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INEQUALITIES FOR POWER SERIES

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Abstract. The aim of this paper is to give several inequalities for power series starting from a generalization of Young's inequality for sequences of complex numbers. Then some inequalities deduced from some variants of the arithmetic-geometric mean inequality will be given.

1. Introduction

We consider an analytic function defined by the power series

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

with real coefficients and convergent on the disk $D(0,R),\ R>0$. As in [4] the weighted version of Hölder's inequality can be stated as below:

$$|f(xy)|=\left|\sum_{n=0}^{\infty}a_nx^ny^n\right|\le\left(\sum_{n=0}^{\infty}|a_n||x|^{pn}\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}|a_n||x|^{qn}\right)^{\frac{1}{q}}=f_A^{\frac{1}{p}}(|x|^p)f_A^{\frac{1}{q}}(|y|^q)$$

for any $x,y\in\mathbf{C}$ with $xy,\ |x|^p,|y|^q\in D(0,R)$ and $f_A(z)$ is a power series defined by $\sum_{n=0}^{\infty}|a_n|z^n$. The power series $f_A(z)$ have the same radius of convergence as the original power series $f(z)$.

We consider the following inequality:

Lemma 1. *([8]) For $0<a,b\le 1$ and $\lambda\in(0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2+A(\lambda)ab\log^2\left(\frac{a}{b}\right)\le\lambda a+(1-\lambda)b-a^{\lambda}b^{1-\lambda}\le$$

$$\le(1-r)(\sqrt{a}-\sqrt{b})^2+B(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

where $r=\min\{\lambda,1-\lambda\}$, $A(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$.

If we take here $\lambda=\frac{1}{p}$ and replace a^{λ} by a and $b^{1-\lambda}$ by b then $1-\lambda=\frac{1}{q}$ and we obtain:

$$ab+r(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+A(\frac{1}{p})a^pb^q\log^2\left(\frac{a^p}{b^q}\right)\le\frac{a^p}{p}+\frac{b^q}{q}\le$$

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$$\leq ab+(1-r)(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+B(\frac{1}{p})a^p b^q\log^2\left(\frac{a^p}{b^q}\right). \tag{1}$$

We also need the inequality from below which is given in [5], Lemma 2.

Lemma 2. *For $a_{ij}\geq 0,\ p_j>0,\ i\in\{1,2,...n\}$ and $j\in\{1,2,...,m\}$ such that $\frac{1}{p_1}+\frac{1}{p_2}+..+\frac{1}{p_m}\geq 1$ we have*

$$\sum_{i=1}^{n}a_{i1}a_{i2}...a_{im}\leq\left(\sum_{i=1}^{n}a_{i1}^{p_1}\right)^{\frac{1}{p_1}}\left(\sum_{i=1}^{n}a_{i2}^{p_2}\right)^{\frac{1}{p_2}}...\left(\sum_{i=1}^{n}a_{im}^{p_m}\right)^{\frac{1}{p_m}}.$$

Next inequality is given in [2], Proposition 5.1 and will be used in Theorem 4.

Proposition 1. *([2]) Let $a_1,...,a_n\geq 0$ and $p_1,...,p_n\geq 0$ with $\sum_{j=1}^{n}p_j=1$ we have*

$$\sum_{i=1}^{n}p_i a_i-a_1^{p_1}...a_n^{p_n}\geq n\lambda\left(\frac{1}{n}\sum_{i=1}^{n}a_i-a_1^{\frac{1}{n}}...a_n^{\frac{1}{n}}\right),$$

with equality if and only if $a_1=...=a_n$, where $\lambda=\min\{p_1,...,p_n\}$.

2. Main results

The following three results were obtained using a refinement of Young's inequality given in [8] for two positive numbers a and b in $(0,1)$ for power series with real coefficients, and the same method as in [4], Theorem 1, 2 and 3.

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}p_n z^n$, $g(z)=\sum_{n=0}^{\infty}q_n z^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0<R<1$. If p,q are real numbers with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $a,b\in\mathbf{C}$, $a,b\neq 0$, $|a|<1$, $|b|<1$ so that $|ab|,\ |a|^p,\ |a|^q,\ |b|^p,\ |b|^q,\ |a|^{\frac{q}{2}}|b|^{\frac{p}{2}},\ |a|^{\frac{p}{2}}|b|^{\frac{q}{2}},\ |a|^p|b|^q,\ |a|^q|b|^p\in D(0,R)$, then we have*

$$|f(ab)g(ab)|+rM_1+A(\frac{1}{p})T_1\leq$$

$$\leq f_A(|ab|)g_A(|ab|)+rM_1+A(\frac{1}{p})T_1\leq$$

$$\leq\frac{1}{p}f_A(|a|^p)g_A(|b|^p)+\frac{1}{q}f_A(|b|^q)g_A(|a|^q)\leq \tag{2}$$

$$\leq f_A(|a||b|)g_A(|a||b|)+(1-r)M_1+B(\frac{1}{p})T_1,$$

and

$$|f(a|b|^{p-1})g(a|b|^{q-1})|+rM_2+A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2\leq \tag{3}$$

$$\leq f_A(|a||b|^{p-1})g_A(|a||b|^{q-1})+rM_2+A\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2\leq$$

$$\leq\frac{1}{p}f_A(|a|^p)g_A(|b|^q)+\frac{1}{q}f_A(|b|^p)g_A(|a|^q),$$

$$\leq f_A(|a||b|^{p-1})g_A(|a||b|^{q-1})+(1-r)M_2+B\left(\frac{1}{p}\right)\log^2\frac{|a|}{|b|}T_2,$$

if in this case $|a| < |b|$, *and* $|a||b|^{p-1}$, $|a||b|^{q-1}$, $|a|^p$, $|a|^q$, $|b|^p$, $|b|^q$, $|a|^{\frac{p}{2}}|b|^{\frac{p}{2}}$, $|a|^{\frac{q}{2}}|b|^{\frac{q}{2}} \in D(0,R)$, *where*

$$M_1 = f_A(|a|^p)g_A(|b|^p) + f_A(|b|^q)g_A(|a|^q) - 2f_A(|a|^{\frac{p}{2}}|b|^{\frac{q}{2}})g_A(|a|^{\frac{q}{2}}|b|^{\frac{p}{2}}),$$

$$M_2 = f_A(|a|^p)g_A(|b|^q) + f_A(|b|^p)g_A(|a|^q) - 2f_A(|a|^{\frac{p}{2}}|b|^{\frac{p}{2}})g_A(|a|^{\frac{q}{2}}|b|^{\frac{q}{2}}),$$

$$T_1 = g_A(|a|^q|b|^p)S_1(|a|^p|b|^q)\log^2\frac{|a|^p}{|b|^q} + f_A(|a|^p|b|^q)S_2(|a|^q|b|^p)\log^2\frac{|a|^q}{|b|^p} -$$

$$-2[pq(\log^2|a| + \log^2|b|) - (p^2+q^2)\log|a|\log|b|]S_3(|a|^p|b|^q)S_4(|a|^q|b|^p),$$

$$T_2 = p^2 g_A(|a|^q)S_1(|a|^p) + q^2 f_A(|a|^p)S_2(|a|^q) - 2pqS_3(|a|^p)S_4(|a|^q),$$

$$S_1(x) = xf_A^{'}(x) + x^2 f_A^{''}(x),\ S_2(x) = xg_A^{'}(x) + x^2 g_A^{''}(x),\ S_3(x) = xf_A^{'}(x),\ S_4(x) = xg_A^{'}(x).$$

Proof. In the first case we replace a by $|a|^j|b|^k$, and b by $|a|^k|b|^j$, $j,k \in \{0,1,...,n\}$ in (1) and then we have

$$|a|^j|b|^k|a|^k|b|^j + r[(|a|^j|b|^k)^{\frac{p}{2}} - (|a|^k|b|^j)^{\frac{q}{2}}]^2 +$$

$$+A_1(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kp}}{|a|^{kq}|b|^{jq}}\right)|a|^{jp}|b|^{kp}|a|^{kq}|b|^{jq} \le$$

$$\le \frac{|a|^{jp}|b|^{kp}}{p} + \frac{|a|^{kq}|b|^{jq}}{q} \le$$

$$\le |a|^j|b|^k|a|^k|b|^j + (1-r)[(|a|^j|b|^k)^{\frac{p}{2}} - (|a|^k|b|^j)^{\frac{q}{2}}]^2 +$$

$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kp}}{|a|^{kq}|b|^{jq}}\right)|a|^{jp}|b|^{kp}|a|^{kq}|b|^{jq}$$

for any $j,k \in \{0,1,2,...,n\}$. We take into account that $|a^jb^ka^kb^j| = |a^jb^j||b^ka^k| = |a|^j|b|^k|a|^k|b|^j$ and if we multiply the inequality with positive quantities $|p_j||q_k|$ and sum over j and k from 0 to n, we obtain

$$\sum_{j=0}^n |p_j||ab|^j\sum_{k=0}^n |q_k||ab|^k + r\sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|[|a|^{jp}|b|^{kp} + |a|^{kq}|b|^{jq} - 2|a|^{j\frac{p}{2}}|a|^{k\frac{q}{2}}|b|^{k\frac{p}{2}}|b|^{j\frac{q}{2}}] +$$

$$+A(\frac{1}{p})\sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k \le$$

$$\le \sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|\left(\frac{|a|^{jp}|b|^{kp}}{p} + \frac{|a|^{kq}|b|^{jq}}{q}\right) \le \tag{4}$$

$$\le \sum_{j=0}^n |p_j||ab|^j\sum_{k=0}^n |q_k||ab|^k + (1-r)\sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|[|a|^{jp}|b|^{kp} + |a|^{kq}|b|^{jq} - 2|a|^{j\frac{p}{2}}|a|^{k\frac{q}{2}}|b|^{k\frac{p}{2}}|b|^{j\frac{q}{2}}] +$$

$$+B(\frac{1}{p})\sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k.$$

Denoting by P_1 the quantity $\sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|\log^2\left(\frac{|a|^{ip-kq}}{|b|^{jq-kp}}\right)(|a|^p|b|^q)^j(|a|^q|b|^p)^k$ by computation we have,

$$P_1 = \sum_{j=0}^n |p_j|\sum_{k=0}^n |q_k|[(jp-kq)\log|a| - (jq-kp)\log|b|]^2(|a|^p|b|^q)^j(|a|^q|b|^p)^k =$$

$$=\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[(jp-kq)^2\log^2|a|+(jq-kp)^2\log^2|b|-$$
$$-2(jp-kq)(jq-kp)\log|a|\log|b|](|a|^p|b|^q)^j(|a|^q|b|^p)^k=$$
$$=\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[j^2(p\log|a|-q\log|b|)^2+k^2(q\log|a|-p\log|b|)^2-$$
$$-2jk(pq(\log^2|a|+\log^2|b|)-(p^2+q^2)\log|a|\log|b|)](|a|^p|b|^q)^j(|a|^q|b|^p)^k=$$
$$=\sum_{j=0}^{n}|p_j|\sum_{k=0}^{n}|q_k|[j^2\log^2\frac{|a|^p}{|b|^q}+k^2\log^2\frac{|a|^q}{|b|^p}-2jk(pq(\log^2|a|+\log^2|b|)-$$
$$-(p^2+q^2)\log|a|\log|b|)](|a|^p|b|^q)^j(|a|^q|b|^p)^k.$$

All the series whose partial sums which appear here in inequality (4) are convergent on the disk $D(0,R)$ therefore we can take the limit when n tends to ∞ in (4) and obtain the inequality (2) taking into account that because T_1 is the limit when n tends of ∞ of P_1.

In the second case, if we replace in (1) a by $\frac{|a|^j}{|b|^j}$ and b by $\frac{|a|^k}{|b|^k}$ then we have

$$\frac{|a|^j|a|^k}{|b|^j|b|^k}+r\left[\frac{|a|^{pj}}{|b|^{pj}}+\frac{|a|^{qk}}{|b|^{qk}}-2\frac{|a|^{\frac{jp}{2}}|a|^{\frac{qk}{2}}}{|b|^{\frac{jp}{2}}|b|^{\frac{qk}{2}}}\right]+A(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kq}}{|b|^{jp}|a|^{kq}}\right)\frac{|a|^{jp}|a|^{kq}}{|b|^{jp}|b|^{kq}}\le$$
$$\le\frac{1}{p}\frac{|a|^{jp}}{|b|^{jp}}+\frac{1}{q}\frac{|a|^{qk}}{|b|^{qk}}\le \tag{5}$$
$$\le\frac{|a|^j|a|^k}{|b|^j|b|^k}+(1-r)\left[\frac{|a|^{pj}}{|b|^{pj}}+\frac{|a|^{qk}}{|b|^{qk}}-2\frac{|a|^{\frac{jp}{2}}|a|^{\frac{qk}{2}}}{|b|^{\frac{jp}{2}}|b|^{\frac{qk}{2}}}\right]+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp}|b|^{kq}}{|b|^{jp}|a|^{kq}}\right)\frac{|a|^{jp}|a|^{kq}}{|b|^{jp}|b|^{kq}}$$

for any $|b|^j,\ |b|^k\neq 0,\ j,\ k\in\{0,1,2,...,n\}$.

Simplifying (5) we get

$$|a|^j|a|^k|b|^{j(p-1)}|b|^{k(q-1)}+r[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+$$
$$+A(\frac{1}{p})\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jp-kq}}\right)|a|^{jp}|a|^{kq}\le\frac{1}{p}|a|^{jp}|b|^{qk}+\frac{1}{q}|a|^{qk}|b|^{jp}\le \tag{6}$$
$$\le|a|^j|a|^k|b|^{j(p-1)}|b|^{k(q-1)}+(1-r)[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+$$
$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{jp-kq}}{|b|^{jp-kq}}\right)|a|^{jp}|a|^{kq}$$

for any $j,k\in\{0,1,2,...,n\}$.

Now we multiply (6) by $|p_j||q_k|\geq 0,\ j,k\in\{0,1,2,...,n\}$ and summing over j and k from 0 to n, we have

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(|a||b|^{p-1})^j(|a|b|^{q-1})^k+r\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+$$
$$+A(\frac{1}{p})\log^2\frac{|a|}{|b|}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}\le$$
$$\le\frac{1}{p}\sum_{j=0}^{n}|p_j||a|^{jp}\sum_{k=0}^{n}|q_k||b|^{qk}+\frac{1}{q}\sum_{k=0}^{n}|q_k||a|^{qk}\sum_{j=0}^{n}|p_j||b|^{jp}\le \tag{7}$$

$$\leq \sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(|a||b|^{p-1})^j(|a|b|^{q-1})^k+(1-r)\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{pj}|b|^{qk}+|a|^{qk}|b|^{jp}-$$

$$-2|a|^{j\frac{p}{2}+k\frac{q}{2}}|b|^{j\frac{p}{2}+k\frac{q}{2}}]+B(\frac{1}{p})\log^2\frac{|a|}{|b|}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}.$$

In this case

$$P_2=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(jp-kq)^2|a|^{jp}|a|^{kq}=\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(j^2p^2+k^2q^2-2pqjk)|a|^{jp}|a|^{kq}.$$

Taking into account that all the series whose partial sums are involved in previous inequality are convergent on the disk $D(0,R)$, and letting n to ∞ in the inequality (7), we notice that the desired inequality (3) takes place, because T_2 is the limit when n tends of ∞ of P_2.

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Theorem 2. *Let* $f(z)=\sum_{n=0}^{\infty}p_nz^n$, $g(z)=\sum_{n=0}^{\infty}q_nz^n$ *be the power series with real coefficients and convergent on the open disk* $D(0,R)$, $0<R<1$. *If* p,q *are real numbers with* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $a,b\in\mathbf{C}$, $a,b\neq 0$, $|a|<1$, $|b|<1$ *such that* $|a||b|$, $|a|^2$, $|a|^q$, $|b|^p$, $|b|^2$, $|a|^{\frac{q}{2}}|b|^{\frac{p}{2}}$, $|a|^{\frac{2}{p}}|b|^{\frac{2}{q}}\in D(0,R)$, *then we have*

$$|f(ab)g(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})|+rM_3+A_1(\frac{1}{p})T_3\leq$$

$$\leq f_A(|ab|)g_A(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})+rM_3+A_1(\frac{1}{p})T_3\leq$$

$$\leq\frac{1}{p}f_A(|b|^p)g_A(|a|^2)+\frac{1}{q}f_A(|a|^q)g_A(|b|^2)\leq \tag{8}$$

$$\leq f_A(|a||b|)g_A(|a|^{\frac{2}{p}}|b|^{\frac{2}{q}})+(1-r)M_3+B(\frac{1}{p})T_3,$$

where

$$M_3=f_A(|a|^2)g_A(|b|^p)+f_A(|a|^q)g_A(|b|^2)-2f_A(|a|^{\frac{q}{2}}|b|^{\frac{p}{2}})g_A(|a||b|),$$

$$T_3=4\log^2\frac{|a|}{|b|}\cdot f_A(|a|^q|b|^p)S_1(|a|^2|b|^2)+\log^2\frac{|b|^p}{|a|^q}g_A(|a|^2|b|^2)S_2(|a|^q|b|^p)+$$

$$+4\log\frac{|a|}{|b|}\log\frac{|b|^p}{|a|^q}S_3(|a|^q|b|^p)S_4(|a|^2|b|^2).$$

Proof. Now, we replace a by $|a|^{k\frac{2}{p}}|b|^j$, and b by $|a|^j|b|^{k\frac{2}{q}}$ in inequality (1), we multiply by $|p_j||q_k|\geq 0$ and then summing over j and k from 0 to n we get

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{k\frac{2}{p}}|b|^j|a|^j|b|^{k\frac{2}{q}}+r\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{2k}|b|^{jp}+|a|^{jq}|b|^{2k}-$$

$$-2|a|^k|b|^{j\frac{p}{2}}|a|^{j\frac{q}{2}}|b|^k]+A_1(\frac{1}{p})\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}\leq$$

$$\leq\frac{1}{p}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{2k}|b|^{jp}+\frac{1}{q}\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{jq}|b|^{2k}\leq \tag{9}$$

$$\leq \sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k||a|^{k\frac{2}{p}}|b|^j|a|^j|b|^{k\frac{2}{q}} + (1-r)\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|[|a|^{2k}|b|^{jp} + |a|^{jq}|b|^{2k} -$$

$$-2|a|^k|b|^{j\frac{p}{2}}|a|^{j\frac{q}{2}}|b|^k] + B(\frac{1}{p})\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}$$

where P_3 is the quantity

$$\sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k}|b|^{jp}}{|a|^{jq}|b|^{2k}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}.$$

By computation, we find,

$$P_3 = \sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|\log^2\left(\frac{|a|^{2k-jq}}{|b|^{2k-jp}}\right)|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k} = \sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(2k\log\frac{|a|}{|b|}-$$

$$-jq\log|a|+jp\log|b|)^2|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k} = \sum_{j=0}^{n}\sum_{k=0}^{n}|p_j||q_k|(4k^2\log^2\frac{|a|}{|b|}+j^2\log^2\frac{|b|^p}{|a|^q}+$$

$$+4jk\log\frac{|a|}{|b|}\log\frac{|b|^p}{|a|^q})|a|^{2k}|b|^{jp}|a|^{jq}|b|^{2k}.$$

Since all the series whose partial sums are involved in the inequality (9) are convergent on the disk $D(0,R)$, letting n tends to ∞ in (9), we deduce the desired inequality, because T_3 is the limit when n tends to ∞ of P_3.

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Theorem 3. *Let $f(z)$ and $g(z)$ be as in Theorem 1. If $|a|^2, |b|^p, |b|^q, |a|^{\frac{2}{p}}|b|, |a|^{\frac{2}{q}}|b|, |a||b|^{\frac{q}{2}}, |a||b|^{\frac{p}{2}} \in D(0,R)$ then one has the following inequality*

$$|f(|a|^{\frac{2}{p}}b)g(|a|^{\frac{2}{q}}b)| + rM_4 + A(\frac{1}{p})T_4 \leq$$

$$\leq f_A(|a|^{\frac{2}{p}}|b|)g_A(|a|^{\frac{2}{q}}|b|) + rM_4 + A(\frac{1}{p})T_4 \leq$$

$$\leq \frac{1}{p}f_A(|a|^2)g_A(|b|^p) + \frac{1}{q}f_A(|b|^q)g_A(|a|^2) \leq$$

$$\leq f_A(|a|^{\frac{2}{p}}|b|)g_A(|a|^{\frac{2}{q}}) + (1-r)M_4 + B(\frac{1}{p})T_4,$$

where

$$M_4 = f_A(|a|^2)g_A(|b|^p) + f_A(|b|^q)g_A(|a|^2) - 2f_A(|a||b|^{\frac{q}{2}})g_A(|a||b|^{\frac{p}{2}}),$$

$$T_4 = \log^2\left(\frac{|a|^2}{|b|^q}\right)g_A(|a|^2|b|^p)S_1(|a|^2|b|^q) + \log^2\left(\frac{|b|^p}{|a|^2}\right)f_A(|a|^2|b|^q)S_2(|a|^2|b|^p)+$$

$$+2S_3(|a|^2|b|^q)S_4(|a|^2|b|^p)\log\left(\frac{|a|^2}{|b|^q}\right)\log\left(\frac{|b|^p}{|a|^2}\right).$$

Proof. Using again the inequality (1) with $|a|^{j\frac{2}{p}}|b|^k$ instead of a and $|a|^{k\frac{2}{q}}|b|^j$ instead of b we obtain for any $j,k \in \{0,1,2,...,n\}$ the following inequality

$$(|a|^{\frac{2}{p}}|b|)^j(|b||a|^{\frac{2}{q}})^k + r[|a|^{2j}|b|^{pk} + |a|^{2k}|b|^{jq} - 2|a|^j|b|^{k\frac{p}{2}}|a|^k|b|^{j\frac{q}{2}}]+$$

$$+A_1(\frac{1}{p})\log^2\left(\frac{|a|^{2j}|b|^{pk}}{|a|^{2k}|b|^{jq}}\right)(|a|^{2j}|b|^{qj})(|a|^{2k}|b|^{kp}) \leq$$

$$\leq \frac{1}{p}|a|^{2j}|b|^{pk} + \frac{1}{q}|a|^{2k}|b|^{jq} \leq \tag{10}$$

$$\leq (|a|^{\frac{2}{p}}|b|)^j(|b||a|^{\frac{2}{q}})^k + (1-r)[|a|^{2j}|b|^{pk} + |a|^{2k}|b|^{jq} - 2|a|^j|b|^{k\frac{p}{2}}|a|^k|b|^{j\frac{q}{2}}]+$$

$$+B(\frac{1}{p})\log^2\left(\frac{|a|^{2j}|b|^{pk}}{|a|^{2k}|b|^{jq}}\right)(|a|^{2j}|b|^{qj})(|a|^{2k}|b|^{kp}).$$

By the same method as in Theorem 1 we find the desired inequality.

∎

Remark 1. *Let $r_1, r_2, ..., r_m \neq 0$ be real numbers such that $\frac{1}{r_1} + \frac{1}{r_2} + ... + \frac{1}{r_m} = 1$ and $f(z) = \sum_{n=0}^{\infty} p_n z^n$, $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0 < R$. If $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m \in \mathbf{C}$, such that $a_1a_2...a_m$, $b_1b_2...b_m$, $|a_i|^{r_i}$, $|b_i|^{r_i} \in D(0,R)$, $i \in \{1,2,...,m\}$ then we have*

$$|f(a_1a_2...a_m)g_A(b_1b_2...b_m)| \leq f_A(|a_1||a_2|...|a_m|)g_A(|b_1||b_2|...|b_m|) \leq$$

$$\leq \frac{1}{r_1}f_A(|a_1|^{r_1})g_A(|b_1|^{r_1}) + \frac{1}{r_2}f_A(|a_2|^{r_2})g_A(|b_2|^{r_2}) + ... + \frac{1}{r_m}f_A(|a_m|^{r_m})g_A(|b_m|^{r_m}).$$

Proof. We use the well-known inequality

$$\alpha_1x_1 + \alpha_2x_2... + \alpha_mx_m \geq x_1^{\alpha_1}x_2^{\alpha_2}...x_m^{\alpha_m}$$

which takes place for any $x_1, x_2, .., x_m > 0$ and $\alpha_1, \alpha_2, ..., \alpha_m$ real numbers such that $\alpha_1 + \alpha_2 + .. + \alpha_m = 1$ and replacing α_i by $\frac{1}{r_i}$ and $x_i^{\frac{1}{r_i}}$ by x_i we obtain

$$\frac{1}{r_1}x_1^{r_1} + \frac{1}{r_2}x_2^{r_2} + ... + \frac{1}{r_m}x_m^{r_m} \geq x_1x_2..x_m.$$

Taking above $x_1 = |a_1|^j|b_1|^k$, $x_2 = |a_2|^j|b_2|^k, ...$ $x_m = |a_m|^j|b_m|^k$ for $j,k \in \{0,1,2,...,n\}$ and using the same method like in Theorem 1 we find the desired inequality.

∎

Proposition 2. *Let a_j be complex numbers and $p_j > 0$, $j \in \{1,2,...,m\}$ such that $\frac{1}{p_1} + \frac{1}{p_2} + .. + \frac{1}{p_m} \geq 1$. If $f(z) = \sum_{n=0}^{\infty} p_n' z^n$ be the power series with real coefficients and convergent on the open disk $D(0,R)$, $0 < R$ and $a_1a_2...a_m, |a_1||a_2|...|a_m|$, $|a_1|^{p_1}, |a_2|^{p_2}, ..., |a_m|^{p_m} \in D(0,R)$, and $|p_i'| \geq 1$ for all $i \in \mathbf{N}$ then the following inequality holds:*

$$|f(a_1a_2...a_m)| \leq f_A(|a_1||a_2|...|a_m|) \leq f^{\frac{1}{p_1}}(|a_1|^{p_1})f^{\frac{1}{p_2}}(|a_2|^{p_2})..f^{\frac{1}{p_m}}(|a_m|^{p_m}). \tag{11}$$

Proof. If we consider $a_{ij} = |p_i^{'}|^{\frac{1}{p_j}} |a_j|^i$ with $i \in \{1, ..., n\}$ and $j \in \{1, ..., m\}$ in Lemma 2, see [5] page 743, the inequality

$$\sum_{i=1}^{n} a_{i1}a_{i2}...a_{im} \leq \left(\sum_{i=1}^{n} a_{i1}^{p_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^{n} a_{i2}^{p_2}\right)^{\frac{1}{p_2}} ... \left(\sum_{i=1}^{n} a_{im}^{p_m}\right)^{\frac{1}{p_m}}$$

becomes:

$$\sum_{i=1}^{n} |p_i^{'}|^{\frac{1}{p_1}+\frac{1}{p_2}+..+\frac{1}{p_m}} |a_1|^i |a_2|^i ... |a_m|^i \leq \left(\sum_{i=1}^{n} |p_i^{'}||a_1|^{ip_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^{n} |p_i^{'}||a_2|^{ip_2}\right)^{\frac{1}{p_2}} ... \left(\sum_{i=1}^{n} |p_i^{'}||a_m|^{ip_m}\right)^{\frac{1}{p_m}}$$

or

$$\sum_{i=1}^{n} |p_i^{'}||a_1 a_2 ... a_m|^i \leq \sum_{i=1}^{n} |p_i^{'}|^{\frac{1}{p_1}+\frac{1}{p_2}+..+\frac{1}{p_m}} |a_1|^i |a_2|^i ... |a_m|^i \leq$$

$$\leq \left(\sum_{i=1}^{n} |p_i^{'}||a_1|^{ip_1}\right)^{\frac{1}{p_1}} \left(\sum_{i=1}^{n} |p_i^{'}||a_2|^{ip_2}\right)^{\frac{1}{p_2}} ... \left(\sum_{i=1}^{n} |p_i^{'}||a_m|^{ip_m}\right)^{\frac{1}{p_m}}.$$

Taking into account that $a_1 a_2 ... a_m$, $|a_1||a_2|...|a_m|$, $|a_1|^{p_1}, |a_2|^{p_2}, ..., |a_m|^{p_m} \in D(0, R)$, when n tends to ∞ we get inequality (11).

∎

Using a refinement of the weighted arithmetic-geometric mean inequality for n real numbers, see [2], we find the following:

Theorem 4. *Let $a_1, a_2, ..., a_n \geq 0$ and $p_1, p_2, ..., p_n > 0$ with $\sum_{j=1}^{n} p_j = 1$ and $\lambda = \min\{p_1, ...p_n\}$. If we assume that the multiplicity attaining λ is 1, then we have the following inequality:*

$$\sum_{i=1}^{n} p_i f_A(|a_i|) g_A(|b_i|) - |f(a_1^{p_1} a_2^{p_2} ... a_n^{p_n}) g(b_1^{p_1} b_2^{p_2} ... b_n^{p_n})| \geq$$

$$\sum_{i=1}^{n} p_i f_A(|a_i|) g_A(|b_i|) - f_A(|a_1|^{p_1} |a_2|^{p_2} ... |a_n|^{p_n}) g_A(|b_1|^{p_1} |b_2|^{p_2} ... |b_n|^{p_n}) \geq$$

$$\geq n\lambda \left(\frac{1}{n} \sum_{i=1}^{n} f_A(|a_i|) g_A(|b_i) - f_A(|a_1|^{\frac{1}{n}} .. |a_n|^{\frac{1}{n}}) g_A(|b_1|^{\frac{1}{n}} ... |b_n|^{\frac{1}{n}})\right),$$

where f, g, f_A and g_A are as in Theorem 1 and $|a_1|^{p_1}...|a_n|^{p_n}$, $|b_1|^{p_1}...|b_n|^{p_n}$, $|a_i|$, $|b_i|$, $|b_1|^{\frac{1}{n}}...|b_n|^{\frac{1}{n}}$, $|a_1|^{\frac{1}{n}}...|a_n|^{\frac{1}{n}} \in D(0, R)$.

Proof. We replace a_i by $|a_i|^j |b_i|^k$ for $j, k \in \{1, 2, ..., m\}$, $i \in \{1, ..., n\}$ in inequality from below and write again this inequality

$$\sum_{i=1}^{n} p_i a_i - a_1^{p_1} a_2^{p_2} ... a_n^{p_n} \geq n\lambda \left(\frac{1}{n} \sum_{i=1}^{n} a_i - a_1^{\frac{1}{n}} ... a_n^{\frac{1}{n}}\right)$$

from Proposition 5.1 (Proposition 1), see [2] obtaining:

$$\sum_{i=1}^{n} p_i |a_i|^j |b_i|^k - |a_1|^{p_1 j} |b_1|^{p_1 k} ... |a_n|^{p_n j} |b_n|^{p_n k} \geq$$

$$\geq n\lambda \left(\frac{1}{n}(|a_1|^j |b_1|^k + ... + |a_n|^j |b_n|^k) - |a_1|^{\frac{j}{n}} |b_1|^{\frac{k}{n}} |a_2|^{\frac{j}{n}} |b_2|^{\frac{k}{n}} ... |a_n|^{\frac{j}{n}} |b_n|^{\frac{k}{n}}\right)$$

which by multiplication by $|p_j^{'}||q_k|$ and summing over j and k will give the desired inequality from conclusion when m tend to infinity. ∎

For finite sequences of real numbers we use the majorization relation from [6]. Let $a = (a_1, a_2, .., a_n)$ and $b = (b_1, b_2, ..., b_n)$ two finite sequences of real numbers. We say that the sequence a majorizes the sequence b and we write

$$a{>>}b \text{ or } \; b{<<}a,$$

if after rearranging terms of the sequence a and b satisfy the following three conditions:

$$a_1 \geq a_2 \geq ... \geq a_n \text{ and } b_1 \geq b_2 \geq ... \geq b_n$$

$$a_1 + a_2 + ... + a_k \geq b_1 + b_2 + .. + b_k, \text{ for each } k, \; 1 \leq k \leq n-1;$$

$$a_1 + a_2 + ... + a_n = a_1 + b_2 + .. + b_n.$$

As in [6], Definition 2, let $F(x_1, x_2, ..., x_n)$ be a function in n nonnegative real variables. Define

$$\sum^{!} F(x_1, x_2, ..., x_n)$$

as the sum of $n!$ summands, obtained from the expression $F(x_1, x_2, ..., x_n)$ as all the possible permutations of the sequence $x = (x_i)_{i=1}^n$.

Particularly, if for some sequence of nonnegative exponents $a = (a_i)_{i=1}^n$, the function F is of the form $F(x_1, x_2, .., x_n) = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$, then instead of

$$\sum^{!} F(x_1, x_2, ..., x_n)$$

we shall write also

$$T[a_1, a_2, ..., a_n](x_1, x_2, ..., x_n)$$

or just $T[a_1, a_2, ..., a_n]$ if it is clear which is the sequence x used here.

Using the technique given in [3] for Muirhead's theorem, we find the following inequality:

Proposition 3. *If* $a << b$ *and* $y_i, z_i \in \mathbf{C}, \; y_i, z_i \neq 0, \; i \in \{1, .., n\}$ *then*

$$\sum^{!} f_A(|y_1|^{a_1}|y_2|^{a_2}...|y_n|^{a_n}) g_A(|z_1|^{a_1}|z_2|^{a_2}...|z_n|^{a_n}) \leq$$

$$\leq \sum^{!} f_A(|y_1|^{b_1}|y_2|^{b_2}...|y_n|^{b_n}) g_A(|z_1|^{b_1}|z_2|^{b_2}...|z_n|^{b_n}),$$

where $f, \; g, \; f_A$ *and* g_A *are as in Theorem 1,* $|y_{\sigma(1)}|^{a_1}|y_{\sigma(2)}|^{a_2}...|y_{\sigma(n)}|^{a_n}$, $|z_{\sigma(1)}|^{b_1}|z_{\sigma(2)}|^{b_2}...|z_{\sigma(n)}|^{b_n} \in D(0, R)$ *for any* σ, σ *being an arbitrary permutation of the numbers* $\{1, 2, ..., n\}$.

Proof. We consider in Muirhead's inequality instead of x_i, $|y_i|^j|z_i|^k, \; i \in \{1, 2, ..., n\}$ we multiply by $|p_j||q_k|$, and summing over $j, k \in \{0, ..., m\}$ we get the desired inequality when m tends to infinity. ∎

Remark 2. *(a) As in* [4]*, there exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that* $Li_n(z)$, ${}_2F_1(a,b;c;z)$, $J_a(z)$ *and* $I_a(z)$ *are power series with real coefficients and convergent on the open disk* $D(0,1)$*. Therefore, like in* [4]*, we can think to rewrite the inequalities given before under conditions from our theorems.*

(b) In addition, as in [4]*, because the functions* $\exp(z)$, $z \in \mathbf{C}$, $\frac{1}{1-z}$, $z \in D(0,1)$, $\ln(\frac{1}{1-z})$, $z \in D(0,1)$, $\sinh(z)$, $z \in \mathbf{C}$ *are power series with real coefficients and convergent on the open disk* $D(0,1)$ *we can think to rewrite the inequalities given before under conditions from our theorems.*

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