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INEQUALITIES FOR POWER SERIES WITH POSITIVE COEFFICIENTS

LOREDANA CIURDARIU AND NICUŞOR MINCULETE

ABSTRACT. In this paper we use a technique given by Ibrahim, Dragomir and Mortici, in order to prove and enunciate several inequalities starting from some classical inequalities.

We present an improvement of Nesbitt's inequality and also a reverse of Nesbitt's inequality. Other important results which appear in the paper are some generalizations of well-known inequalities obtained by convergent power series with positive coefficients.

1. Introduction

In [8], Ibrahim and Dragomir found some inequalities for power series via Buzano's result and some applications for several fundamental complex functions.

Ibrahim, Dragomir and Darus established in [9] some inequalities for power series with real coefficients by utilizing Young's inequality for sequences of complex numbers.

In [15], Mortici used the technique, by power series, for proving the well-known Nesbitt's inequality

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2},\ a,\ b,\ c>0,$$

which is equivalent to inequality,

$$\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}\geq\frac{3}{2},\ a,\ b,\ c>0,$$

where $a+b+c=1$.

In demonstration, he used Jensen's inequality for the convex function $g(x)=x^n$ and geometric series,

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\ |x|<1.$$

It is easy to see that

$$\frac{x}{1-x}=\sum_{n=1}^{\infty}x^n,\ |x|<1.$$

In Theorem 1 are presented two inequalities which are used in Corollary 2 for an improvement of Nesbitt's inequality. Also another reverse of Nesbit's inequality is given in Corollary 3.

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Other important results which appear in this paper are some generalizations for convergent power series with positive coefficients of complements of Cauchy's inequality given by (7) and (8) starting from technique introduced in [15]. These results were obtained in Theorems 5 and 6 and then in Theorem 7 a generalization of inequalities (9) and (10) from Theorem 6, see [13] was given. By a similary technique we can also find a variant of reverse inequality of Young for functions which are sums of power series with positive coefficients in Proposition 3, using a result from [6].

In addition, in Theorem 11 is given a variant of a generalization of the Cauchy-Schwarz inequality from [2] in the case of the convergent power series with positive coefficients. As applications, in Remark 2 and Remark 4 we can see what would become several refinements of Radon's inequality from [16] and the classical Cauchy's and Holder's inequalities for such functions.

2. **Main results**

Theorem 1. *For any* $a \geq b \geq c > 0$ *and* $a+b+c=1$, *there is the inequality*

$$\frac{1}{(1-c)^3}(a^2+b^2+c^2-\frac{1}{3}) \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq$$

$$\leq \frac{1}{(1-a)^3}(a^2+b^2+c^2-\frac{1}{3}). \tag{1}$$

Proof. According to [7], if $g: I \to \mathbb{R}$ is a twice differentiable function such that there exist real constants γ and Γ so that $0 \leq \gamma \leq g^{''}(x) \leq \Gamma$ for any $\in I$ we find the inequality

$$\frac{\gamma}{2}\sum_{j=1}^{3}p_j\left(x_j-\sum_{i=1}^{3}p_ix_i\right)^2 \leq \sum_{i=1}^{3}p_ig(x_i)-g\left(\sum_{i=1}^{3}p_ix_i\right) \leq \frac{\Gamma}{2}\sum_{j=1}^{3}p_j\left(x_j-\sum_{i=1}^{3}p_ix_i\right)^2,$$

where $p_i > 0$ for all $i \in \{1,2,3\}$ and $\sum_{i=1}^{3} p_i = 1$.

Since $a \geq b \geq c > 0$ and the function $g(x) = x^n,\ n \geq 2$, is convex and $p_1 = p_2 = p_3 = \frac{1}{3}$, implies

$$0 \leq \gamma = g^{''}(c) = n(n-1)c^{n-2} \leq g^{''}(x) \leq \Gamma = g^{''}(a) = n(n-1)a^{n-2}.$$

Therefore we have the following inequality

$$\frac{n(n-1)c^{n-2}}{6}\sum_{cyclic}\left(a-\frac{a+b+c}{3}\right)^2 \leq \frac{a^n+b^n+c^n}{3}-\left(\frac{a+b+c}{3}\right)^n \leq$$

$$\leq \frac{n(n-1)a^{n-2}}{6}\sum_{cyclic}\left(a-\frac{a+b+c}{3}\right)^2.$$

Because $a,\ b,\ c > 0$ and $a+b+c=1$, we deduce the inequality

$$\frac{n(n-1)c^{n-2}}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2 \leq a^n+b^n+c^n-3\left(\frac{1}{3}\right)^n \leq \frac{n(n-1)a^{n-2}}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2.$$

By passing to power series we obtain

$$\frac{1}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\sum_{n=1}^{\infty}n(n-1)c^{n-2}\le\sum_{n=1}^{\infty}a^n+\sum_{n=1}^{\infty}b^n+\sum_{n=1}^{\infty}c^n-3\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\le$$

$$\le\frac{1}{2}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\sum_{n=1}^{\infty}n(n-1)a^{n-2}.$$

But, we know the power series

$$\frac{x}{1-x}=\sum_{n=1}^{\infty}x^n,$$

and

$$\frac{2}{(1-x)^3}=\sum_{n=1}^{\infty}n(n-1)x^n,\ |x|<1.$$

Therefore, the above inequality becomes

$$\frac{1}{(1-c)^3}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2\le\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2}\le\frac{1}{(1-a)^3}\sum_{cyclic}\left(a-\frac{1}{3}\right)^2,$$

which is equivalent to the inequality

$$\frac{1}{(1-c)^3}\left(a^2+b^2+c^2-\frac{1}{3}\right)\le\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2}\le$$

$$\le\frac{1}{(1-a)^3}\left(a^2+b^2+c^2-\frac{1}{3}\right).$$

∎

The below inequality represents an improvement of Nesbitt's inequality.

Corollary 1. *For any* $a\ge b\ge c>0$, *there is the inequality*

$$\frac{a+b+c}{3(a+b)^3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\le$$

$$\le\frac{a+b+c}{3(b+c)^3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]. \tag{2}$$

Proof. In Theorem 1 we assume, without loss of generality, that $a+b+c=1$. By replacement in inequality (2), we deduce inequality (1). Therefore, the requirement is true. ∎

Another reverse inequality of Nesbitt's inequality is the following:

Corollary 2. *For any* $a\ge b\ge c>0$, *there is the inequality*

$$0\le\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}-\frac{3}{2}\le 3\left(\frac{a}{b+c}+\frac{c}{a+b}-2\frac{a+c}{a+2b+c}\right). \tag{3}$$

Proof. We assume, without loss of generality, that $a+b+c=1$. By replacement in inequality (3), we deduce inequality

$$0 \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq 3\left(\frac{a}{1-a}+\frac{c}{1-c}-2\frac{a+c}{2-a-c}\right).$$

Simić showed in [17], that if $(x_i)_{i=1}^n \in [a,b]^n$, then

$$0 \leq \sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right) \leq g(a)+g(b)-2g\left(\frac{a+b}{2}\right),$$

where $p_i > 0$ for all $i \in \{1,...,n\}$ to see that $\sum_{i=1}^{n} p_i = 1$.

Since $a \geq b \geq c > 0$ and the function $g(x) = x^n$ is convex, and $p_1 = p_2 = p_3 = \frac{1}{3}$, it follows the inequality

$$0 \leq \frac{a^n+b^n+c^n}{3} - \left(\frac{a+b+c}{3}\right)^n \leq a^n + c^n - 2\left(\frac{a+c}{2}\right)^n,$$

so, by passing to power series, we deduce

$$0 \leq \sum_{n=1}^{\infty} a^n + \sum_{n=1}^{\infty} b^n + \sum_{n=1}^{\infty} c^n - 3\sum_{n=1}^{\infty}\left(\frac{a+b+c}{3}\right)^n \leq 3\left[\sum_{n=1}^{\infty} a^n + \sum_{n=1}^{\infty} c^n - 2\sum_{n=1}^{\infty}\left(\frac{a+c}{2}\right)^n\right],$$

which is equivalent to

$$0 \leq \frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}-\frac{3}{2} \leq 3\left(\frac{a}{1-a}+\frac{c}{1-c}-2\frac{a+c}{2-a-c}\right),$$

where $a,\ b,\ c > 0$ and $a+b+c=1$.

∎

Theorem 2. *Let R be the convergence radius of the power series $\sum_{n=0}^{\infty} a_n x^n$ with positive coefficients, which has the sum $f(x)$ on $(-R, R)$.*

(a) If $a,\ b,\ c \in (-\sqrt{R}, \sqrt{R})$ then there is the inequality

$$f(a^2)+f(b^2)+f(c^2) \geq f(ab)+f(bc)+f(ca). \tag{4}$$

(b) If $x_1,\ x_2, ..., x_n \in (-\sqrt{R}, \sqrt{R})$ then there is the inequality

$$f(x_1^2)+f(x_2^2)+...+f(x_n^2) \geq f(x_1x_2)+f(x_2x_3)+...+f(x_{n-1}x_n)+f(x_nx_1). \tag{5}$$

Proof. (a) If $a,b,c \in (-\sqrt{R}, \sqrt{R})$, then taking into account the well-known inequality, $a^2+b^2+c^2 \geq ab+bc+ca$, which is true for all a,b,c for every $n \in \mathbb{N}$ we have,

$$a^{2n}+b^{2n}+c^{2n} \geq (ab)^n + (bc)^n + (ca)^n$$

and then

$$\sum_{n=1}^{\infty} a_n(a^{2n}+b^{2n}+c^{2n}) \geq \sum_{n=1}^{\infty} a_n((ab)^n + (bc)^n + (ca)^n).$$

Therefore we obtain,

$$f(a^2)+f(b^2)+f(c^2) \geq f(ab)+f(bc)+f(ca),$$

if $a,b,c \in (-\sqrt{R}, \sqrt{R})$, where R is the convergence radius of the power series $\sum_{n=1}^{\infty} a_n x^n$ with positive coefficients, $a_n \geq 0$, which has the sum $f(x)$ on $(-R, R)$.

(b) We use the well-known inequality, $x_1^2+x_2^2+...+x_n^2 \geq x_1x_2+x_2x_3+...+x_{n-1}x_n+x_nx_1$ and we will apply previous method.

$\blacksquare$

Remark 1. *(a) We can notice that*

$$\sum_{n=1}^{\infty} na_n x^n = x\sum_{n=1}^{\infty} na_n x^{n-1} = xf^{'}(x).$$

Multiplying this time by $na_n \geq 0$ the inequality

$$a^{2n} + b^{2n} + c^{2n} \geq (ab)^n + (bc)^n + (ca)^n,$$

we have,

$$na_n a^{2n} + na_n b^{2n} + na_n c^{2n} \geq na_n(ab)^n + na_n(bc)^n + na_n(ca)^n$$

for all $n \in \mathbb{N}^*$ and by addition we obtain,

$$\sum_{i=1}^{\infty}(na_n a^{2n} + na_n b^{2n} + na_n c^{2n}) \geq \sum_{n=1}^{\infty}(na_n(ab)^n + na_n(bc)^n + na_n(ca)^n)$$

which means

$$a^2 f^{'}(a^2) + b^2 f^{'}(b^2) + c^2 f^{'}(c^2) \geq abf^{'}(ab) + bcf^{'}(bc) + caf^{'}(ca)$$

for all $a, b, c \in (-\sqrt{R}, \sqrt{R})$, where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $f(x) = \sum_{n=1}^{\infty} a_n x^n,\ a_n \geq 0,\ (\forall)\ n \in \mathbb{N}^*$, f being a differentiable mapping on $(-R, R)$.

(b) By applying the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, twice we obtain,

$$a^{4n} + b^{4n} + c^{4n} \geq (ab)^{2n} + (bc)^{2n} + (ca)^{2n} \geq (a^2bc)^n + (ab^2c)^n + (abc^2)^n.$$

Using the summation method with respect to n again, we get

$$a_n(a^{4n} + b^{4n} + c^{4n}) \geq a_n((a^2bc)^n + (ab^2c)^n + (abc^2)^n)$$

and therefore

$$f(a^4) + f(b^4) + f(c^4) \geq f(a^2bc) + f(ab^2c) + f(abc^2),$$

where $a, b, c \in (-\sqrt{R}, \sqrt{R})$, R is the convergence radius of the power series $\sum_{n=1}^{\infty} a_n x^n$ with positive coefficients and the sum $f(x)$, with $a\ b,\ c \in (-R^{\frac{1}{4}}, R^{\frac{1}{4}})$.

As in [15], if we multiply the inequality,

$$a^{4n} + b^{4n} + c^{4n} \geq (a^2bc)^n + (ab^2c)^n + (abc^2)^n$$

by $na_n \geq 0$ before summing, we will find

$$a^4 f^{'}(a^4) + b^4 f^{'}(b^4) + c^4 f^{'}(c^4) \geq a^2bcf^{'}(a^2bc) + ab^2cf^{'}(ab^2c) + abc^2 f^{'}(abc^2),$$

for all $a\ b,\ c \in (-R^{\frac{1}{4}}, R^{\frac{1}{4}})$ where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $f(x) = \sum_{n=1}^{\infty} a_n x^n$ on $(-R, R)$, f being a differentiable mapping on $(-R, R)$.

Now, like in [15], we use this method in order to use results related to convexity.

Proposition 1. *Let $f : [0, \infty) \to \mathbb{R}$ is any increasing and concave function, $0 < x \leq y \leq z$, and n is a positive integer. If $g(x) = \sum_{n=1}^{\infty} a_n x^n$ with the convergence radius $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $a_n \geq 0,\ (\forall)\ n \in \mathbb{N}^*$, then the following inequalities hold:*

$$(a) \qquad g(z)f(y) + g(y)f(x) + g(x)f(z) \geq g(x)f(y) + g(z)f(x) + g(y)f(z);$$

(b) $zg^{'}(z)f(y)+yg^{'}(y)f(x)+xg^{'}(x)f(z)\geq xg^{'}(x)f(y)+zg^{'}(z)f(x)+yg^{'}(y)f(z)$,

if in addition, g is a differentiable mapping on $(-R,R)$.

Proof. In [10], the author showed that

$$(z^n-x^n)f(y)\geq(z^n-y^n)f(x)+(y^n-x^n)f(z),$$

where $f:[0,\infty)\to\mathbb{R}$ is any increasing and concave function, $0<x\leq y\leq z$, and n is a positive integer.

Multiplying by $a_n\geq 0$ last inequality for $n=1,2,...$ with $0<x\leq y\leq z<R$, we get

$$(g(z)-g(x))f(y)\geq(g(z)-g(y))f(x)+(g(y)-g(x))f(z),$$

or

$$g(z)f(y)+g(y)f(x)+g(x)f(z)\geq g(x)f(y)+g(z)f(x)+g(y)f(z),$$

where $g(x)=\sum_{n=1}^{\infty}a_nx^n$ with the convergence radius $R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ and $a_n\geq 0$, $(\forall)\ n\in\mathbb{N}^*$.

Now if we multiply the inequality given in [10] by $na_n\geq 0$ for $n=1,2,...$ and then summing with respect to n, we have

$$(zg^{'}(z)-xg^{'}(x))f(y)\geq(zg^{'}(z)-yg^{'}(y))f(x)+(yg^{'}(y)-xg^{'}(x))f(z)$$

or

$$zg^{'}(z)f(y)+yg^{'}(y)f(x)+xg^{'}(x)f(z)\geq xg^{'}(x)f(y)+zg^{'}(z)f(x)+yg^{'}(y)f(z),$$

where $g(x)=\sum_{n=1}^{\infty}a_nx^n$ with the convergence radius $R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ and $a_n\geq 0$, $(\forall)\ n\in\mathbb{N}^*$, g being a differentiable mapping on $(-R,R)$.

∎

It is known the Schur's inequality, see [11].

If x,y,z are non-negative real numbers and t is a positive number then

$$x^t(x-y)(x-z)+y^t(y-z)(y-x)+z^t(z-x)(z-y)\geq 0,$$

with inequality if and only if $x=y=z$ or two of them are equal and the other is zero. When t is an even positive integer, the inequality holds for all real numbers x,y and z.

According to [18], a generalization of Schur's inequality is the following: Suppose a,b,c are positive real numbers. If the triples (a,b,c) and (x,y,z) are similarly sorted, then the following inequality holds:

$$a(x-y)(x-z)+b(y-z)(y-x)+c(z-x)(z-y)\geq 0.$$

Moreover, according to [18], in 2007 Vornicu showed that a yet further generalized form of Schur's inequality holds:

Consider $a,b,c,x,y,z\in\mathbb{R}$, where $a\geq b\geq c$ and either $x\geq y\geq z$ or $z\geq y\geq x$. Let $k\in\mathbb{Z}^+$ and let $f:\mathbb{R}\to\mathbb{R}_0^+$ be either convex or monotonic. Then,

$$f(x)(a-b)^k(a-c)^k+f(y)(b-a)^k(b-c)^k+f(z)(c-a)^k(c-b)^k\geq 0.$$

Proposition 2. *Under previous conditions, if g is the sum of the power series $\sum_{k=1}^{\infty} a_k x^k$ with the convergence radius $R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}}$, $a_k \geq 0$, $(\forall)\ k \in \mathbb{N}^*$ and $a - c < \sqrt{R}$ then we have:*

$$(a) \quad f(x)g((a-b)(a-c)) + f(y)g((b-a)(b-c)) + f(z)g((c-a)(c-b)) \geq 0,$$

and

$$(a-b)(a-c)f(x)g^{'}((a-b)(a-c)) + (b-a)(b-c)f(y)g^{'}((b-a)(b-c))+$$

$$(b) \qquad +(c-a)(c-b)f(z)g^{'}((c-a)(c-b)) \geq 0,$$

when g is a differentiable mapping on $(-R, R)$.

Proof. We multiply the previous inequality by $a_k \geq 0$ for $k = 1, 2, ...$ and then summing with respect to k we get,

$$f(x)g((a-b)(a-c)) + f(y)g((b-a)(b-c)) + f(z)g((c-a)(c-b)) \geq 0,$$

where $g(x) = \sum_{k=1}^{\infty} a_k x^k$ with the convergence radius $R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}}$, $a_k \geq 0$, $(\forall)\ k \in \mathbb{N}^*$ and $a - c < \sqrt{R}$.

In addition, if we multiply the inequality by $ka_k \geq 0$ for $k = 1, 2, ...$ and then summing with respect to k we get,

$$(a-b)(a-c)f(x)g^{'}((a-b)(a-c)) + (b-a)(b-c)f(y)g^{'}((b-a)(b-c))+$$

$$+(c-a)(c-b)f(z)g^{'}((c-a)(c-b)) \geq 0,$$

where $g(x) = \sum_{k=1}^{\infty} a_k x^k$ with the convergence radius $R = \lim_{k\to\infty} \frac{a_k}{a_{k+1}}$, $a_k \geq 0$, $(\forall)\ k \in \mathbb{N}^*$ and $a - c < \sqrt{R}$, g being a differentiable mapping on $(-R, R)$.

∎

Also, in [6], in Corollary 2.2, (ii) the authors presented as an alternative reverse inequality for Young's inequality the following inequality, the difference-type reverse inequality:

Corollary 3. *([6]) For $a, b > 0$ and $\lambda \in [0, 1]$, the following inequalities hold:*

(i) Ratio-type revese inequality

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^{\lambda} \exp\{\frac{\lambda 1-\lambda)(a-b)^2}{d_1^2}\},$$

where $d_1 = \min\{a, b\}$.

(ii) Difference-type reverse inequality

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^{\lambda} + \lambda(1-\lambda)log^2\left(\frac{a}{b}\right)d_2,$$

where $d_2 = \max\{a, b\}$.

Using previous result, see [6], we can find the following inequality for functions which are sums of power series with positive coefficients.

Proposition 3. *Let R be the convergence radius of the power series $\sum_{n=1}^{\infty} a_n x^n$, with $a_n \geq 0$ for all $n \in \mathbb{N}^*$ which is convergent and has the sum $f(x)$, when $x \in (-R, R)$, where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $R \neq 0$,.*

For $a, b \in (0, R)$ and $\lambda \in [0, 1]$, the following inequality holds:

$$f(a^{1-\lambda}b^{\lambda}) \leq (1-\lambda)f(a) + \lambda f(b) \leq$$

$$\leq f(a^{1-\lambda}b^{\lambda}) + \lambda(1-\lambda)\left[\log\left(\frac{a}{b}\right)\right]^2 \cdot [df^{'}(d) + d^2 f^{''}(d)],$$

where $d = \max\{a, b\}$.

Proof. We can suppose without loss of generality that $d = \max\{a, b\} = a$. Then for every $n \in \mathbb{N}^*$ we see that $d^n = \max\{a^n, b^n\}$. By replacing a with a^n, b with b^n and d with d^n in $a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^{\lambda} + \lambda(1-\lambda)\left[log\left(\frac{a}{b}\right)\right]^2 d$ and then multiplying by $a_n \geq 0$ for every $n \in \mathbb{N}^*$ we get

$$a_n a^{n(1-\lambda)}b^{n\lambda} \leq a_n(1-\lambda)a^n + a_n\lambda b^n \leq a_n a^{n(1-\lambda)}b^{n\lambda} + a_n\lambda(1-\lambda)\left[log\left(\frac{a^n}{b^n}\right)\right]^2 d^n,$$

for every $n \in \mathbb{N}^*$. Then by adding previous inequalities when $n \in \{1, 2, ...m\}$ and $m \in \mathbb{N}^*$ we obtain,

$$\sum_{n=1}^{m} a_n[a^{(1-\lambda)}b^{\lambda}]^n \leq \sum_{n=1}^{m} a_n(1-\lambda)a^n + \sum_{n=1}^{m} a_n\lambda b^n \leq$$

$$\leq \sum_{n=1}^{m} a_n[a^{(1-\lambda)}b^{\lambda}]^n + \lambda(1-\lambda)\left[log\left(\frac{a}{b}\right)\right]^2 \sum_{n=1}^{m} a_n n^2 d^n.$$

When m tends to infinity we have

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b \leq a^{1-\lambda}b^{\lambda} + \lambda(1-\lambda)\left[log\left(\frac{a}{b}\right)\right]^2 S(d),$$

because $0 < a < R,\ 0 < b < R,\ 0 < a^{1-\lambda}b^{\lambda} < R$ and $0 < d < R$.

In this case $S(x)$ is the sum of the convergent series $\sum_{n=1}^{\infty} a_n n^2 x^n$ for $x \in (-R, R)$ and is $xf^{'}(x) + x^2 f^{''}(x)$. This series has the same convergence radius as series which has the sum $f(x)$ and the calculus result from the following: $\frac{S(x)}{x} = \sum_{n=1}^{\infty} a_n n^2 x^{n-1}$ and by integration we have, $\int \frac{S(x)}{x}dx = \sum_{n=1}^{\infty} a_n n x^n$ and we denote by $D(x)$, $\int \frac{S(x)}{x}dx$. Again by integation from $\frac{D(x)}{x} = \sum_{n=1}^{\infty} a_n n x^{n-1}$ we get $\int \frac{D(x)}{x}dx = \sum_{n=1}^{\infty} a_n x^n = f(x)$ and then by derivation we obtain the result.
∎

3. Generalizations of several well-known inequalities

We need to recall the following three results which are enunciated in [13]:

Let a and b be two positive n-tuples, with

$$0 < m_1 \leq a_i \leq M_1 \text{ and } 0 < m_2 \leq b_i \leq M_2 \ (i = 1, ..., n) \tag{6}$$

for some constants m_1, m_2, M_1, and M_2.

The following complement of Cauchy's inequality is valid:

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} a_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^{n} a_k b_k. \tag{7}$$

Theorem 3. *(*[13]*) Let u, v be real numbers such that $0 < v \leq u < 1,\ u + v = 1$, and w, a, b be positive n-tuples such that*

$$m \leq \frac{a_k}{b_k} \leq M \ (k = 1, ..., n).$$

Then

$$u \sum_{k=1}^{n} w_k b_k^2 + vMm \sum_{k=1}^{n} w_k a_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k a_k b_k. \tag{8}$$

In the following we will give a generalization of a complement of Cauchy's inequality given in [5] and [13] by J. B. Diaz and F. T. Metcalf for power series with positive coefficients.

Theorem 4. *Let the power series $\sum_{n=1}^{\infty} a_n x^n$, with $a_n \geq 0$ for all $n \in \mathbb{N}^*$ which is convergent and has the sum $f(x)$, when $x \in (-R, R)$, where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $R \neq 0$,.*

(i) If d, b are positive n-tuples which satisfy conditions $0 < m_1 \leq d_i \leq M_1$ and $0 < m_2 \leq b_i \leq M_2$ $(i = \overline{1,n})$ for some constants $m_1,\ m_2,\ M_1$ and M_2 then we have,

$$\sum_{k=1}^{n} f(b_k^2) + \sum_{k=1}^{n} f(\frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2) \leq \sum_{k=1}^{n} f(\frac{M_2}{m_1} d_k b_k) + \sum_{k=1}^{n} f(\frac{m_2}{M_1} d_k b_k).$$

when $M_2^2 \frac{M_1}{m_1} < R$.

(ii) Under previous conditions, if in addition f is a differentiable mapping on $(-R, R)$ we obtain,

$$\sum_{k=1}^{n} b_k^2 f'(b_k^2) + \sum_{k=1}^{n} \frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2 f'(\frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2) \leq$$

$$\leq \sum_{k=1}^{n} \frac{M_2}{m_1} d_k b_k f'(\frac{M_2}{m_1} d_k b_k) + \sum_{k=1}^{n} \frac{m_2}{M_1} d_k b_k f'(\frac{m_2}{M_1} d_k b_k).$$

Proof. (i) Taking into account inequality,

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2}{M_1}\frac{M_2}{m_1} \sum_{k=1}^{n} d_k^2 \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \sum_{k=1}^{n} d_k b_k,$$

see [13], inequality (7), where d_k is replaced by d_k^p, b_k is replaced by b_k^p, m_1 is replaced by m_1^p, M_1 by M_1^p, m_2 by m_2^p and M_2 by M_2^p, $p \in \mathbb{N}^*$ and multiplying by a_p the inequality before summing we obtain

$$\sum_{p=1}^{\infty} a_p \left(\sum_{k=1}^{n} b_k^{2p} + \frac{m_2^p}{M_1^p}\frac{M_2^p}{m_1^p} \sum_{k=1}^{n} d_k^{2p}\right) \leq \sum_{p=1}^{\infty} a_p \left[\left(\frac{M_2^p}{m_1^p} + \frac{m_2^p}{M_1^p}\right) \sum_{k=1}^{n} (d_k b_k)^p\right].$$

Using hypothesis $0 < m_1 \leq d_i \leq M_1$ and $0 < m_2 \leq b_i \leq M_2$ $(i = \overline{1,n})$, when $M_2^2 \frac{M_1}{m_1} < R$ we notice that b_k^2, $\frac{m_2}{M_1}\frac{M_2}{m_1} d_k^2$, $\frac{M_2}{m_1} d_k b_k$ and $\frac{m_2}{M_1} d_k b_k$ are in $(0, R)$ and then the power series being convergent, we obtain the inequality from conclusion.

(ii) The proof will be as in (i), the difference being that the inequality will be obtained by multiplication by pa_p instead of a_p.

∎

An improvement of last theorem, using Theorem 2 from [13], will be also presented below:

Theorem 5. *Let the power series* $\sum_{n=1}^{\infty} a_n x^n$, *with* $a_n \geq 0$ *for all* $n \in \mathbb{N}^*$ *which is convergent and has the sum* $f(x)$, *when* $x \in (-R, R)$, *where* $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ *and* $R \neq 0$,.

Let u, v *be real numbers such that* $0 < v \leq u < 1,\ u+v = 1$ *and* w, d, b *be positive* n*-tuples such that*

$$0 \leq m \leq \frac{b_k}{d_k} \leq M, \ \ \text{and} \ d_k^2 M^2 < R \ (k = 1, ..., n).$$

Then

$$u \sum_{k=1}^{n} w_k f(b_k^2) + v \sum_{k=1}^{n} w_k f(Mmd_k^2) \leq v \sum_{k=1}^{n} w_k f(md_k b_k) + u \sum_{k=1}^{n} w_k f(Md_k b_k).$$

Proof. Using the inequality,

$$u \sum_{k=1}^{n} w_k b_k^2 + vMm \sum_{k=1}^{n} w_k d_k^2 \leq (vm + uM) \sum_{k=1}^{n} w_k d_k b_k,$$

see [13], inequality (8), where d_k is replaced by d_k^p, b_k is replaced by b_k^p, m is replaced by m^p, M by M^p, $p \in \mathbb{N}^*$ and multiplying by a_p the inequality obtained before summing we obtain

$$\sum_{p=1}^{\infty} a_p \left(u \sum_{k=1}^{n} w_k b_k^{2p} + vM^p m^p \sum_{k=1}^{n} w_k d_k^{2p} \right) \leq \sum_{p=1}^{\infty} a_p \left[(vm^p + uM^p) \sum_{k=1}^{n} w_k (d_k b_k)^p \right].$$

By hypothesis, $m \leq \frac{d_k}{b_k} \leq M$, and $d_k^2 M^2 < R$, $(k = \overline{1,n})$ we notice that $b_k^2 < R$, $Mmd_k^2 < R$, $md_k b_k < R$ and $Md_k b_k < R$ $(k = \overline{1,n})$ and thus previous inequality becomes

$$u \sum_{k=1}^{n} w_k f(b_k^2) + v \sum_{k=1}^{n} w_k f(Mmd_k^2) \leq v \sum_{k=1}^{n} w_k f(md_k b_k) + u \sum_{k=1}^{n} w_k f(Md_k b_k).$$

∎

Theorem 6. *(*[13]*) Let* a *and* b *be two positive* n*-tuple,* $p^{-1} + q^{-1} = 1,\ 0 < m < M,\ m \leq \frac{a_i}{b_i^{\frac{q}{p}}} \leq M (i = 1, ..., n),\ p_i \geq 0 (i = 1, ..., n)$. *Then*

(i) if $p > 1$ *(or* $p < 0$*), we have*

$$(9) \qquad (M - m) \sum_{k=1}^{n} p_k a_k^p + (mM^p - Mm^p) \sum_{k=1}^{n} p_k b_k^q \leq (M^p - m^p) \sum_{k=1}^{n} p_k a_k b_k$$

and if $0 < p < 1$, *then reverse inequality in (9) holds.*

(ii) if $p > 1$, *we have*

$$(10) \qquad \left(\sum_{k=1}^{n} p_k a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} p_k b_k^q \right)^{\frac{1}{q}} \leq \lambda \sum_{k=1}^{n} p_k a_k b_k,$$

where

$$(11) \qquad \lambda = |M^p - m^p| |p(M - m)|^{-\frac{1}{p}} |q(Mm^p - mM^p)|^{-\frac{1}{q}},$$

and if $p < 1 (\neq 0)$ *the reverse inequality in (11) holds.*

Theorem 7. *Let d and b be two positive n-tuples, $p^{-1}+q^{-1}=1,\ 0<m<M$, $0\le m\le \frac{d_i}{b_i^{\frac{q}{p}}}\le M,\ \ (i=1,...,n),\ p_i\ge 0,\ (i=1,...,n)$. If $p>1$ then we have,*

$$\sum_{k=1}^{n} p_k[f(Md_k^p)-f(md_k^p)]+\sum_{k=1}^{n} p_k[f(mM^p b_k^q)-f(Mm^p b_k^q)]\le$$

$$\le \sum_{k=1}^{n} p_k[f(M^p d_k b_k)-f(m^p d_k b_k)],$$

where $f(x)$ is the sum of the power series $\sum_{n=1}^{\infty} a_n x^n$, with $a_n\ge 0$ for all $n\in\mathbb{N}^*$ which is convergent when $x\in(-R,R)$, where $R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$, $R\neq 0$ and $M^{p+1}b_i^q<R,\ (i=1,...,n)$.

If $0<p<1$, the reverse inequality holds.

Proof. This time we will use the inequality (9) from Theorem 5, see [13] where d_k will be replaced by d_k^r, b_k by b_k^r, m by m^r and M by M^r, $r\in\{1,2,...\}$. Then we will multiply by $a_r\ge 0$ the corresponding inequalities and sum, and will obtain,

$$\sum_{r=1}^{\infty} a_r[(M^r-m^r)\sum_{k=1}^{n} p_k d_k^{pr}+(m^r M^{pr}-M^r m^{pr})\sum_{k=1}^{n} p_k b_k^{qr}]\le$$

$$\le \sum_{r=1}^{\infty} a_r[(M^{pr}-m^{pr})\sum_{k=1}^{n} p_k d_k^r b_k^r]$$

Using conditions from hypothesis, $p^{-1}+q^{-1}=1,\ 0<m<M,\ 0\le m\le \frac{d_i}{b_i^{\frac{q}{p}}}\le M,\ \ (i=1,...,n)$ and $M^{p+1}b_i^q<R,\ (i=1,...,n)$ we notice that $0<Md_k^pR,\ 0<md_k^p<R,\ 0<mM^p b_k^q<R,\ 0<Mm^p b_k^q<R,\ 0<M^p d_k b_k<R$ and $0<m^p d_k b_k<R$ and that means the corresponding power series are convergent and thus we obtain the inequality from the conclusion.

∎

Starting from Theorem 2.3, Theorem 2.7 and Theorem 2.9 by the power series method we will obtain the following inequalities. These inequalities can be also obtained from Theorem 2.2, Theorem 2.6 and Theorem 2.8, see [16], if we take $x_i=\frac{a_i}{b_i},\ p_i=\frac{b_i}{\sum_{i=1}^{n} b_i},\ i=\overline{1,n}$.

Remark 2. *Let the power series $\sum_{n=1}^{\infty} a_n x^n$, with $a_n\ge 0$ for all $n\in\mathbb{N}^*$ which is convergent and has the sum $f(x)$, when $x\in(-R,R)$, where $R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ and $R\neq 0$, f being a differentiable mapping on $(-R,R)$ in the case of first inequality.*

If $n\ge 2,\ \{d_1,d_2,...,d_n\},\ a_i\in\mathbb{R}_+$ and $\{b_1,b_2,...,b_n\},\ b_i>0$ for all $i\in\{1,2,...,n\}$ are n real, positive numbers with $0<m\le\frac{d_i}{b_i}\le M<R,\ i\in\{1,...,n\}$ then we have:

(12)

$$\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\cdot\sum_{i=1}^{n} b_i f'\left(\frac{d_i}{b_i}\right)\le\sum_{i=1}^{n}\left[d_i f'\left(\frac{d_i}{b_i}\right)-b_i f\left(\frac{d_i}{b_i}\right)\right]+\sum_{i=1}^{n} b_i f\left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right),$$

$$(13)\quad 0\le\sum_{i=1}^{n} b_i f\left(\frac{d_i}{b_i}\right)-\sum_{i=1}^{n} b_i\cdot f\left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right)\le\frac{M-m}{4}(f'(M)-f'(m))\sum_{i=1}^{n} b_i.$$

(14)
$$\sum_{i=1}^{n} b_i f\left(\frac{d_i}{b_i}\right) - \sum_{i=1}^{n} b_i f\left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right) \geq$$

$$\geq \max_{1\leq i<j\leq n}\left[b_i f\left(\frac{d_i}{b_i}\right) + b_j f\left(\frac{d_j}{b_j}\right) - (b_i+b_j) f\left(\frac{d_i+d_j}{b_i+b_j}\right)\right],$$

(15)
$$\sum_{i=1}^{n} b_i f\left(\frac{d_i}{b_i}\right) - \sum_{i=1}^{n} b_i f\left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right) \leq \left[f(M)+f(m)-2f\left(\frac{M+m}{2}\right)\right]\cdot\left(\sum_{i=1}^{n} b_i\right).$$

(16)
$$0 \leq \sum_{i=1}^{n} b_i f\left(\frac{d_i}{b_i}\right) - \sum_{i=1}^{n} b_i f\left(\frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right) \leq$$

$$\leq \left(\sum_{i=1}^{n} b_i\right) f\left(\frac{(M+m)\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} b_i}\right) - 2f\left(\frac{M+m}{2}\right)\left(\sum_{i=1}^{n} b_i\right) + \sum_{i=1}^{n} b_i f\left(\frac{d_i}{b_i}\right).$$

It is necessary also to enunciate below a result from [1] which will be used then in the proof of the following theorem.

Theorem 8. *([1]) If $n \in \mathbb{N}^* - \{1\}$, $a \in \mathbb{R}_+, b, c, d, x_k \in \mathbb{R}_+^*$, $X_n = \sum_{k=1}^{n} x_k$, $cX_n > d\max_{1\leq k\leq n} x_k$ and $m \in [1,\infty), p \in \mathbb{R}_+^*$, then:*

$$\sum_{k=1}^{n} \frac{(aX_n+bx_k)^m}{(cX_n-dx_k)^p} \geq \frac{(an+b)^m}{(cn-d)^p} n^{p-m+1} X_n^{m-p}.$$

Theorem 9. *Let the power series $\sum_{n=1}^{\infty} a_n x^n$ with $a_n \geq 0$, $(\forall)\ n \in \mathbb{N}^*$ which is convergent and has the sum $f(x)$, when $x \in (-R, R)$, where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $R \neq 0$.*

If $n \in \mathbb{N}^* - \{1\}$, $a, b, c, d, x_k \in \mathbb{R}_+^*$, $k \in \{1, ..., n\}$, $X_n = \sum_{k=1}^{n} x_k$ such that $cX_n > d\max_{1\leq k\leq n} x_k^t$, $p \in \mathbb{R}_+^*$, $an+b<1$ and $x_k < R$, $k \in \{1, ..., n\}$ then:

(17)
$$\sum_{k=1}^{n} \frac{f(aX_n+bx_k)}{(cX_n-dx_k)^p} \geq \frac{n^{p+1}}{(cn-d)^p X_n^p} f\left(\frac{an+b}{n} X_n\right).$$

Proof. Demonstration will result by the same reason like before. ∎

Remark 3. *For example, under the conditions of previous theorems, these inequalities can be stated for the functions like, e^x, $\sinh(x)$, $\cosh(x)$, $\frac{1}{2}\ln\frac{1+x}{1-x}, |x| < 1$ or $\frac{1+x}{4x} + \frac{1-x^2}{2x^2} ln(1-x)$, $x \in (-1,0)\cup(0,1)$.*

4. **Applications**

From classical Cauchy-Schwarz's inequality and Holder's inequality by using the power series method in some particular conditions we can state next inequalities for functions which are sums of power series with positive coefficients:

1. Let the power series $\sum_{n=1}^{\infty} a_n x^n$ with $a_n \geq 0,\ (\forall)\ n \in \mathbb{N}^*$ which is convergent and has the sum $f(x)$, when $x \in (-R, R)$, where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ and $R \neq 0$.

(i) If d, b are real numbers with $|d| < \sqrt{R}$ and $|b| < \sqrt{R}$ then,

$$f^2(db) < f(d^2)f(b^2).$$

(ii) If d, b are positive numbers with $d^p < R,\ b^q < R$ and $p > 1,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$ then

$$f(db) \leq f^{\frac{1}{p}}(d^p) f^{\frac{1}{q}}(b^q).$$

(iii) If $a, b, c \in R_+$ with $a, b, c < \frac{R}{2}$ then

$$f(a+b) + f(b+c) + f(c+a) \leq f(2a) + f(2b) + f(2c);$$

(iv) If $x_1, x_2, ..., x_n \in R_+$ and $x_i < R^{\frac{1}{n+1}},\ i \in \{1, ..., n\}$ where $n \in N^*$ then we have:

$$f(x_1^2 x_2 ... x_n) + f(x_1 x_2^2 ... x_n) + ... + f(x_1 x_2 ... x_n^2) \leq f(x_1^{n+1}) + f(x_2^{n+1}) + ... + f(x_n^{n+1});$$

(v) If $a, b, c \in R_+$ with $\left(\frac{a}{b}\right)^2 < R,\ \left(\frac{b}{c}\right)^2 < R,\ \left(\frac{c}{a}\right)^2 < R,\ \left|\frac{a}{c}\right| < R,\ \left|\frac{b}{a}\right| < R,\ \left|\frac{c}{b}\right| < R$, then we have

$$f(\frac{a^2}{b^2}) + f(\frac{b^2}{c^2}) + f(\frac{c^2}{a^2}) \geq f(\frac{a}{c}) + f(\frac{b}{a}) + f(\frac{c}{b}).$$

Proof. (i) We take $d_k = \sqrt{a_k} d^k$ and $b_k = \sqrt{a_k} b^k,\ k \in \{1, ..., n\}$ in Cauchy's inequality,

$$(\sum_{k=1}^{n} d_k b_k)^2 \leq (\sum_{k=1}^{n} d_k^2)(\sum_{k=1}^{n} b_k^2),$$

we get

$$(\sum_{k=1}^{n} a_k d^k b^k)^2 \leq \sum_{k=1}^{n} a_k d^{2k} \cdot \sum_{k=1}^{n} a_k b^{2k}.$$

When $n \to \infty$ we obtain,

$$f^2(db) < f(d^2)f(b^2)$$

taking into account that $|db| < R, |d^2| < R$ and $|b^2| < R$.

(ii) We take $d_k = a_k^{\frac{1}{p}} d^k$ and $b_k = a_k^{\frac{1}{q}} b^k,\ k \in \{1, ..., n\}$ in Holder's inequality,

$$\sum_{k=1}^{n} d_k b_k \leq (\sum_{k=1}^{n} d_k^p)^{\frac{1}{p}} (\sum_{k=1}^{n} b_k^q)^{\frac{1}{q}},$$

when $n \to \infty$ we obtain,

$$f(db) < f^{\frac{1}{p}}(d^p) f^{\frac{1}{q}}(b^q)$$

taking into account that $db < R, d^p < R$ and $b^q < R$.

For (iii) we use the elementary inequality,

$$(a+b)^m + (b+c)^m + (c+a)^m \leq 2^m(a^m + b^m + c^m)$$

when $a, b, c \in R_+,\ m \in N^*$.

(iv) We use the inequality,

$$\left(\sum_{i=1}^{n} x_i\right) x_1 ... x_n \leq \sum_{i=1}^{n} x_i^{n+1},$$

for $x_i > 0,\ i \in \{1, ..., n\},\ n \in N^*$.

(v) The inequality result from the elementary inequality,

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{c} + \frac{b}{a} + \frac{c}{b},$$

when $a, b, c \in R^*$.

▮

2. We know the following power series:

$$\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} \frac{1}{n} x^n,$$

for any $|x| < 1$.

Therefore, if we take the function $f(x) = \log\left(\frac{1}{1-x}\right)$, then using inequality (4), we obtain the following inequality:

$$(1 - ab)(1 - bc)(1 - ca) \geq (1 - a^2)(1 - b^2)(1 - c^2), \tag{18}$$

for every $a, b, c \in (-1, 1)$.

This inequality is equivalent to the inequality

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq ab + bc + ca + a^2b^2 + b^2c^2 + c^2a^2, \tag{19}$$

for every $a, b, c \in (-1, 1)$, which implies the following inequality:

$$a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{1}{2}[a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2], \tag{20}$$

for every $a, b, c \in (-1, 1)$.

Inequality (20) proved the inequality

$$a^2 + b^2 + c^2 \geq ab + bc + ca + \frac{a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2}{2(a + b + c)^2}, \tag{21}$$

for every $a, b, c > 0$, which implies the following inequality

$$3(a^2 + b^2 + c^2) \geq (a + b + c)^2 + \frac{a^2(b - c)^2 + b^2(c - a)^2 + c^2(a - b)^2}{(a + b + c)^2}, \tag{21}$$

for any $a, b, c > 0$.

Remark 4. *As in* [9]*, there exist some inequalities for special functions such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind. It is known that* $Li_n(z)$, ${}_2F_1(a, b; c; z)$, $J_a(z)$ *and* $I_a(z)$ *are power series with real coefficients and convergent on the open disk* $D(0, 1)$*. Therefore, like in* [9]*, we can think to rewrite the inequalities given before under conditions from our theorems.*

Corollary 4. *If $Li_n(z)$ is the polylogarithm function, then we have*

$$Li_n(a^{1-\lambda}b^{\lambda}) \leq (1-\lambda)Li_n(a) + \lambda Li_n(b) \leq$$
$$\leq Li_n(a^{1-\lambda}b^{\lambda}) + \lambda(1-\lambda)\left[\log\frac{a}{b}\right]^2\left[dLi_n^{'}(d) + d^2 Li_n^{''}(d)\right],$$

where $d = \max\{a,b\}$, $a,b \in D(0,1)$, $\lambda \in [0,1]$.

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Loredana Ciurdariu: Department of Mathematics, "Politehnica" University of Timişoara, P-ta. Victoriei, No. 2, 300006-Timişoara

Nicuşor Minculete: Transilvania University of Braşov, Street Iuliu Maniu 50, Braşov, 500091, Romania. E-mail: minculeten@yahoo.com