))\right]^{\frac{1}{q}}$$

when $\beta > 0$; and

$$\Psi^{-}_{q,\alpha}(s,t) = e^{-t\beta}(-q\beta)^{-\frac{q+1}{q}}\left[\Gamma(q+1) - \Gamma(q+1, -q\beta(t-s))\right]^{\frac{1}{q}}$$

when $\beta < 0$.

Theorem 10. *Let $f : [a,b] \to \mathbb{C}$ an absolutely continuous function on $[a,b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$ and $1 < p < \infty$. Let $q > 1$ be a real number such that $\frac{1}{p} + \frac{1}{q} = 1$.*

If $\beta \neq 0$, then

$$\begin{aligned} &\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\le \Psi^{+}_{q,\alpha}(a,x)\|f' - \alpha f\|_{[a,x],p} + \Psi^{-}_{q,\alpha}(x,b)\|f' - \alpha f\|_{[x,b],p} \\ &\le \left[\Psi^{+}_{q,\alpha}(a,x) + \Psi^{-}_{q,\alpha}(x,b)\right]\|f' - \alpha f\|_{[a,b],p}, \end{aligned} \tag{19}$$

for $x \in [a,b]$.

If $\beta = 0$, then

$$\begin{aligned} &\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[x,b],p} \\ &\le \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f' - i\gamma f\|_{[a,b],\infty}, \end{aligned} \tag{20}$$

for $x \in [a,b]$.

The inequalities in (20) *are sharp.*

Proof. Take the modulus of (11) and make use of the Hölder's inequality to obtain the following inequalities for $1 < p < \infty$ and its Hölder's conjugate q:

$$\begin{aligned} \left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| &\le \left(\int_a^x \left((t-a)|e^{-\alpha t}|\right)^q\,dt\right)^{\frac{1}{q}}\|f' - \alpha f\|_{[a,x],p} \\ &\quad + \left(\int_x^b \left((b-t)|e^{\alpha t}|\right)^q\,dt\right)^{\frac{1}{q}}\|f' - \alpha f\|_{[x,b],p} \\ &= \left(\int_a^x (t-a)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}}\|f' - \alpha f\|_{[a,x],p} \\ &\quad + \left(\int_x^b (b-t)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}}\|f' - \alpha f\|_{[x,b],p}. \end{aligned}$$

We evaluate the integral

$$\begin{aligned} \int_a^x (t-a)^q e^{-tq\beta}\,dt &= e^{-aq\beta}(q\beta)^{-q-1}\int_0^{q\beta(x-a)} z^q e^{-z}\,dz \\ &= e^{-aq\beta}(q\beta)^{-q-1}\left[\Gamma(q+1) - \Gamma(q+1, q\beta(x-a))\right]; \end{aligned}$$

by letting $z = (t-a)q\beta$; and thus

$$\left(\int_a^x (t-a)^q e^{-tq\beta}\, dt \right)^{\frac{1}{q}}$$
$$= e^{-a\beta} (q\beta)^{-\frac{q+1}{q}} \left[\Gamma(q+1) - \Gamma(q+1, q\beta(x-a)) \right]^{\frac{1}{q}} = \Psi^{+}_{q,\alpha}(a,x).$$

Now, we evaluate the integral

$$\begin{aligned} \int_x^b (b-t)^q e^{-tq\beta}\, dt &= e^{-bq\beta}(-q\beta)^{-q-1} \int_0^{q(-\beta)(b-x)} z^q e^{-z}\, dz \\ &= e^{-bq\beta}(-q\beta)^{-q-1} \left[\Gamma(q+1) - \Gamma(q+1, -q\beta(b-x)) \right], \end{aligned}$$

by letting $z = -(b-t)q\beta$; and thus

$$\left(\int_x^b (b-t)^q e^{-tq\beta}\, dt \right)^{\frac{1}{q}}$$
$$= e^{-b\beta} (-q\beta)^{-\frac{q+1}{q}} \left[\Gamma(q+1) - \Gamma(q+1, -q\beta(b-x)) \right]^{\frac{1}{q}} = \Psi^{-}_{q,\alpha}(x,b).$$

Therefore, for any $x \in [a,b]$, we have

$$\begin{aligned} &\left| \frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\, dt \right| \\ &\leq \left(\int_a^x (t-a)^q e^{-tq\beta}\, dt \right)^{\frac{1}{q}} \|f' - \alpha f\|_{[a,x],p} \\ &\quad + \left(\int_x^b (b-t)^q e^{-tq\beta}\, dt \right)^{\frac{1}{q}} \|f' - \alpha f\|_{[x,b],p} \\ &= \Psi^{+}_{q,\alpha}(a,x) \|f' - \alpha f\|_{[a,x],p} + \Psi^{-}_{q,\alpha}(x,b) \|f' - \alpha f\|_{[x,b],p} \\ &\leq \left[\Psi^{+}_{q,\alpha}(a,x) + \Psi^{-}_{q,\alpha}(x,b) \right] \|f' - \alpha f\|_{[a,b],p}, \end{aligned}$$

If $\beta = 0$, then

$$\begin{aligned} &\left| \frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\, dt \right| \\ &\leq \left(\int_a^x (t-a)^q\, dt \right)^{\frac{1}{q}} \|f' - i\gamma f\|_{[a,x],p} + \left(\int_x^b (b-t)^q\, dt \right)^{\frac{1}{q}} \|f' - i\gamma f\|_{[x,b],p} \\ &= \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[x,b],p} \\ &\leq \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,b],\infty}, \end{aligned}$$

for any $x \in [a,b]$. This completes the proof. The sharpness of the inequalities in (20) is given by Corollary 12. □

Remark 11. If $\alpha = 0$ in Theorem 10, then we have a refinement for the Ostrowski inequality:

$$\begin{aligned}
&\left| f(x)(b-a) - \int_a^b f(t)\,dt \right| \\
&\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[x,b],p} \\
&\le \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,b],\infty},
\end{aligned} \tag{21}$$

for any $x \in [a,b]$. The above inequalities are sharp (cf. Corollary 12).

Let $h(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h(t) = g(t)e^{\alpha t}$ in Theorem 10, then we have the Ostrowski inequalities: If $\beta \neq 0$, then

$$\begin{aligned}
\left| g(x)(b-a) - \int_a^b g(t)\,dt \right| &\le \Psi^+_{q,\alpha}(a,x)\|g'h\|_{[a,x],p} + \Psi^-_{q,\alpha}(x,b)\|g'h\|_{[x,b],p} \\
&\le \left[\Psi^+_{q,\alpha}(a,x) + \Psi^-_{q,\alpha}(x,b)\right] \|g'h\|_{[a,b],p},
\end{aligned} \tag{22}$$

for $x \in [a,b]$. If $\beta = 0$, then

$$\begin{aligned}
\left| g(x)(b-a) - \int_a^b g(t)\,dt \right| &\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h\|_{[x,b],p} \\
&\le \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h\|_{[a,b],\infty},
\end{aligned} \tag{23}$$

for $x \in [a,b]$.

Corollary 12. *Under the assumptions of Theorem 10, we have the special cases as follows:*

If $\beta \neq 0$, then

$$\begin{aligned}
&\left| \frac{f\left(\frac{a+b}{2}\right)}{e^{\frac{\alpha(a+b)}{2}}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \qquad (24) \\
&\le \Psi^+_{q,\alpha}\left(a, \frac{a+b}{2}\right) \|f' - \alpha f\|_{[a,\frac{a+b}{2}],p} + \Psi^-_{q,\alpha}\left(\frac{a+b}{2}, b\right) \|f' - \alpha f\|_{[\frac{a+b}{2},b],p} \\
&\le \left[\Psi^+_{q,\alpha}\left(a, \frac{a+b}{2}\right) + \Psi^-_{q,\alpha}\left(\frac{a+b}{2}, b\right)\right] \|f' - \alpha f\|_{[a,b],p}.
\end{aligned}$$

If $\beta = 0$, then

$$\begin{aligned}
&\left| \frac{f\left(\frac{a+b}{2}\right)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt \right| \\
&\le \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}} \left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],p} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],p} \right] \\
&\le \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,b].\infty}.
\end{aligned} \tag{25}$$

The inequalities in (25) *are sharp.*

Proof. We obtain (24) and (25) by taking $x = (a+b)/2$ in (19) and (20), respectively. To prove the sharpness of the constants in (25), we assume the inequalities hold for

$C, D > 0$ instead of $\frac{1}{2}$ and 1, respectively:

$$\begin{aligned}&\left|\frac{f\left(\frac{a+b}{2}\right)}{e^{ix\gamma}}(b-a)-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\le C\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],p}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],p}\right] \\ &\le D\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\ \|f'-i\gamma f\|_{[a,b].\infty}.\end{aligned}$$

Let $\gamma = 0$ and take $f(x) = |x - \frac{a+b}{2}|$ on $[a, b]$, and we now have

$$\frac{(b-a)^2}{4} \le 2C\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \le D\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}.$$

Take $q \to 1$, then we have $\frac{(b-a)^2}{4} \le 2C\frac{(b-a)^2}{4} \le D\frac{(b-a)^2}{4}$, which yields $C \ge \frac{1}{2}$ and $D \ge 1$; and the proof is completed. □

The case for the 1-norm is as follows:

Theorem 13. *Let $f : [a, b] \to \mathbb{C}$ an absolutely continuous function on $[a, b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$. If $\beta \neq 0$, then*

$$\begin{aligned}&\left|\frac{f(x)}{e^{\alpha x}}(b-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\le \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|f'-\alpha f\|_{[a,x],1} \\ \quad +(b-x)e^{-x\beta}\|f'-\alpha f\|_{[x,b],1}, & \beta>0 \\ (x-a)e^{-x\beta}\|f'-\alpha f\|_{[a,x],1} \\ \quad +\frac{1}{-\beta}e^{-(b\beta+1)}\|f'-\alpha f\|_{[x,b],1}, & \beta<0. \end{cases} \\ &\le \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)}+(b-x)e^{-x\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta>0 \\ \left[(x-a)e^{-x\beta}+\frac{1}{-\beta}e^{-(b\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta<0; \end{cases}\end{aligned} \tag{26}$$

for any $x \in [a, b]$. If $\beta = 0$, then

$$\begin{aligned}&\left|\frac{f(x)}{e^{ix\gamma}}(b-a)-\int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\le (x-a)\|f'-i\gamma f\|_{[a,x],1}+(b-x)\|f'-i\gamma f\|_{[x,b],1} \\ &\le (b-a)\|f'-i\gamma f\|_{[a,b],1},\end{aligned} \tag{27}$$

for any $x \in [a, b]$.

Proof. Take the modulus of (11) and make use of the Hölder's inequality to obtain:

$$\begin{aligned}&\left|\frac{f(x)}{e^{\alpha x}}(b-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\le \sup_{t\in[a,x]}(t-a)e^{-t\beta}\|f'-\alpha f\|_{[a,x],1}+\sup_{t\in[x,b]}(b-t)e^{-t\beta}\|f'-\alpha f\|_{[x,b],1}.\end{aligned}$$

Now, we evaluate the supremum of the function $A(t) = (t - a)e^{-t\beta}$ and $B(t) = (b - t)e^{-t\beta}$. We have $A'(t) = e^{-\beta t}(1 - \beta(t - a))$. If $\beta < 0$, then $A'(t) \ge 0$ for all $t \in [a, x]$, thus, the supremum is attained at $t = x$. The stationary point of A is $t_a = (a\beta + 1)/\beta$. Furthermore, when $\beta > 0$, we have $A''(t_a) = -\beta e^{-(a\beta+1)} \le 0$; thus, the supremum is attained at $t = t_a$.

We have $B'(t) = -e^{-\beta t}(1+\beta(b-t))$. If $\beta > 0$, then $B'(t) \le 0$ for all $t \in [x,b]$, thus, the supremum is attained at $t = x$. The stationary point of B is $t_b = (b\beta+1)/\beta$. Furthermore, when $\beta < 0$, we have $A''(t_s) = \beta e^{-(b\beta+1)} \le 0$; thus, the supremum is attained at $t = t_b$.

Therefore,

$$\left|\frac{f(x)}{e^{\alpha x}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \le \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|f'-\alpha f\|_{[a,x],1} \\ \quad +(b-x)e^{-x\beta}\|f'-\alpha f\|_{[x,b],1}, & \beta > 0 \\ (x-a)e^{-x\beta}\|f'-\alpha f\|_{[a,x],1} \\ \quad +\frac{1}{-\beta}e^{-(b\beta+1)}\|f'-\alpha f\|_{[x,b],1}, & \beta < 0. \end{cases}$$

If $\beta = 0$, then

$$\begin{aligned} &\left|\frac{f(x)}{e^{ix\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt\right| \\ &\le \sup_{t\in[a,x]}(t-a)\|f'-i\gamma f\|_{[a,x],1} + \sup_{t\in[x,b]}(b-t)\|f'-i\gamma f\|_{[x,b],1} \\ &\le (x-a)\|f'-i\gamma f\|_{[a,x],1} + (b-x)\|f'-i\gamma f\|_{[x,b],1} \\ &\le (b-a)\|f'-i\gamma f\|_{[a,b],1}. \end{aligned}$$

This completes the proof. □

Remark 14. If $\alpha = 0$ in Theorem 13, then we have the Ostrowski inequality:

$$\begin{aligned} \left|f(x)(b-a) - \int_a^b f(t)\,dt\right| &\le (x-a)\|f'\|_{[a,x],1} + (b-x)\|f'\|_{[x,b],1} \\ &\le (b-a)\|f'\|_{[a,b],1}, \end{aligned} \tag{28}$$

for any $x \in [a,b]$.

Let $h(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h(t) = g(t)e^{\alpha t}$ in Theorem 13, then we have the Ostrowski inequalities: If $\beta \ne 0$, then

$$\begin{aligned} &\left|g(x)(b-a) - \int_a^b g(t)\,dt\right| \\ &\le \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|g'h\|_{[a,x],1} + (b-x)e^{-x\beta}\|g'h\|_{[x,b],1}, & \beta > 0 \\ (x-a)e^{-x\beta}\|g'h\|_{[a,x],1} + \frac{1}{-\beta}e^{-(b\beta+1)}\|g'h\|_{[x,b],1}, & \beta < 0. \end{cases} \\ &\le \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)} + (b-x)e^{-x\beta}\right]\|g'h\|_{[a,b],1}, & \beta > 0 \\ \left[(x-a)e^{-x\beta} + \frac{1}{-\beta}e^{-(b\beta+1)}\right]\|g'h\|_{[a,b],1}, & \beta < 0; \end{cases} \end{aligned} \tag{29}$$

for any $x \in [a,b]$. If $\beta = 0$, then

$$\begin{aligned} \left|g(x)(b-a) - \int_a^b g(t)\,dt\right| &\le (x-a)\|g'h\|_{[a,x],1} + (b-x)\|g'h\|_{[x,b],1} \\ &\le (b-a)\|g'h\|_{[a,b],1}, \end{aligned} \tag{30}$$

for any $x \in [a,b]$.

Corollary 15. *Under the assumptions of Theorem 13, we have the special cases as follows: If $\beta \neq 0$, then*

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{e^{\frac{\alpha(a+b)}{2}}}(b-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \tag{31}$$

$$\leq \begin{cases} \frac{1}{\beta} e^{-(a\beta+1)} \|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} + \frac{b-a}{2} e^{-\frac{a+b}{2}\beta} \|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta > 0 \\ \frac{b-a}{2} e^{-\frac{a+b}{2}\beta} \|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} + \frac{1}{-\beta} e^{-(b\beta+1)} \|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta < 0. \end{cases}$$

$$\leq \begin{cases} \left[\frac{1}{\beta} e^{-(a\beta+1)} + \frac{b-a}{2} e^{-\frac{a+b}{2}\beta}\right] \|f' - \alpha f\|_{[a,b],1}, & \beta > 0 \\ \left[\frac{b-a}{2} e^{-\frac{a+b}{2}\beta} + \frac{1}{-\beta} e^{-(b\beta+1)}\right] \|f' - \alpha f\|_{[a,b],1}, & \beta < 0. \end{cases}$$

If $\beta = 0$, then

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{e^{i\frac{a+b}{2}\gamma}}(b-a) - \int_a^b \frac{f(t)}{e^{it\gamma}}\,dt \right| \leq \frac{b-a}{2}\left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],1} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],1}\right]$$

$$\leq (b-a)\|f' - i\gamma f\|_{[a,b],1}. \tag{32}$$

3. Trapezoid type inequalities

The following trapezoid type inequalities with complex weights holds:

Theorem 16. *Let $f : [a,b] \to \mathbb{C}$ an absolutely continuous function on $[a,b]$ and $\alpha = \beta + i\gamma \in \mathbb{C}$.*

If $\beta \neq 0$, then we have

$$\left| \frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right|$$

$$\leq \frac{e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta}}{\beta^2}\|f' - \alpha f\|_{[a,x],\infty}$$

$$+ \frac{e^{-x\beta} - [(b-x)\beta + 1]e^{-b\beta}}{\beta^2}\|f' - \alpha f\|_{[x,b],\infty} \tag{33}$$

$$\leq \frac{1}{\beta^2}\left[2e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta} - [(b-x)\beta + 1]e^{-b\beta}\right]\|f' - \alpha f\|_{[a,b],\infty},$$

for any $x \in [a,b]$.

If $\beta = 0$, then we have

$$\left| \frac{f(b)}{e^{ix\gamma}}(b-x) + \frac{f(a)}{e^{ix\gamma}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right|$$

$$\leq \frac{1}{2}(x-a)^2\|f' - i\gamma f\|_{[a,x],\infty} + \frac{1}{2}(b-x)^2\|f' - i\gamma f\|_{[x,b],\infty} \tag{34}$$

$$\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\|f' - i\gamma f\|_{[a,b],\infty},$$

for any $x \in [a,b]$.

The constants $\frac{1}{2}$ and $\frac{1}{4}$ in (34) are sharp.

Proof. We have the trapezoid identity for absolutely continuous function $g : [a,b] \to \mathbb{C}$

$$g(b)(b-x) + g(a)(x-a) - \int_a^b g(t)\,dt = \int_a^b (t-x)g'(t)\,dt \tag{35}$$

where $x \in [a,b]$. If $g(t) = f(t)/e^{\alpha t}$, then $g'(t) = (f'(t) - \alpha f(t))/e^{\alpha t}$; and with this choice of g, (35) becomes:

$$\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt = \int_a^b (t-x)\frac{f'(t) - \alpha f(t)}{e^{\alpha t}}\,dt \tag{36}$$

Taking the modulus in (36) we get

$$\begin{aligned}
&\left|\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \int_a^b |t-x||e^{-\alpha t}||f'(t) - \alpha f(t)|\,dt \\
&= \int_a^x (x-t)|e^{-\alpha t}||f'(t) - \alpha f(t)|\,dt + \int_x^b (t-x)|e^{-\alpha t}||f'(t) - \alpha f(t)|\,dt \\
&\le \int_a^x (x-t)e^{-t\beta}\,dt\|f' - \alpha f\|_{[a,x],\infty} + \int_x^b (t-x)e^{-t\beta}\,dt\|f' - \alpha f\|_{[x,b],\infty}
\end{aligned}$$

We have,

$$\begin{aligned}
\int_a^x (x-t)e^{-t\beta}\,dt &= -\frac{x-t}{\beta}e^{-t\beta}\Big|_a^x - \frac{1}{\beta}\int_a^x e^{-t\beta}\,dt \\
&= \frac{x-a}{\beta}e^{-x\beta} + \frac{1}{\beta^2}\left(e^{-x\beta} - e^{-a\beta}\right) \\
&= \frac{e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta}}{\beta^2};
\end{aligned}$$

and

$$\begin{aligned}
\int_x^b (t-x)e^{-t\beta}\,dt &= -\frac{t-x}{\beta}e^{-t\beta}\Big|_x^b + \frac{1}{\beta}\int_x^b e^{-t\beta}\,dt \\
&= -\frac{b-x}{\beta}e^{-x\beta} - \frac{1}{\beta^2}\left(e^{-b\beta} - e^{-x\beta}\right) \\
&= \frac{e^{-x\beta} - [(b-x)\beta + 1]e^{-b\beta}}{\beta^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \int_a^x (x-t)e^{-t\beta}\,dt\|f' - \alpha f\|_{[a,x],\infty} + \int_x^b (t-x)e^{-t\beta}\,dt\|f' - \alpha f\|_{[x,b],\infty} \\
&= \frac{e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta}}{\beta^2}\|f' - \alpha f\|_{[a,x],\infty} \\
&\quad + \frac{e^{-x\beta} - [(b-x)\beta + 1]e^{-b\beta}}{\beta^2}\|f' - \alpha f\|_{[x,b],\infty} \\
&\le \frac{1}{\beta^2}\left[2e^{-x\beta} + [(x-a)\beta - 1]e^{-a\beta} - [(b-x)\beta + 1]e^{-b\beta}\right]\|f' - \alpha f\|_{[a,b],\infty},
\end{aligned}$$

and this proves (33).

If $\beta = 0$, then

$$\begin{aligned}
&\left|\frac{f(b)}{e^{ix\gamma}}(b-x)+\frac{f(a)}{e^{ix\gamma}}(x-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \int_a^x (x-t)\,dt\;\|f'-i\gamma f\|_{[a,x],\infty}+\int_x^b (t-x)\,dt\;\|f'-i\gamma f\|_{[x,b],\infty} \\
&= \frac{(x-a)^2}{2}\;\|f'-i\gamma f\|_{[a,x],\infty}+\frac{(b-x)^2}{2}\;\|f'-i\gamma f\|_{[x,b],\infty} \\
&\le \frac{(x-a)^2+(b-x)^2}{2}\;\|f'-i\gamma f\|_{[a,b],\infty} \\
&= \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\;\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned}$$

This completes the proof. The sharpness of the constants in (34) is given by Corollary 19. □

Remark 17. Note the similarity of the bounds in Theorems 7 and 16. The first set of upper bounds in (8) and (9) can be obtained by letting $a = x$, $x = b$ in the first term, and $x = a$, $b = x$ in the second term in (33) and (34), respectively.

Remark 18. If $\alpha = 0$ in Theorem 16, then we have a refinement for the trapezoid inequality:

$$\begin{aligned}
&\left|f(a)(x-a)+f(b)(b-x)-\int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2}(x-a)^2\|f'\|_{[a,x],\infty}+\frac{1}{2}(b-x)^2\|f'\|_{[x,b],\infty} \qquad (37)\\
&\le \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty},
\end{aligned}$$

for any $x \in [a,b]$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are sharp (cf. Corollary 19).

Let $h(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h(t) = g(t)e^{\alpha t}$ in Theorem 16, then we have the trapezoid inequalities: If $\beta \neq 0$, then we have

$$\begin{aligned}
\left|g(a)(x-a)+g(b)(b-x)-\int_a^b g(t)\,dt\right| &\le \frac{e^{-x\beta}+[(x-a)\beta-1]e^{-a\beta}}{\beta^2}\|g'h|_{[a,x],\infty} \\
&\quad +\frac{e^{-x\beta}-[(b-x)\beta+1]e^{-b\beta}}{\beta^2}\|g'h\|_{[x,b],\infty} \qquad (38)\\
&\le \frac{1}{\beta^2}\left[2e^{-x\beta}+[(x-a)\beta-1]e^{-a\beta}\right. \\
&\quad \left.-[(b-x)\beta+1]e^{-b\beta}\right]\|g'h\|_{[a,b],\infty},
\end{aligned}$$

for any $x \in [a,b]$. If $\beta = 0$, then we have

$$\begin{aligned}
&\left|g(a)(x-a)+g(b)(b-x)-\int_a^b g(t)\,dt\right| \\
&\le \frac{1}{2}(x-a)^2\|g'h\|_{[a,x],\infty}+\frac{1}{2}(b-x)^2\|g'h\|_{[x,b],\infty} \qquad (39)\\
&\le \left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|g'h\|_{[a,b],\infty},
\end{aligned}$$

for any $x \in [a,b]$.

Corollary 19. *Under the assumptions of Theorem 16, we have the special cases as follows:*

If $\beta \neq 0$, then we have

$$\begin{aligned}
&\left|(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2}-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leq \frac{e^{-\frac{a+b}{2}\beta}+[\frac{b-a}{2}\beta-1]e^{-a\beta}}{\beta^2}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],\infty} \\
&\quad+\frac{e^{-\frac{a+b}{2}\beta}-[\frac{b-a}{2}\beta+1]e^{-b\beta}}{\beta^2}\|f'-\alpha f\|_{[\frac{a+b}{2},b],\infty} \\
&\leq \frac{1}{\beta^2}\Big[2e^{-\frac{a+b}{2}\beta}-e^{-a\beta}-e^{-b\beta} \\
&\quad+\frac{b-a}{2}\beta\left(e^{-a\beta}-e^{-b\beta}\right)\Big]\|f'-\alpha f\|_{[a,b],\infty}.
\end{aligned} \tag{40}$$

If $\beta = 0$, then we have

$$\begin{aligned}
&\left|(b-a)e^{-i\frac{a+b}{2}\gamma}\frac{f(a)+f(b)}{2}-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leq \frac{1}{8}(b-a)^2\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],\infty}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right] \\
&\leq \frac{1}{4}(b-a)^2\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned} \tag{41}$$

The constant $\frac{1}{8}$ and $\frac{1}{4}$ in (41) are sharp.

Proof. We obtain (40) and (41) by taking $x = (a+b)/2$ in (33) and (34), respectively. To prove the sharpness of the constants in (41), we assume that the inequalities hold for $E, F > 0$ instead of $\frac{1}{8}$ and $\frac{1}{4}$, respectively:

$$\begin{aligned}
&\left|(b-a)e^{-i\frac{a+b}{2}\gamma}\frac{f(a)+f(b)}{2}-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leq E(b-a)^2\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],\infty}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],\infty}\right] \\
&\leq F(b-a)^2\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned}$$

Let $\gamma = 0$ and choose $f(x) = |x - \frac{a+b}{2}|$ on $[a, b]$, thus we have

$$\frac{(b-a)^2}{4} \leq 2E(b-a)^2 \leq F(b-a)^2,$$

which yields $E \geq \frac{1}{8}$ and $F \geq \frac{1}{4}$. □

We recall the notation $\Psi^+_{q,\alpha}$ and $\Psi^-_{q,\alpha}$ from Section 2 for the next results:

Theorem 20. *Let $f : [a, b] \to \mathbb{C}$ an absolutely continuous function on $[a, b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$ and $1 < p < \infty$. Let $q > 1$ be a real number such that $\frac{1}{p} + \frac{1}{q} = 1$.*

If $\beta \neq 0$, then

$$\begin{aligned}
&\left|\frac{f(b)}{e^{\alpha x}}(b-x)+\frac{f(a)}{e^{\alpha x}}(x-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\leq \Psi^-_{q,\alpha}(a,x)\|f'-\alpha f\|_{[a,x],p}+\Psi^+_{q,\alpha}(x,b)\|f'-\alpha f\|_{[x,b],p} \\
&\leq \left[\Psi^-_{q,\alpha}(a,x)+\Psi^+_{q,\alpha}(x,b)\right]\|f'-\alpha f\|_{[a,b],p},
\end{aligned} \tag{42}$$

for any $x \in [a, b]$.

If $\beta = 0$, then

$$\begin{aligned}
&\left| \frac{f(b)}{e^{ix\gamma}}(b-x) + \frac{f(a)}{e^{ix\gamma}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \\
&\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \left\| f' - i\gamma f \right\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \left\| f' - i\gamma f \right\|_{[x,b],p} \\
&\le \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \left\| f' - i\gamma f \right\|_{[a,b],\infty},
\end{aligned} \tag{43}$$

for any $x \in [a,b]$.

The inequalities in (43) *are sharp.*

Proof. Take the modulus of (36) and make use of the Hölder's inequality to obtain the following inequalities for $1 < p < \infty$ and its Hölder's conjugate q:

$$\begin{aligned}
&\left| \frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \\
&\le \int_a^b |t-x| |e^{-\alpha t}| |f'(t) - \alpha f(t)|\,dt \\
&= \int_a^x (x-t) e^{-t\beta} |f'(t) - \alpha f(t)|\,dt + \int_x^b (t-x) e^{-t\beta} |f'(t) - \alpha f(t)|\,dt \\
&\le \left(\int_a^x (x-t)^q e^{-tq\beta}\,dt \right)^{\frac{1}{q}} \left\| f' - \alpha f \right\|_{[a,x],p} \\
&\quad + \left(\int_x^b (t-x)^q e^{-tq\beta}\,dt \right)^{\frac{1}{q}} \left\| f' - \alpha f \right\|_{[x,b],p}.
\end{aligned}$$

We evaluate the integral:

$$\begin{aligned}
\int_a^x (x-t)^q e^{-tq\beta}\,dt &= e^{-xq\beta}(-q\beta)^{-q-1} \int_0^{q(-\beta)(a-x)} z^q e^{-z}\,dz \\
&= e^{-xq\beta}(-q\beta)^{-q-1} \left[\Gamma(q+1) - \Gamma(q+1, -q\beta(x-a)) \right]
\end{aligned}$$

by letting $z = -(x-t)q\beta$ and thus

$$\begin{aligned}
&\left(\int_a^x (x-t)^q e^{-tq\beta}\,dt \right)^{\frac{1}{q}} \\
&= e^{-x\beta}(-q\beta)^{-\frac{q+1}{q}} \left[\Gamma(q+1) - \Gamma(q+1, -q\beta(x-a)) \right]^{\frac{1}{q}} = \Psi^-_{q,\alpha}(a,x).
\end{aligned}$$

Now, we evaluate the integral:

$$\begin{aligned}
\int_x^b (t-x)^q e^{-tq\beta}\,dt &= e^{-xq\beta}(q\beta)^{-q-1} \int_0^{q\beta(b-x)} z^q e^{-z}\,dz \\
&= e^{-xq\beta}(q\beta)^{-q-1} \left[\Gamma(q+1) - \Gamma(q+1, q\beta(b-x)) \right]
\end{aligned}$$

by letting $z = (t-x)q\beta$ and thus

$$\begin{aligned}
&\left(\int_x^b (t-x)^q e^{-tq\beta}\,dt \right)^{\frac{1}{q}} \\
&= e^{-x\beta}(q\beta)^{-\frac{q+1}{q}} \left[\Gamma(q+1) - \Gamma(q+1, q\beta(b-x)) \right]^{\frac{1}{q}} = \Psi^+_{q,\alpha}(x,b).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{f(b)}{e^{\alpha x}}(b-x)+\frac{f(a)}{e^{\alpha x}}(x-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \left(\int_a^x (x-t)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[a,x],p} \\
&\quad + \left(\int_x^b (t-x)^q e^{-tq\beta}\,dt\right)^{\frac{1}{q}} \|f'-\alpha f\|_{[x,b],p} \\
&= \Psi^-_{q,\alpha}(a,x)\|f'-\alpha f\|_{[a,x],p}+\Psi^+_{q,\alpha}(x,b)\|f'-\alpha f\|_{[x,b],p} \\
&\le \left[\Psi^-_{q,\alpha}(a,x)+\Psi^+_{q,\alpha}(x,b)\right]\|f'-\alpha f\|_{[a,b],p}.
\end{aligned}$$

If $\beta=0$, then

$$\begin{aligned}
&\left|\frac{f(b)}{e^{ix\gamma}}(b-x)+\frac{f(a)}{e^{ix\gamma}}(x-a)-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\
&\le \left(\int_a^x (x-t)^q\,dt\right)^{\frac{1}{q}} \|f'-i\gamma f\|_{[a,x],\infty} \\
&\quad + \left(\int_x^b (t-x)^q\,dt\right)^{\frac{1}{q}} \|f'-i\gamma f\|_{[x,b],\infty} \\
&= \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'-i\gamma f\|_{[a,x],p}+\frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'-i\gamma f\|_{[x,b],p} \\
&\le \frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'-i\gamma f\|_{[a,b],\infty}.
\end{aligned}$$

This completes the proof. The sharpness of inequalities in (43) is given by Corollary 23. □

Remark 21. Note the similarity of the bounds in Theorems 10 and 20. The first set of upper bounds in (19) and (20) can be obtained by letting $a=x$, $x=b$ in the first term, and $x=a$, $b=x$ in the second term in (42) and (43), respectively.

Remark 22. If $\alpha=0$ in Theorem 20, then we have a refinement for the trapezoid inequality:

$$\begin{aligned}
&\left|f(a)(x-a)+f(b)(b-x)-\int_a^b f(t)\,dt\right| \\
&\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,x],p}+\frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[x,b],p} \\
&\le \frac{(x-a)^{\frac{q+1}{q}}+(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],\infty},
\end{aligned} \tag{44}$$

for any $x\in[a,b]$. The above inequalities are sharp (cf. Corollary 23).

Let $h(t)=e^{\alpha t}$ for $t\in[a,b]$. If $f(t)=g(t)h(t)=g(t)e^{\alpha t}$ in Theorem 20, then we have the trapezoid inequalities: If $\beta\neq 0$, then

$$\begin{aligned}
&\left|g(a)(x-a)+g(b)(b-x)-\int_a^b g(t)\,dt\right| \\
&\le \Psi^-_{q,\alpha}(a,x)\|g'h\|_{[a,x],p}+\Psi^+_{q,\alpha}(x,b)\|g'h\|_{[x,b],p} \\
&\le \left[\Psi^-_{q,\alpha}(a,x)+\Psi^+_{q,\alpha}(x,b)\right]\|g'h\|_{[a,b],p},
\end{aligned} \tag{45}$$

for any $x\in[a,b]$.

If $\beta = 0$, then

$$\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\,dt \right| \tag{46}$$
$$\le \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h\|_{[a,x],p} + \frac{(b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h|_{[x,b],p}$$
$$\le \frac{(x-a)^{\frac{q+1}{q}} + (b-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}} \|g'h\|_{[a,b],\infty},$$

for any $x \in [a,b]$.

Corollary 23. *Under the assumptions of Theorem 20, we have the special cases as follows:*

If $\beta \neq 0$, then

$$\left| (b-a)e^{-\frac{\alpha(a+b)}{2}} \frac{f(a)+f(b)}{2} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \tag{47}$$
$$\le \Psi^-_{q,\alpha}\left(a, \frac{a+b}{2}\right) \|f' - \alpha f\|_{[a,\frac{a+b}{2}],p} + \Psi^+_{q,\alpha}\left(\frac{a+b}{2}, b\right) \|f' - \alpha f\|_{[\frac{a+b}{2},b],p}$$
$$\le \left[\Psi^-_{q,\alpha}\left(a, \frac{a+b}{2}\right) + \Psi^+_{q,\alpha}\left(\frac{a+b}{2}, b\right)\right] \|f' - \alpha f\|_{[a,b],p}.$$

If $\beta = 0$, then

$$\left| (b-a)e^{-i\frac{a+b}{2}\gamma} \frac{f(a)+f(b)}{2} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right| \tag{48}$$
$$\le \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}} \left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],p} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],p} \right]$$
$$\le \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,b],\infty}.$$

The inequalities in (48) *are sharp.*

Proof. We obtain (47) and (48) by taking $x = (a+b)/2$ in (42) and (43), respectively. To prove the sharpness of the inequalities in (48), we assume that the inequalities hold for $G, H > 0$ instead of $\frac{1}{2}$ and 1, respectively:

$$\left| (b-a)e^{-i\frac{a+b}{2}\gamma} \frac{f(a)+f(b)}{2} - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt \right|$$
$$\le G\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \left[\|f' - i\gamma f\|_{[a,\frac{a+b}{2}],p} + \|f' - i\gamma f\|_{[\frac{a+b}{2},b],p} \right]$$
$$\le H\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \|f' - i\gamma f\|_{[a,b],\infty}.$$

Let $\gamma = 0$ and choose $f(x) = |x - \frac{a+b}{2}|$ on $[a,b]$, thus we have

$$\frac{(b-a)^2}{4} \le 2G\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}} \le H\frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}.$$

Let $q \to 1$, and now we have $\frac{(b-a)^2}{4} \le 2G\frac{(b-a)^2}{4} \le H\frac{(b-a)^2}{4}$ which yields $G \ge \frac{1}{2}$ and $H \ge 1$; and the proof is completed. □

The case for the 1-norm is as follows:

Theorem 24. *Let $f : [a,b] \to \mathbb{C}$ an absolutely continuous function on $[a,b]$, $\alpha = \beta + i\gamma \in \mathbb{C}$.*

If $\beta \neq 0$, then

$$\left|\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \tag{49}$$

$$\leq \begin{cases} (x-a)e^{-a\beta}\|f'-\alpha f\|_{[a,x],1} \\ \quad +\frac{1}{\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[x,b],1}, & \beta > 0 \\ \frac{1}{-\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[a,x],1} \\ \quad +(b-x)e^{-b\beta}\|f'-\alpha f\|_{[x,b],1}, & \beta < 0. \end{cases}$$

$$\leq \begin{cases} \left[(x-a)e^{-a\beta} + \frac{1}{\beta}e^{-(x\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta > 0 \\ \left[\frac{1}{-\beta}e^{-(x\beta+1)} + (b-x)e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta < 0, \end{cases}$$

for $x \in [a,b]$.

If $\beta = 0$, then

$$\begin{aligned} &\left|\frac{f(b)}{e^{ix\gamma}}(b-x) + \frac{f(a)}{e^{ix\gamma}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\leq (x-a)\|f'-i\gamma f\|_{[a,x],1} + (b-x)\|f'-i\gamma f\|_{[x,b],1} \\ &\leq (b-a)\|f'-i\gamma f\|_{[a,b],1}, \end{aligned} \tag{50}$$

for $x \in [a,b]$.

Proof. Take the modulus of (36) and make use of the Hölder's inequality to obtain:

$$\begin{aligned} &\left|\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\leq \int_a^b |t-x||e^{-\alpha t}||f'(t)-\alpha f(t)|\,dt \\ &\leq \sup_{t\in[a,x]}(x-t)e^{-t\beta}\|f'-\alpha f\|_{[a,x],1} + \sup_{t\in[x,b]}(t-x)e^{-t\beta}\|f'-\alpha f\|_{[x,b],p}. \end{aligned}$$

Now, we evaluate the supremum of the function $C(t) = (x-t)e^{-t\beta}$ and $D(t) = (t-x)e^{-t\beta}$.

We have $C'(t) = -e^{-\beta t}(1+\beta(x-t))$. If $\beta > 0$, then $C'(t) \leq 0$ for all $t \in [a,x]$, thus, the supremum is attained at $t = a$. The stationary point of C is $t_c = (x\beta+1)/\beta$. Furthermore, when $\beta < 0$, we have $C'(t_c) = \beta e^{-(x\beta+1)} \leq 0$; thus the supremum is attained at $t = t_c$.

We have $D'(t) = e^{-\beta t}(1+\beta(t-x))$. If $\beta < 0$, then $D'(t) \geq 0$ for all $t \in [x,b]$, thus, the supremum is attained at $t = b$. The stationary point of D is $t_d = (x\beta+1)/\beta$. Furthermore, when $\beta > 0$, we have $D''(t_d) = -\beta e^{-(x\beta+1)} \leq 0$; thus the supremum is attained at $t = t_d$.

Therefore,

$$\left|\frac{f(b)}{e^{\alpha x}}(b-x) + \frac{f(a)}{e^{\alpha x}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right|$$

$$\leq \begin{cases} (x-a)e^{-a\beta}\|f'-\alpha f\|_{[a,x],1} \\ \quad +\frac{1}{\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[x,b],1}, & \beta > 0 \\ \frac{1}{-\beta}e^{-(x\beta+1)}\|f'-\alpha f\|_{[a,x],1} \\ \quad +(b-x)e^{-b\beta}\|f'-\alpha f\|_{[x,b],1}, & \beta < 0. \end{cases}$$

If $\beta = 0$, then

$$\begin{aligned}
&\left| \frac{f(b)}{e^{ix\gamma}}(b-x) + \frac{f(a)}{e^{ix\gamma}}(x-a) - \int_a^b \frac{f(t)}{e^{\alpha t}}\, dt \right| \\
&\le \sup_{t\in[a,x]} (x-t)\|f' - i\gamma f\|_{[a,x],1} + \sup_{t\in[x,b]} (t-x)\|f' - i\gamma f\|_{[x,b],1} \\
&\le (x-a)\|f' - i\gamma f\|_{[a,x],1} + (b-x)\|f' - i\gamma f\|_{[x,b],1} \\
&\le (b-a)\|f' - i\gamma f\|_{[a,b],1}.
\end{aligned}$$

This completes the proof. □

Remark 25. Note the similarity of the bounds in Theorems 13 and 24. The first set of upper bounds in (26) and (27) can be obtained by letting $a = x$, $x = b$ in the first term, and $x = a$, $b = x$ in the second term in (49) and (55), respectively.

Remark 26. If $\alpha = 0$ in Theorem 24, then we have the trapezoid inequality:

$$\begin{aligned}
&\left| f(a)(x-a) + f(b)(b-x) - \int_a^b f(t)\, dt \right| \\
&\le (x-a)\|f'\|_{[a,x],1} + (b-x)\|f'\|_{[x,b],1} \\
&\le (b-a)\|f'\|_{[a,b],1},
\end{aligned} \tag{51}$$

for any $x \in [a,b]$.

Let $h(t) = e^{\alpha t}$ for $t \in [a,b]$. If $f(t) = g(t)h(t) = g(t)e^{\alpha t}$ in Theorem 24, then we have the trapezoid inequalities: If $\beta \neq 0$, then

$$\begin{aligned}
&\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\, dt \right| \\
&\le \begin{cases} (x-a)e^{-a\beta}\|g'h\|_{[a,x],1} + \frac{1}{\beta}e^{-(x\beta+1)}\|g'h\|_{[x,b],1}, & \beta > 0 \\ \frac{1}{-\beta}e^{-(x\beta+1)}\|g'h\|_{[a,x],1} + (b-x)e^{-b\beta}\|g'h\|_{[x,b],1}, & \beta < 0. \end{cases} \\
&\le \begin{cases} \left[(x-a)e^{-a\beta} + \frac{1}{\beta}e^{-(x\beta+1)}\right]\|g'h\|_{[a,b],1}, & \beta > 0 \\ \\ \left[\frac{1}{-\beta}e^{-(x\beta+1)} + (b-x)e^{-b\beta}\right]\|g'h\|_{[a,b],1}, & \beta < 0, \end{cases}
\end{aligned} \tag{52}$$

for $x \in [a,b]$. If $\beta = 0$, then

$$\begin{aligned}
&\left| g(a)(x-a) + g(b)(b-x) - \int_a^b g(t)\, dt \right| \\
&\le (x-a)\|g'h\|_{[a,x],1} + (b-x)\|g'h\|_{[x,b],1} \\
&\le (b-a)\|g'h\|_{[a,b],1},
\end{aligned} \tag{53}$$

for $x \in [a,b]$.

Corollary 27. *Under the assumptions of Theorem 24, we have the special cases as follows:*

If $\beta \neq 0$, then

$$\left|(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2}-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \tag{54}$$
$$\leq \begin{cases} \frac{b-a}{2}e^{-a\beta}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad +\frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta>0 \\ \frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f'-\alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad +\frac{b-a}{2}e^{-b\beta}\|f'-\alpha f\|_{[\frac{a+b}{2},b],1}, & \beta<0. \end{cases}$$
$$\leq \begin{cases} \left[\frac{b-a}{2}e^{-a\beta}+\frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\right]\|f'-\alpha f\|_{[a,b],1}, x & \beta>0 \\ \left[\frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)}+\frac{b-a}{2}e^{-b\beta}\right]\|f'-\alpha f\|_{[a,b],1}, & \beta<0, \end{cases}$$

If $\beta = 0$, then

$$\begin{aligned} &\left|(b-a)e^{-i\frac{a+b}{2}\gamma}\frac{f(a)+f(b)}{2}-\int_a^b \frac{f(t)}{e^{\alpha t}}\,dt\right| \\ &\leq \frac{b-a}{2}\left[\|f'-i\gamma f\|_{[a,\frac{a+b}{2}],1}+\|f'-i\gamma f\|_{[\frac{a+b}{2},b],1}\right] \\ &\leq (b-a)\|f'-i\gamma f\|_{[a,b],1}. \end{aligned} \tag{55}$$

4. Applications to approximations of Fourier and Laplace Integral Transform

The Fourier transform has been a principal analytical tool in many fields of research, such as probability theory, quantum physics and boundary-value problems [3]. The approximations of the finite Fourier transform of different classes of functions have been considered by employing integral inequalities of Ostrowski type. We refer to Barnett and Dragomir [4] for the approximations of the Fourier transform of absolutely continuous functions; to Barnett, Dragomir and Hanna [5] for functions of bounded variation; and to Dragomir, Cho and Kim [13] for Lebesgue integrable mappings. Using a pre-Grüss type inequality, Dragomir, Hanna and Roumeliotis [14] obtained some approximations of the finite Fourier transform for complex-valued functions.

In this section, we provide the applications of the inequalities obtained in Sections 2 and 3 to give approximations of the finite Fourier transform. We also apply these inequalities to obtain approximations of the finite Laplace transform.

Let $f : [a,b] \to \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) be a Lebesgue integrable mapping defined on the finite interval $[a,b]$. Let $\mathcal{L}(f)$ and $\mathcal{F}(f)$ be their finite Laplace and Fourier transforms, respectively, defined by

$$\begin{aligned} \mathcal{L}(f)(\alpha) &:= \int_a^b f(s)e^{-\alpha s}\,ds, \quad \alpha \in \mathbb{C}, \\ \mathcal{F}(f)(t) &:= \int_a^b f(s)e^{-2\pi its}\,ds, \quad t \in \mathbb{R}. \end{aligned}$$

4.1. **Laplace transform.** If $\beta \neq 0$, then by Corollary 9, we have the approximation of $\mathcal{L}(f)(\alpha)$ by $f(\frac{a+b}{2})(b-a)e^{-\frac{\alpha(a+b)}{2}}$, with the following error bounds:

$$\begin{aligned}
&\left|\frac{f(\frac{a+b}{2})(b-a)}{e^{\frac{\alpha(a+b)}{2}}} - \mathcal{L}(f)(\alpha)\right| \\
&\le \frac{e^{-a\beta} - [\frac{b-a}{2}\beta + 1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],\infty} \\
&\quad + \frac{e^{-b\beta} + [\frac{b-a}{2}\beta - 1]e^{-\frac{a+b}{2}\beta}}{\beta^2}\|f' - \alpha f\|_{[\frac{a+b}{2},b],\infty} \\
&\le \frac{1}{\beta^2}\left[e^{-a\beta} + e^{-b\beta} - 2e^{-\frac{a+b}{2}\beta}\right]\|f' - \alpha f\|_{[a,b],\infty}.
\end{aligned} \tag{56}$$

If $\beta \neq 0$, then by Corollary 12, we have the approximation of $\mathcal{L}(f)(\alpha)$ by $f(\frac{a+b}{2})(b-a)e^{-\frac{\alpha(a+b)}{2}}$, with the error bounds:

$$\begin{aligned}
&\left|\frac{f(\frac{a+b}{2})(b-a)}{e^{\frac{\alpha(a+b)}{2}}} - \mathcal{L}(f)(\alpha)\right| \qquad (57)\\
&\le \Psi^+_{q,\alpha}\left(a, \frac{a+b}{2}\right)\|f' - \alpha f\|_{[a,\frac{a+b}{2}],p} + \Psi^-_{q,\alpha}\left(\frac{a+b}{2}, b\right)\|f' - \alpha f\|_{[\frac{a+b}{2},b],p} \\
&\le \left[\Psi^+_{q,\alpha}\left(a, \frac{a+b}{2}\right) + \Psi^-_{q,\alpha}\left(\frac{a+b}{2}, b\right)\right]\|f' - \alpha f\|_{[a,b],p}.
\end{aligned}$$

If $\beta \neq 0$, then by Corollary 15, then we have the approximation of $\mathcal{L}(f)(\alpha)$ by $f(\frac{a+b}{2})(b-a)e^{-\frac{\alpha(a+b)}{2}}$, with the error bounds:

$$\begin{aligned}
&\left|\frac{f(\frac{a+b}{2})(b-a)}{e^{\frac{\alpha(a+b)}{2}}} - \mathcal{L}(f)(\alpha)\right| \qquad (58)\\
&\le \begin{cases} \frac{1}{\beta}e^{-(a\beta+1)}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad + \frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta > 0 \\ \frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad + \frac{1}{-\beta}e^{-(b\beta+1)}\|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta < 0. \end{cases} \\
&\le \begin{cases} \left[\frac{1}{\beta}e^{-(a\beta+1)} + \frac{b-a}{2}e^{-\frac{a+b}{2}\beta}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta > 0 \\ \left[\frac{b-a}{2}e^{-\frac{a+b}{2}\beta} + \frac{1}{-\beta}e^{-(b\beta+1)}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta < 0. \end{cases}
\end{aligned}$$

If $\beta \neq 0$, then by Corollary 19, then we have the approximation of $\mathcal{L}(f)(\alpha)$ by $(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$\begin{aligned}
&\left|(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2} - \mathcal{L}(f)(\alpha)\right| \\
&\le \frac{e^{-\frac{a+b}{2}\beta} + [\frac{b-a}{2}\beta - 1]e^{-a\beta}}{\beta^2}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],\infty} \\
&\quad + \frac{e^{-\frac{a+b}{2}\beta} - [\frac{b-a}{2}\beta + 1]e^{-b\beta}}{\beta^2}\|f' - \alpha f\|_{[\frac{a+b}{2},b],\infty} \\
&\le \frac{1}{\beta^2}\Big[2e^{-\frac{a+b}{2}\beta} - e^{-a\beta} - e^{-b\beta} \\
&\quad + \frac{b-a}{2}\beta\left(e^{-a\beta} - e^{-b\beta}\right)\Big]\|f' - \alpha f\|_{[a,b],\infty}.
\end{aligned} \tag{59}$$

If $\beta \neq 0$, then by Corollary 23, then we have the approximation of $\mathcal{L}(f)(\alpha)$ by $(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$
\begin{aligned}
(60)\quad & \left|(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2} - \mathcal{L}(f)(\alpha)\right| \\
& \leq \Psi^{-}_{q,\alpha}\left(a, \frac{a+b}{2}\right)\|f' - \alpha f\|_{[a,\frac{a+b}{2}],p} + \Psi^{+}_{q,\alpha}\left(\frac{a+b}{2}, b\right)\|f' - \alpha f\|_{[\frac{a+b}{2},b],p} \\
& \leq \left[\Psi^{-}_{q,\alpha}\left(a, \frac{a+b}{2}\right) + \Psi^{+}_{q,\alpha}\left(\frac{a+b}{2}, b\right)\right]\|f' - \alpha f\|_{[a,b],p}.
\end{aligned}
$$

If $\beta \neq 0$, then by Corollary 27, then we have the approximation of $\mathcal{L}(f)(\alpha)$ by $(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$
\begin{aligned}
(61)\quad & \left|(b-a)e^{-\frac{\alpha(a+b)}{2}}\frac{f(a)+f(b)}{2} - \mathcal{L}(f)(\alpha)\right| \\
& \leq \begin{cases} \frac{b-a}{2}e^{-a\beta}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad +\frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta > 0 \\ \frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)}\|f' - \alpha f\|_{[a,\frac{a+b}{2}],1} \\ \quad +\frac{b-a}{2}e^{-b\beta}\|f' - \alpha f\|_{[\frac{a+b}{2},b],1}, & \beta < 0. \end{cases} \\
& \leq \begin{cases} \left[\frac{b-a}{2}e^{-a\beta} + \frac{1}{\beta}e^{-(\frac{a+b}{2}\beta+1)}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta > 0 \\ \left[\frac{1}{-\beta}e^{-(\frac{a+b}{2}\beta+1)} + \frac{b-a}{2}e^{-b\beta}\right]\|f' - \alpha f\|_{[a,b],1}, & \beta < 0. \end{cases}
\end{aligned}
$$

4.2. **Fourier transform.** Let $\gamma = 2\pi t$ in Corollary 9. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $f(\frac{a+b}{2})(b-a)e^{-i\pi(a+b)t}$, with the following error bounds:

$$
\begin{aligned}
& \left|\frac{f(\frac{a+b}{2})(b-a)}{e^{i\pi(a+b)t}} - \mathcal{F}(f)(t)\right| \\
(62)\quad & \leq \frac{1}{8}(b-a)^2\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],\infty} + \|f' - 2\pi itf\|_{[\frac{a+b}{2},b],\infty}\right] \\
& \leq \frac{1}{4}(b-a)^2\|f' - 2\pi itf\|_{[a,b],\infty}.
\end{aligned}
$$

Let $\gamma = 2\pi t$ in Corollary 12. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $f(\frac{a+b}{2})(b-a)e^{-i\pi(a+b)t}$, with the following error bounds:

$$
\begin{aligned}
& \left|\frac{f(\frac{a+b}{2})(b-a)}{e^{i\pi(a+b)t}} - \mathcal{F}(f)(t)\right| \\
(63)\quad & \leq \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}}\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],p} + \|f' - 2\pi itf\|_{[\frac{a+b}{2},b],p}\right] \\
& \leq \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\|f' - 2\pi itf\|_{[a,b],p}.
\end{aligned}
$$

Let $\gamma = 2\pi t$ in Corollary 15. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $f(\frac{a+b}{2})(b-a)e^{-i\pi(a+b)t}$, with the following error bounds:

$$
\begin{aligned}
& \left|\frac{f(\frac{a+b}{2})(b-a)}{e^{i\pi(a+b)t}} - \mathcal{F}(f)(t)\right| \\
(64)\quad & \leq \frac{b-a}{2}\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],1}\|f' - 2\pi itf\|_{[\frac{a+b}{2},b],1}\right] \\
& \leq (b-a)\|f' - 2\pi itf\|_{[a,b],1}.
\end{aligned}
$$

Let $\gamma = 2\pi t$ in Corollary 19. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$\begin{aligned}
&\left|(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2} - \mathcal{F}(f)(t)\right| \\
&\leq \frac{1}{8}(b-a)^2\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],\infty} + \|f' - 2\pi itf\|_{[\frac{a+b}{2},b],\infty}\right] \\
&\leq \frac{1}{4}(b-a)^2\|f' - 2\pi itf\|_{[a,b],\infty}.
\end{aligned} \tag{65}$$

Let $\gamma = 2\pi t$ in Corollary 23. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$\begin{aligned}
&\left|(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2} - \mathcal{F}(f)(t)\right| \\
&\leq \frac{(b-a)^{\frac{q+1}{q}}}{2(2q+2)^{\frac{1}{q}}}\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],p} + \|f' - 2\pi itf\|_{[\frac{a+b}{2},b],p}\right] \\
&\leq \frac{(b-a)^{\frac{q+1}{q}}}{(2q+2)^{\frac{1}{q}}}\|f' - 2\pi itf\|_{[a,b],p}.
\end{aligned} \tag{66}$$

Let $\gamma = 2\pi t$ in Corollary 27. If $\beta = 0$, then we have the approximation of $\mathcal{F}(f)(t)$ by $(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2}$, with the following error bounds:

$$\begin{aligned}
&\left|(b-a)e^{-i\pi(a+b)t}\frac{f(a)+f(b)}{2} - \mathcal{F}(f)(t)\right| \\
&\leq \frac{b-a}{2}\left[\|f' - 2\pi itf\|_{[a,\frac{a+b}{2}],1} + \|f' - 2\pi itf\|_{[\frac{a+b}{2},b],1}\right] \\
&\leq (b-a)\|f' - 2\pi itf\|_{[a,b],1}.
\end{aligned} \tag{67}$$

References

1. A. M. Acu and F. D. Sofonea, *On an inequality of Ostrowski type*, J. Sci. Arts, No. 3(16) (2011), 281–287.
2. A. M. Acu, A. Baboş and F. D. Sofonea, *The mean value theorems and inequalities of Ostrowski type*, Sci. Stud. Res. Ser. Math. Inform., **21**, No. 1 (2011), 5–16.
3. E. O. Brigham, *The Fast Fourier Transform and its Applications*, Englewood Cliffs, NJ: Prentice-Hall Inc., 1988.
4. N. S. Barnett and S.S. Dragomir, *An approximation for the Fourier transform of absolutely continuous mappings*, Proc. 4th Int. Conf. on Modelling and Simulation, Victoria University, Melbourne, 2002, 351–355. RGMIA Res. Rep. Coll., **5**, (2002), Supplement, Article 33. [Online: http://rgmia.org/papers/v5e/FourierTransform.pdf]
5. N. S. Barnett, S.S. Dragomir and G. Hanna, *Error estimates for approximating the Fourier transform of functions of bounded variation*, RGMIA Res. Rep. Coll., **7**, No. 1 (2004), Article 11. [Online: http://rgmia.org/papers/v7n1/FoTransform.pdf]
6. P. Cerone, *On relationships between Ostrowski, trapezoidal and Chebychev identities and inequalities*, Soochow J. Math., **28**, No. 3 (2002), 311–328.
7. P. Cerone and S. S. Dragomir, *Trapezoidal-type rules from an inequalities point of view, Handbook of analytic-computational methods in applied mathematics*, Chapman & Hall/CRC, Boca Raton, FL, 2000, 65–134.
8. P. Cerone, S. S. Dragomir and E. Kikianty, *Ostrowski and Trapezoid Type Inequalities Related to Pompeiu's Mean Value Theorem*, RGMIA Res. Rep. Coll., **17** (2014), No. 58 [Online: http://rgmia.org/papers/v17/v17a58.pdf]
9. S. S. Dragomir, *An inequality of Ostrowski type via Pompeiu's mean value theorem*, J. Inequal. Pure Appl. Math., **6**, No. 3 (2005), Article 83, 9 pp.
10. S. S. Dragomir, *Another Ostrowski type inequality via Pompeiu's mean value theorem*, RGMIA Res. Rep. Coll. **16** (2013), No. 64 [Online: http://rgmia.org/papers/v16/v16a64.pdf]
11. S. S. Dragomir, *Inequalities of Pompeiu's type for absolutely continuous functions with applications to Ostrowski's inequality*, RGMIA Res. Rep. Coll. **16** (2013), No. 66 [Online: http://rgmia.org/papers/v16/v16a66.pdf]

12. S. S. Dragomir, *Exponential Pompeiu's type inequalities with applications to Ostrowski's inequality* RGMIA Res. Rep. Coll., **16**, No. 70 (2013) [Online: http://rgmia.org/papers/v16/v16a70.pdf]
13. S. S. Dragomir, Y. J. Cho and S. S. Kim, *An approximation for the Fourier transform of Lebesgue integrable mappings*, Fixed point Theory and Applications, **4**, Y. J. Cho, J. K. Kim, and S. M. Kong (Eds.), Nova Science Publishers Inc., 2003, pp. 67–76.
14. S. S. Dragomir, G. Hanna and J. Roumeliotis, *A reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality for complex-valued functions and applications for Fourier transform* Bull. Korean Math. Soc., **42**, No. 4 (2005), 725–738.
15. S. S. Dragomir and T. M. Rassias, *Generalisations of the Ostrowski inequality and applications* (English summary) Ostrowski type inequalities and applications in numerical integration, 1–63, Kluwer Acad. Publ., Dordrecht, 2002.
16. D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives* Mathematics and its Applications (East European Series), 53. Kluwer Academic Publishers Group, Dordrecht, 1991.
17. A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Helv., **10** (1938), 226–227.
18. J. Pečarić and Š. Ungar, *On an inequality of Ostrowski type*, J. Ineq. Pure Appl. Math., **7**, No. 4 (2006), Art. 151.
19. D. Pompeiu, *Sur une proposition analogue au théorème des accroissements finis*, Mathematica, Timişoara, **22** (1946), 143–146.
20. E. C. Popa, *An inequality of Ostrowski type via a mean value theorem*, Gen. Math., **15**, No. 1 (2007), 93–100.
21. P. K. Sahoo and T. Riedel, *Mean Value Theorems and Functional Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, 2000.

[1]Department of Mathematics and Statistics, La Trobe University, Bundoora 3086, Australia, E-mail: p.cerone@latrobe.edu.au

[2]School of Engineering and Science, Victoria University, PO Box 14428, Melbourne 8001, Victoria, Australia, E-mail: sever.dragomir@vu.edu.au
[3]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa

[4]Department of Pure and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park 2006, South Africa
E-mail: eder.kikianty@gmail.com; ekikianty@uj.ac.za
* Corresponding Author