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Some Trace Inequalities of Čebyšev Type for Functions of Operators in Hilbert Spaces

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Abstract. Some trace operator inequalities for synchronous functions that are related to the Čebyšev inequality for sequences of real numbers are given.

1. Introduction

For $\mathbf{p} = (p_1, ..., p_n)$, $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ n-tuples of real numbers, consider the *Čebyšev functional*

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := P_n \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i, \tag{1.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

In 1882-1883, Čebyšev [**4**] and [**5**] proved that, if $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the same (opposite) sense and $\mathbf{p}$ is nonnegative, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq (\leq)\, 0. \tag{1.2}$$

The inequality (1.2) was mentioned by Hardy, Littlewood and Polya in their book [**16**] in 1934 in the more general setting of *synchronous sequences*, i.e., if $\mathbf{a}, \mathbf{b}$ are synchronous (asynchronous), this means that

$$(a_i - a_j)(b_i - b_j) \geq (\leq)\, 0 \text{ for each } i, j \in \{1, ..., n\}, \tag{1.3}$$

then (1.2) holds true.

For general real weights $\mathbf{p}$, Mitrinović and Pečarić has shown in [**21**] that the inequality (1.2) holds true if

$$0 \leq P_k \leq P_n \text{ for } k \in \{1, ..., n-1\}, \tag{1.4}$$

and $\mathbf{a}, \mathbf{b}$ are monotonic in the same (opposite) sense.

We say that the functions f, $g : [a, b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a, b]$ if they satisfy the following condition:

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \text{ for each } t, s \in [a, b].$$

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It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a, b]$, then they are synchronous on $[a, b]$ while if they have opposite monotonicity, they are asynchronous.

For some extensions of the discrete *Čebyšev inequality* for *synchronous (asynchronous)* sequences of vectors in an inner product space, see [**12**] and [**13**].

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\operatorname{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\operatorname{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

For any $f, g \in C(\operatorname{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f) \Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \operatorname{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \operatorname{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(\operatorname{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\operatorname{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \operatorname{Sp}(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\operatorname{Sp}(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \text{ for any } t \in \operatorname{Sp}(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$.

The following result provides an inequality of Čebyšev type for functions of one selfadjoint operator:

Let A be a selfadjoint operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with $\operatorname{Sp}(A) \subseteq [m, M]$ for some real numbers $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and synchronous (asynchronous) on $[m, M]$, then [**9**]

$$\langle f(A) g(A) x, x \rangle \geq (\leq) \langle f(A) x, x \rangle \langle g(A) x, x \rangle \tag{1.5}$$

for any $x \in H$ with $\|x\| = 1$.

As a particular case of interest we notice that if A is a positive selfadjoint operator on H, then

$$\left\langle A^{p+q} x, x \right\rangle \geq \langle A^p x, x \rangle \langle A^q x, x \rangle \tag{1.6}$$

for any $x \in H$ with $\|x\| = 1$ and $p, q > 0$.

It is known, see for instance [**22**, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator S on H and two bounded functions φ and ψ such that $A = \varphi(S)$ and $B = \psi(S)$. Moreover, if $\{E_\lambda\}$ is the spectral family over the closed interval $[0, 1]$ for the selfadjoint operator S, then $S = \int_{0-}^{1} \lambda dE_\lambda$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_\lambda\}$ on $[0, 1]$ and

$$A = \varphi(S) = \int_{0-}^{1} \varphi(\lambda) dE_\lambda \text{ and } B = \psi(S) = \int_{0-}^{1} \psi(\lambda) dE_\lambda. \tag{1.7}$$

Now, if A and B are as above with $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \to \mathbb{C}$ we have the representations

$$f(A) = \int_{0-}^{1} (f \circ \varphi)(\lambda)\, dE_\lambda \text{ and } g(B) = \int_{0-}^{1} (g \circ \psi)(\lambda)\, dE_\lambda. \tag{1.8}$$

Definition 1. *We say that the continuous functions $f, g : J \to \mathbb{R}$ are operator synchronous (asynchronous) on J, if for any A and B two commuting bounded selfadjoint operators on the complex Hilbert space H with $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subseteq J$ we have*

$$(f(A) - f(B))(g(A) - g(B)) \geq (\leq)\, 0 \tag{1.9}$$

in the operator order.

In what follows, unless specified, H will be a complex Hilbert space.

In [**10**] we proved the following basic result:

Theorem 1. *The continuous functions $f, g : J \to \mathbb{R}$ are synchronous (asynchronous) on J if and only if they are operator synchronous (asynchronous) on J.*

The case of monotonic functions is as follows:

Corollary 1. *If the continuous functions $f, g : J \to \mathbb{R}$ have the same monotonicity on J, then for any A and B two commuting bounded selfadjoint operators on the Hilbert space H with $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subseteq J$ we have*

$$f(A)\, g(A) + f(B)\, g(B) \geq g(A)\, f(B) + f(A)\, g(B) \tag{1.10}$$

in the operator order.

Remark 1. *We observe that the above inequality (1.10) can provide numerous inequalities of interest for two commuting selfadjoint operators.*

For instance, if A and B are positive commuting operators on H then for any $p, q > 0$ we have

$$A^{p+q} + B^{p+q} \geq B^p A^q + A^p B^q. \tag{1.11}$$

If the commuting operators A and B are positive definite on H, then also

$$A \ln(A) + B \ln(B) \geq B \ln(A) + A \ln(B).$$

Also, if A and B are commuting operators on H with $0 \leq A, B \leq \frac{\pi}{2} 1_H$, then

$$\sin(A)\cos(A) + \sin(B)\cos(B) \leq \sin(B)\cos(A) + \sin(A)\cos(B). \tag{1.12}$$

In order to obtain some similar results for trace of operators in Hilbert spaces we need some preliminary facts as follows.

2. Some Facts on Trace of Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 2. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \ \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

PROPOSITION 1. *If $A \in \mathcal{B}(H)$,then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;
(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

THEOREM 3. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{2.8}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \|B\| \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

THEOREM 4. *We have:*
(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [**23**]

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{2.12}$$

where $\alpha \in (0,1)$ and $A,\, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha},\, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq \left[\operatorname{tr}\left(|A|^{2}\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^{2}\right)\right]^{1/2} \tag{2.13}$$

with $A,\, B \in \mathcal{B}_2(H)$.

If $A \geq 0$ and $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

$$0 \leq \operatorname{tr}(PA) \leq \|A\| \operatorname{tr}(P). \tag{2.14}$$

Indeed, since $A \geq 0$, then $\langle Ax, x\rangle \geq 0$ for any $x \in H$. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, then

$$0 \leq \left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \leq \|A\| \left\|P^{1/2}e_i\right\|^2 = \|A\| \langle Pe_i, e_i\rangle$$

for any $i \in I$. Summing over $i \in I$ we get

$$0 \leq \sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle \leq \|A\| \sum_{i\in I}\langle Pe_i, e_i\rangle = \|A\| \operatorname{tr}(P)$$

and since

$$\sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle = \sum_{i\in I}\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle = \operatorname{tr}\left(P^{1/2}AP^{1/2}\right) = \operatorname{tr}(PA)$$

we obtain the desired result (2.14).

This obviously imply the fact that, if A and B are selfadjoint operators with $A \leq B$ and $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then

$$\operatorname{tr}(PA) \leq \operatorname{tr}(PB). \tag{2.15}$$

Now, if A is a selfadjoint operator, then we know that

$$|\langle Ax, x\rangle| \leq \langle |A|\, x, x\rangle \text{ for any } x \in H.$$

This inequality follows by Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$\begin{aligned} |\operatorname{tr}(PA)| &= \left|\sum_{i\in I}\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \leq \sum_{i\in I}\left|\left\langle AP^{1/2}e_i, P^{1/2}e_i\right\rangle\right| \\ &\leq \sum_{i\in I}\left\langle |A|\, P^{1/2}e_i, P^{1/2}e_i\right\rangle = \operatorname{tr}(P\,|A|), \end{aligned} \tag{2.16}$$

for any A a selfadjoint operator and $P \in \mathcal{B}_1(H)$ with $P \geq 0$.

For the theory of trace functionals and their applications the reader is referred to [**26**].

For some classical trace inequalities see [**6**], [**8**], [**20**] and [**30**], which are continuations of the work of Bellman [**2**]. For related works the reader can refer to [**1**], [**3**], [**6**], [**15**], [**17**], [**18**], [**19**], [**24**] and [**27**].

3. Trace Inequalities for Synchronous Functions

We start with the following simple result:

Proposition 2. *Let A and B be two commuting bounded selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on J. If $P \in \mathcal{B}_1(H)$ with $P \geq 0$, then*

$$\operatorname{tr}[Pf(A)g(A)] + \operatorname{tr}[Pf(B)g(B)] \geq \operatorname{tr}[Pg(A)f(B)] + \operatorname{tr}[Pf(A)g(B)]. \tag{3.1}$$

The proof follows from the inequality (1.10) for synchronous functions and the property (2.15) for the trace functional.

Theorem 5. *Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$ and assume that the continuous functions $f, g : J \to \mathbb{R}$ are synchronous on J. If $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$, then*

$$\begin{aligned} &\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} \\ &\geq \left(\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\right)\left(g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\right). \end{aligned} \tag{3.2}$$

Proof. If we write the inequality (3.1) for $B = \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H$, then we get

$$\begin{aligned} &\operatorname{tr}[Pf(A)g(A)] + f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}(P) \\ &\geq f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}[Pg(A)] + g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}[Pf(A)], \end{aligned}$$

i.e.

$$\begin{aligned} &\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} + f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\geq f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} + g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}, \end{aligned}$$

or, by subtracting in both sides the quantity $\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}$,

$$\begin{aligned} &\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} \\ &\geq f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} + g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\ &- f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} &f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} + g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\ &- f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} \\ &= \left(\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\right)\left(g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\right), \end{aligned}$$

then we get from (3.3) the desired result (3.2). □

Corollary 2. *With the assumptions of Theorem 5 and if one of the functions f and g is convex while the other is concave, then we have*

$$\begin{aligned}(3.4)\qquad & \frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}-\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} \\ & \geq\left(\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\right)\left(g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)-\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\right) \\ & \geq 0.\end{aligned}$$

Proof. If f is convex and g is concave, then by Jensen's inequality for trace [**11**] we have

$$\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\geq f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)$$

and

$$g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\geq\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}$$

and the last inequality in (3.4) is proved. □

The following result also holds:

Theorem 6. *Let A and B be two selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subseteq J$ and assume that the continuous functions $f,g:J\to\mathbb{R}$ are synchronous on J. If $P,Q\in\mathcal{B}_1(H)\setminus\{0\}$ with $P,Q\geq 0$, then*

$$\begin{aligned}(3.5)\qquad & \frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Qf(B)g(B)]}{\operatorname{tr}(Q)} \\ & \geq\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(B)]}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(B)]}{\operatorname{tr}(Q)}\end{aligned}$$

and, in particular

$$\begin{aligned}(3.6)\qquad & \frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Pf(B)g(B)]}{\operatorname{tr}(P)} \\ & \geq\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(B)]}{\operatorname{tr}(P)}+\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pf(B)]}{\operatorname{tr}(P)}.\end{aligned}$$

Proof. We consider only the case of synchronous functions. In this case we have then

$$(3.7)\qquad f(t)g(t)+f(s)g(s)\geq f(t)g(s)+f(s)g(t)$$

for each $t,s\in[a,b]$.

If we fix $s\in[a,b]$ and apply the property (P) for the inequality (1.8) then we have

$$(3.8)\qquad f(A)g(A)+f(s)g(s)1_H\geq g(s)f(A)+f(s)g(A)$$

in the operator order of $\mathcal{B}(H)$.

Utilising the property (2.15) we have

$$\operatorname{tr}[P(f(A)g(A)+f(s)g(s)1_H)]\geq\operatorname{tr}[P(g(s)f(A)+f(s)g(A))],$$

which is equivalent to

$$\operatorname{tr}[Pf(A)g(A)]+f(s)g(s)\operatorname{tr}(P)\geq g(s)\operatorname{tr}[Pf(A)]+f(s)\operatorname{tr}[Pg(A)],$$

for any $s \in [a,b]$.

This inequality implies the following inequality in the order of $\mathcal{B}(H)$

$$\operatorname{tr}[Pf(A)g(A)]1_H + \operatorname{tr}(P)f(B)g(B) \geq \operatorname{tr}[Pf(A)]g(B) + \operatorname{tr}[Pg(A)]f2(B).$$

Utilising again the property (2.15) we have

$$\begin{aligned} &\operatorname{tr}(Q)\operatorname{tr}[Pf(A)g(A)] + \operatorname{tr}(P)\operatorname{tr}[Qf(B)g(B)] \\ &\geq \operatorname{tr}[Pf(A)]\operatorname{tr}[Qg(B)] + \operatorname{tr}[Pg(A)]\operatorname{tr}[Qf(B)] \end{aligned}$$

and the inequality (3.5) is proved. □

Corollary 3. *Let A be a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$ and assume that the continuous functions $f,g : J \to \mathbb{R}$ are synchronous on J. If $P,Q \in \mathcal{B}_1(H) \setminus \{0\}$ with $P,Q \geq 0$, then*

$$\begin{aligned} &\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} + \frac{\operatorname{tr}[Qf(A)g(A)]}{\operatorname{tr}(Q)} \\ &\geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qg(A)]}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Qf(A)]}{\operatorname{tr}(Q)} \end{aligned} \tag{3.9}$$

and, in particular

$$\frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)}. \tag{3.10}$$

The inequality (3.10) is a trace version of the Čebyšev inequality.

We can improve this inequality as follows.

Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$ and assume that the continuous functions $f,g : J \to \mathbb{R}$ are synchronous on J. For $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$, we can define the functional

$$\mathcal{C}_{(f,g)}(A,P) := \frac{\operatorname{tr}[Pf(A)g(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pg(A)]}{\operatorname{tr}(P)} \geq 0.$$

We have the following result:

Theorem 7. *Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq J$, $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$ and assume that the continuous functions $f,g : J \to \mathbb{R}$ are synchronous on J. Then we have*

$$\begin{aligned} \mathcal{C}_{(f,g)}(A,P) &\geq \max\left\{\left|\mathcal{C}_{(|f|,g)}(A,P)\right|, \left|\mathcal{C}_{(f,|g|)}(A,P)\right|, \left|\mathcal{C}_{(|f|,|g|)}(A,P)\right|\right\} \\ &\geq 0. \end{aligned} \tag{3.11}$$

Proof. Utilising the continuity of modulus property, we have

$$\begin{aligned} (f(t)-f(s))(g(t)-g(s)) &= |(f(t)-f(s))(g(t)-g(s))| \\ &\geq |(|f(t)|-|f(s)|)(g(t)-g(s))| \end{aligned}$$

for any $t,s \in J$.

This is equivalent to

$$\begin{aligned} &f(t)g(t)+f(s)g(s)-f(t)g(s)-f(s)g(t) \\ &\geq ||f(t)|g(t)+|f(s)|g(s)-|f(t)|g(s)-|f(s)|g(t)| \end{aligned} \tag{3.12}$$

for any $t,s \in J$.

This implies in the order of $\mathcal{B}(H)$ that

$$f(A)g(A)+f(s)g(s)1_H-g(s)f(A)-f(s)g(A) \geq \left||f(A)|g(A)+|f(s)|g(s)1_H-g(s)|f(A)|-|f(s)|g(A)\right| \tag{3.13}$$

for any $s\in J$.

Applying the property (2.15) we have

$$\begin{aligned}&\operatorname{tr}[Pf(A)g(A)]+f(s)g(s)\operatorname{tr}(P)-g(s)\operatorname{tr}[Pf(A)]-f(s)\operatorname{tr}[Pg(A)]\\ &\geq \operatorname{tr}\left[P\left||f(A)|g(A)+|f(s)|g(s)1_H-g(s)|f(A)|-|f(s)|g(A)\right|\right]\end{aligned} \tag{3.14}$$

for any $s\in J$.

Using the property (2.16) we also have

$$\begin{aligned}&\operatorname{tr}\left[P\left||f(A)|g(A)+|f(s)|g(s)1_H-g(s)|f(A)|-|f(s)|g(A)\right|\right]\\ &\geq \left|\operatorname{tr}\left[P|f(A)|g(A)+|f(s)|g(s)1_H-g(s)|f(A)|-|f(s)|g(A)\right]\right|\\ &=\left|\operatorname{tr}[P|f(A)|g(A)]+|f(s)|g(s)\operatorname{tr}(P)\right.\\ &\left.-g(s)\operatorname{tr}[P|f(A)|]-|f(s)|\operatorname{tr}[Pg(A)]\right|\end{aligned} \tag{3.15}$$

for any $s\in J$.

By (3.14) and (3.15) we have

$$\begin{aligned}&\operatorname{tr}[Pf(A)g(A)]+f(s)g(s)\operatorname{tr}(P)\\ &-g(s)\operatorname{tr}[Pf(A)]-f(s)\operatorname{tr}[Pg(A)]\\ &\geq \left|\operatorname{tr}[P|f(A)|g(A)]+|f(s)|g(s)\operatorname{tr}(P)\right.\\ &\left.-g(s)\operatorname{tr}[P|f(A)|]-|f(s)|\operatorname{tr}[Pg(A)]\right|\end{aligned}$$

for any $s\in J$.

This inequality implies in the order of $\mathcal{B}(H)$ that

$$\begin{aligned}&\operatorname{tr}[Pf(A)g(A)]1_H+\operatorname{tr}(P)f(A)g(A)\\ &-\operatorname{tr}[Pf(A)]g(A)-\operatorname{tr}[Pg(A)]f(A)\\ &\geq \left|\operatorname{tr}[P|f(A)|g(A)]1_H+\operatorname{tr}(P)|f(A)|g(A)\right.\\ &\left.-\operatorname{tr}[P|f(A)|]g(A)-\operatorname{tr}[Pg(A)]|f(A)|\right|.\end{aligned} \tag{3.16}$$

Taking the trace and repeating the reason, we deduce

$$\begin{aligned}&\operatorname{tr}(P)\operatorname{tr}[Pf(A)g(A)]+\operatorname{tr}(P)\operatorname{tr}[Pf(A)g(A)]\\ &-\operatorname{tr}[Pf(A)]\operatorname{tr}[Pg(A)]-\operatorname{tr}[Pg(A)]\operatorname{tr}[Pf(A)]\\ &\geq \left|\operatorname{tr}[P|f(A)|g(A)]\operatorname{tr}(P)+\operatorname{tr}(P)\operatorname{tr}[|f(A)|g(A)]\right.\\ &\left.-\operatorname{tr}[P|f(A)|]\operatorname{tr}[Pg(A)]-\operatorname{tr}[Pg(A)]\operatorname{tr}[P|f(A)|]\right|,\end{aligned} \tag{3.17}$$

which is equivalent to

$$\mathcal{C}_{(f,g)}(A,P)\geq\left|\mathcal{C}_{(|f|,g)}(A,P)\right|.$$

The other inequalities follows in a similar way and the details are omitted. □

4. Some Examples

If we take the functions f, $g : [0,\infty) \to [0,\infty)$, $f(t) = t^p$ and $g(t) = t^q$ with $p, q > 0$ then by (3.2) we have

$$\begin{aligned}\frac{\operatorname{tr}(PA^{p+q})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)} \\ \geq \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\right)\left(\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^q - \frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)}\right),\end{aligned} \tag{4.1}$$

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

If $p > 0$ and $q \in (0,1)$, then we have a better result:

$$\begin{aligned}\frac{\operatorname{tr}(PA^{p+q})}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)} \\ \geq \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\right)\left(\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^q - \frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)}\right) \\ \geq 0,\end{aligned} \tag{4.2}$$

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

By using (3.5) we have for $p, q > 0$ that

$$\frac{\operatorname{tr}(PA^{p+q})}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(QB^{p+q})}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^q)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QB^p)}{\operatorname{tr}(Q)} \tag{4.3}$$

for any A, $B \geq 0$ and P, $Q \in \mathcal{B}_1(H) \setminus \{0\}$ with P, $Q \geq 0$.

In particular, we have

$$\frac{\operatorname{tr}(PA^{p+q})}{\operatorname{tr}(P)} + \frac{\operatorname{tr}(QA^{p+q})}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QA^q)}{\operatorname{tr}(Q)} + \frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QA^p)}{\operatorname{tr}(Q)} \tag{4.4}$$

for any $A \geq 0$ and P, $Q \in \mathcal{B}_1(H) \setminus \{0\}$ with P, $Q \geq 0$.

Also

$$\frac{\operatorname{tr}(PA^{p+q})}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^q)}{\operatorname{tr}(P)} \tag{4.5}$$

for any $A \geq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

Moreover, if in (4.5) we choose $A = P$, then from (4.5) we get

$$\frac{\operatorname{tr}\left(P^{p+q+1}\right)}{\operatorname{tr}(P)} \geq \frac{\operatorname{tr}\left(P^{p+1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(P^{q+1}\right)}{\operatorname{tr}(P)} \tag{4.6}$$

for $p, q > 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$ with $P \geq 0$.

If we take the functions $f, g : [0,\infty) \to [0,\infty)$, $f(t) = t^p$ and $g(t) = \ln t$ with $p \geq 1$ then by (3.4) we have

$$\begin{aligned}\frac{\operatorname{tr}(PA^p \ln A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\ \geq \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\right)\left(\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}\right) \\ \geq 0,\end{aligned} \tag{4.7}$$

for any positive definite operators A and P with $P \in \mathcal{B}_1(H) \setminus \{0\}$.

If we use (3.9), then we have for $p>0$

$$\frac{\operatorname{tr}(PA^p\ln A)}{\operatorname{tr}(P)}+\frac{\operatorname{tr}(QA^p\ln A)}{\operatorname{tr}(Q)} \geq \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(Q\ln A)}{\operatorname{tr}(Q)}+\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(QA^p)}{\operatorname{tr}(Q)} \tag{4.8}$$

for any positive definite operators A, P and Q with $P,\ Q\in\mathcal{B}_1(H)\setminus\{0\}$.

In particular

$$\frac{\operatorname{tr}(PA^p\ln A)}{\operatorname{tr}(P)}\geq\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}, \tag{4.9}$$

for any positive definite operators A and P with $P\in\mathcal{B}_1(H)\setminus\{0\}$.

If we apply the inequality (3.11), then we have an improvement of (4.9) as follows

$$\frac{\operatorname{tr}(PA^p\ln A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \geq\left|\frac{\operatorname{tr}(PA^p|\ln A|)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P|\ln A|)}{\operatorname{tr}(P)}\right|\geq 0, \tag{4.10}$$

for any positive definite operators A and P with $P\in\mathcal{B}_1(H)\setminus\{0\}$.

If we use the inequality (3.10) for the $f,\ g:[0,\infty)\to[0,\infty)$, $f(t)=t$ and $g(t)=-t^{-1}$, then we get

$$1\leq\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}, \tag{4.11}$$

for any positive definite operators A and P with $P\in\mathcal{B}_1(H)\setminus\{0\}$.

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