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SOME GRÜSS' TYPE INEQUALITIES FOR TRACE OF OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of Grüss' type for trace of operators in Hilbert spaces, under suitable assumptions for the involved operators, are given.

1. Introduction

In 1935, G. Grüss [31] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [38, Chapter X] established the following discrete version of Grüss' inequality:

Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the book [38].

For other related results see the papers [1]-[3], [8]-[10], [11]-[13], [17]-[24], [29], [40], [50] and the references therein.

In [18], in order to generalize the above result in abstract structures the author has proved the following Grüss' type inequality in real or complex inner product spaces.

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Theorem 1 (Dragomir, 1999, [18])**.** *Let* $(H, \langle ., . \rangle)$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ *and* $e \in H, \|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.4}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \, |\Gamma - \gamma| . \tag{1.5}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [21] and the references therein.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(\operatorname{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\operatorname{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

For any $f, g \in C(\operatorname{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t\in \operatorname{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \operatorname{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(\operatorname{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\operatorname{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \operatorname{Sp}(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\operatorname{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \operatorname{Sp}(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

In the recent paper [26], we obtained amongst other the following refinement of the Grüss inequality:

Theorem 2 (Dragomir, 2009, [26])**.** *Let* A *be a selfadjoint operator on the Hilbert space* $(H; \langle ., . \rangle)$ *and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$ *and* $\gamma := \min_{t\in[m,M]} f(t)$ *and* $\Gamma := \max_{t\in[m,M]} f(t)$ *then*

$$\begin{aligned} &|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle| \\ &\leq \frac{1}{2}(\Gamma - \gamma)\left[\|g(A) x\|^2 - \langle g(A) x, x\rangle^2\right]^{1/2} \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta) \end{aligned} \tag{1.6}$$

for each $x \in H$ *with* $\|x\| = 1$, *where* $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$.

In order to state some Grüss' type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

2. Some Facts on Trace of Operators

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 3. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If $A\in\mathcal{B}(H)$, then the following are equivalent:*
(i) $A\in\mathcal{B}_1(H)$;
(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$;
(ii) A (or $|A|$) is the product of two elements of $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 4. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{2.8}$$

for any $A\in\mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\left\{\langle A,B\rangle_2 \mid B\in\mathcal{B}_2(H),\ \|B\|\le 1\right\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ is a Banach space.
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H)\cong K(H)^* \text{ and } \mathcal{B}_1(H)^*\cong\mathcal{B}(H),$$

where $K(H)^*$ is the dual space of $K(H)$ and $\mathcal{B}_1(H)^*$ is the dual space of $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 5. *We have*
(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\|=1$;
(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A,B\rangle_2=\operatorname{tr}(B^*A)=\operatorname{tr}(AB^*)\ \text{ and }\ \|A\|_2^2=\operatorname{tr}(A^*A)=\operatorname{tr}\left(|A|^2\right)$$

for any $A,\ B\in\mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [42]

$$|\operatorname{tr}(AB)|\le\operatorname{tr}(|AB|)\le\left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha}\tag{2.12}$$

where $\alpha\in(0,1)$ and $A,B\in\mathcal{B}(H)$ with $|A|^{1/\alpha},|B|^{1/(1-\alpha)}\in\mathcal{B}_1(H)$.

In particular, for $\alpha=\frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)|\le\operatorname{tr}(|AB|)\le\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2}\tag{2.13}$$

with $A,\ B\in\mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [45].

For some classical trace inequalities see [14], [16], [39] and [49], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [30], [33], [34], [36], [43] and [46].

3. Some Grüss' Type Trace Inequalities

We denote by $\mathcal{B}_1^+(H):=\{P:\ P\in\mathcal{B}_1(H)\text{ and }P\ge0\}$.

We have the following result:

Theorem 6. *For any $A,\ C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\&\le\inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&\le\inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot1_H\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}\tag{3.1}$$

where $\|\cdot\|$ is the operator norm.

Proof. We observe that, for any $\lambda\in\mathbb{C}$ we have

$$\begin{aligned}&\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[P(A-\lambda1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\\&=\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[PA\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\\&-\frac{\lambda}{\operatorname{tr}(P)}\operatorname{tr}\left[P\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\\&=\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}.\end{aligned}\tag{3.2}$$

Taking the modulus in (3.2) and utilizing the properties of the trace, we have

$$\begin{aligned}(3.3)\quad & \left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ & =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\right| \\ & =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right]\right| \\ & \le \|A-\lambda 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\end{aligned}$$

for any $\lambda\in\mathbb{C}$, where for the last inequality we used the inequality (2.11).

Utilising Schwarz's inequality (2.13) we also have

$$\begin{aligned}(3.4)\quad & \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ & =\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}P^{1/2}\right|\right) \\ & \le\left[\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\right]^{1/2}[\operatorname{tr}(P)]^{1/2}.\end{aligned}$$

Observe that

$$\begin{aligned}(3.5)\quad & \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right) \\ & =\operatorname{tr}\left(\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right) \\ & =\operatorname{tr}\left(P^{1/2}\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right) \\ & =\operatorname{tr}\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right) \\ & =\operatorname{tr}\left(\left(C^*-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right) \\ & =\operatorname{tr}\left[\left(|C|^2-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}C^*+\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 1_H\right)P\right] \\ & =\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)\operatorname{tr}(P).\end{aligned}$$

By (3.4) and (3.5) we get

$$\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\le\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}\operatorname{tr}(P)$$

and by (3.3) we have

$$\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{3.6}$$
$$\leq \|A-\lambda\cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leq \|A-\lambda\cdot 1_H\| \left(\frac{\operatorname{tr}\left(|C|^2 P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}$$

for any $\lambda\in\mathbb{C}$.

Taking the infimum over $\lambda\in\mathbb{C}$ in (3.6) we get the desired result (3.1). □

Corollary 1. *For any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$0\leq \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \tag{3.7}$$
$$\leq \inf_{\mu\in\mathbb{C}}\|C-\mu\cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leq \inf_{\mu\in\mathbb{C}}\|C-\mu\cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}.$$

In particular, we have

$$0\leq \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \leq \inf_{\mu\in\mathbb{C}}\|C-\mu\cdot 1_H\|^2. \tag{3.8}$$

Proof. If we take in (3.1) $A=C^*$ then we get

$$\left|\frac{\operatorname{tr}(PC^*C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PC^*)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|$$
$$\leq \inf_{\lambda\in\mathbb{C}}\|C^*-\lambda\cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leq \inf_{\lambda\in\mathbb{C}}\|C^*-\lambda\cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},$$

which is clearly equivalent to (3.7).

The inequality (3.8) follows from the inequality between the second and fourth term in (3.7). □

Corollary 2. *For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\left| \frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \right| \tag{3.9}$$
$$\leq \inf_{\lambda \in \mathbb{C}} \|C - \lambda \cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right)$$
$$\leq \inf_{\lambda \in \mathbb{C}} \|C - \lambda \cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}.$$

Following [27], for the complex numbers α, β and the bounded linear operator T we define the following transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \overline{\alpha} I)(\beta I - T),$$

where by T^* we denote the adjoint of T.

We list some properties of the transform $\mathcal{C}_{\alpha,\beta}(\cdot)$ that are useful in the following:

(i) For any $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ we have:

$$\mathcal{C}_{\alpha,\beta}(I) = (1 - \overline{\alpha})(\beta - 1) I, \qquad \mathcal{C}_{\alpha,\alpha}(T) = -(\alpha I - T)^*(\alpha I - T),$$
$$\mathcal{C}_{\alpha,\beta}(\gamma T) = |\gamma|^2 \mathcal{C}_{\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}}(T) \quad \text{for each } \gamma \in \mathbb{C} \setminus \{0\},$$
$$[\mathcal{C}_{\alpha,\beta}(T)]^* = \mathcal{C}_{\beta,\alpha}(T)$$

and

$$\mathcal{C}_{\overline{\beta},\overline{\alpha}}(T^*) - \mathcal{C}_{\alpha,\beta}(T) = T^*T - TT^*.$$

(ii) The operator $T \in B(H)$ is normal if and only if $\mathcal{C}_{\overline{\beta},\overline{\alpha}}(T^*) = \mathcal{C}_{\alpha,\beta}(T)$ for each $\alpha, \beta \in \mathbb{C}$.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y \rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\operatorname{Re}\langle \mathcal{C}_{\alpha,\beta}(T) x, x \rangle = \operatorname{Re}\langle \mathcal{C}_{\beta,\alpha}(T) x, x \rangle \tag{3.10}$$
$$= \frac{1}{4}|\beta - \alpha|^2 - \left\| \left(T - \frac{\alpha + \beta}{2} I \right) x \right\|^2$$

that holds for any scalars α, β and any vector $x \in H$ with $\|x\| = 1$ we can give a simple characterization result that is useful in the following:

Lemma 1. *For $\alpha, \beta \in \mathbb{C}$ and $T \in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}(T)$ (or, equivalently $\mathcal{C}_{\beta,\alpha}(T)$) is accretive;*

(ii) *The transform $\mathcal{C}_{\overline{\alpha},\overline{\beta}}(T^*)$ $\left(\text{or, equivalently } \mathcal{C}_{\overline{\beta},\overline{\alpha}}(T^*)\right)$ is accretive;*

(iii) *We have the norm inequality*

$$\left\| T - \frac{\alpha + \beta}{2} \cdot I \right\| \leq \frac{1}{2}|\beta - \alpha| \tag{3.11}$$

or, equivalently,

$$\left\| T^* - \frac{\bar{\alpha} + \bar{\beta}}{2} \cdot I \right\| \leq \frac{1}{2}|\beta - \alpha|. \tag{3.12}$$

Remark 1. *In order to give examples of operators $T \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $\mathcal{C}_{\alpha,\beta}(T)$ is accretive, it suffices to select a bounded linear operator S and the complex numbers z, w with the property that $\|S - zI\| \leq |w|$ and, by choosing $T = S$, $\alpha = \frac{1}{2}(z + w)$ and $\beta = \frac{1}{2}(z - w)$ we observe that T satisfies (3.11), i.e., $\mathcal{C}_{\alpha,\beta}(T)$ is accretive.*

Corollary 3. *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ such that the transform $\mathcal{C}_{\alpha,\beta}(A)$ is accretive, or, equivalently*

$$\left\| A - \frac{\alpha + \beta}{2} \cdot I \right\| \leq \frac{1}{2} |\beta - \alpha| .$$

For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leq \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leq \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} . \end{aligned} \tag{3.13}$$

In particular, if $C \in \mathcal{B}(H)$ is such that $\mathcal{C}_{\alpha,\beta}(C)$ is accretive, then

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \\ &\leq \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leq \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} \leq \frac{1}{4} |\beta - \alpha|^2 . \end{aligned} \tag{3.14}$$

Also

$$\begin{aligned} &\left| \frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \right| \\ &\leq \frac{1}{2} |\beta - \alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leq \frac{1}{2} |\beta - \alpha| \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2} \leq \frac{1}{4} |\beta - \alpha|^2 . \end{aligned} \tag{3.15}$$

We have the following Grüss type inequality:

Corollary 4. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $A, C \in B(H)$ such that the transforms $\mathcal{C}_{\alpha,\beta}(A)$ and $\mathcal{C}_{\gamma,\delta}(C)$ are accretive. Then for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \leq \frac{1}{4} |\beta - \alpha| |\gamma - \delta| . \tag{3.16}$$

Remark 2. *In the case when A is a selfadjoint operator and $m1_H \leq A \leq M1_H$ for some real numbers $m < M$, then*

$$\left| A - \frac{m+M}{2} 1_H \right| \leq \frac{1}{2} (M-m) 1_H,$$

which implies that

$$\left\| A - \frac{m+M}{2} 1_H \right\| \leq \frac{1}{2} (M-m).$$

Then by (3.13) we have

$$\begin{aligned} &\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leq \frac{1}{2} (M-m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leq \frac{1}{2} (M-m) \left[\frac{\operatorname{tr}\left(P |C|^2\right)}{\operatorname{tr}(P)} - \left| \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right|^2 \right]^{1/2}, \end{aligned} \tag{3.17}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and $C \in \mathcal{B}(H)$.

If C is a selfadjoint operator and $k1_H \leq C \leq K1_H$ for some real numbers $k < K$, then

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \\ &\leq \frac{1}{2} (K-k) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left| \left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H \right) P \right| \right) \\ &\leq \frac{1}{2} (K-k) \left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right)^2 \right]^{1/2} \leq \frac{1}{4} (K-k)^2, \end{aligned} \tag{3.18}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

We have the following Grüss type inequality

$$\left| \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \leq \frac{1}{4} (M-m)(K-k) \tag{3.19}$$

provided that $m1_H \leq A \leq M1_H$ and $k1_H \leq C \leq K1_H$.

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. Then for any $C \in \mathcal{M}_n(\mathbb{C})$ we have

$$\begin{aligned} \left| \frac{\operatorname{tr}(AC)}{n} - \frac{\operatorname{tr}(A)}{n} \frac{\operatorname{tr}(C)}{n} \right| &\leq \frac{1}{2} (M-m) \frac{1}{n} \operatorname{tr}\left(\left| C - \frac{\operatorname{tr}(C)}{n} I_n \right| \right) \\ &\leq \frac{1}{2} (M-m) \left[\frac{\operatorname{tr}\left(|C|^2\right)}{n} - \left| \frac{\operatorname{tr}(C)}{n} \right|^2 \right]^{1/2}, \end{aligned} \tag{3.20}$$

where I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$.

If C is a Hermitian matrix such that $\operatorname{Sp}(C) \subseteq [k, K]$ for some scalars k, K with $k < K$, then

$$0 \le \frac{\operatorname{tr}(C^2)}{n} - \left(\frac{\operatorname{tr}(C)}{n}\right)^2 \le \frac{1}{2}(K-k)\frac{1}{n}\operatorname{tr}\left(\left|C - \frac{\operatorname{tr}(C)}{n}1_H\right|\right) \le \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}(C^2)}{n} - \left(\frac{\operatorname{tr}(C)}{n}\right)^2\right]^{1/2} \le \frac{1}{4}(K-k)^2. \tag{3.21}$$

In the case when the operator A is a function of selfadjoint operators we have the following result as well.

Theorem 7. *Let S be a selfadjoint operator with $\operatorname{Sp}(S) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{C}$ a continuous function of bounded variation on $[m, M]$. For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\begin{aligned} &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \frac{1}{2}\bigvee_m^M(f)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \frac{1}{2}\bigvee_m^M(f)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}, \end{aligned} \tag{3.22}$$

where $\bigvee_m^M(f)$ is the total variation of f on the interval.

If the function $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.

$$|f(t) - f(s)| \le L|t - s|$$

for any $t, s \in [m, M]$, then

$$\begin{aligned} &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \end{aligned} \tag{3.23}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. From the inequality (3.3) we have

$$\begin{aligned} &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \|f(S) - \lambda 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \end{aligned} \tag{3.24}$$

for any $\lambda \in \mathbb{C}$.

From (3.24) we get

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{3.25}$$
$$\leq\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right).$$

Since f is of bounded variation on $[m,M]$, then we have

$$\left|f(t)-\frac{f(m)+f(M)}{2}\right|=\left|\frac{f(t)-f(m)+f(t)-f(M)}{2}\right| \tag{3.26}$$
$$\leq\frac{1}{2}\left[|f(t)-f(m)|+|f(M)-f(t)|\right]$$
$$\leq\frac{1}{2}\bigvee_m^M(f)$$

for any $t\in[m,M]$.

From (3.26) we get in the order $\mathcal{B}(H)$ that

$$\left|f(S)-\frac{f(m)+f(M)}{2}1_H\right|\leq\frac{1}{2}\bigvee_m^M(f)\,1_H,$$

which implies that

$$\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\|\leq\frac{1}{2}\bigvee_m^M(f)\,1_H. \tag{3.27}$$

Making use of (3.25) and (3.27) we get the first inequality (3.22). The second part is obvious.

From (3.24) we have

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{3.28}$$
$$\leq\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$

any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Since

$$|f(t)-f(s)|\leq L|t-s|$$

for any $t,s\in[m,M]$, then we have in the order $\mathcal{B}(H)$ that

$$|f(S)-f(s)\,1_H|\leq L|S-s1_H|$$

for any $s\in[m,M]$. In particular, we have

$$\left|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right|\leq L\left|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right|,$$

which implies that

$$\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\|\leq L\left\|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right\|$$

and by (3.28) we get the first inequality in (3.23).

The second part is obvious. □

Remark 3. *If we take $f(t) = t$ in (3.22), then we get the inequality (3.17) while from (3.23) we obtain*

$$\begin{aligned}(3.29)\quad &\left|\frac{\operatorname{tr}(PSC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\end{aligned}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The case of selfadjoint operators C is as follows:

Corollary 5. *Let S be a selfadjoint operator with $\operatorname{Sp}(S) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{C}$ a continuous function of bounded variation on $[m, M]$. If C is selfadjoint with $\operatorname{Sp}(C) \subseteq [n, N]$ for some real numbers $n < N$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then we have the inequality*

$$\begin{aligned}(3.30)\quad &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le \frac{1}{2}\bigvee_m^M(f) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \frac{1}{2}\bigvee_m^M(f) \left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \le \frac{1}{4}(M-n)\bigvee_m^M(f).\end{aligned}$$

If the function $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, then

$$\begin{aligned}(3.31)\quad &\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\le L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\le \frac{1}{2}(M-n)L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|.\end{aligned}$$

4. Some Examples

If we write the inequality (3.22) for the function $f:[m,M]\subset[0,\infty)\to[0,\infty)$, $f(t)=t^r$, $r>0$, then we get

$$\begin{aligned}\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| &\le \frac{1}{2}(M^r-m^r)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\ &\le \frac{1}{2}(M^r-m^r)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}\tag{4.1}$$

while from (3.23) we have

$$\begin{aligned}\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| &\le \Delta_r\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\ &\le \Delta_r\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\end{aligned}\tag{4.2}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$, any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, where

$$\Delta_r:=\begin{cases} rM^{r-1} \text{ if } r\ge 1,\\ \\ rm^{r-1} \text{ if } r\in(0,1).\end{cases}$$

If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ then from (4.1) and (4.2) we get the power inequalities

$$\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\le\frac{1}{4}(M^r-m^r)(N-n)\tag{4.3}$$

and

$$\left|\frac{\operatorname{tr}(PS^rC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^r)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\le\frac{1}{2}\Delta_r(N-n)\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|.\tag{4.4}$$

If we write the inequality (3.22) for the function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$, then we have

$$\begin{aligned}\left|\frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| &\le \frac{1}{2}\ln\left(\frac{M}{m}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\ &\le \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}\tag{4.5}$$

while from (3.23) we have

$$\begin{aligned}(4.6)\qquad &\left|\frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leq \frac{1}{m}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leq \frac{1}{m}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\left[\frac{\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$, any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ then from (4.5) and (4.6) we have

$$(4.7)\qquad \left|\frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leq\frac{1}{4}(N-n)\ln\left(\frac{M}{m}\right)$$

and

$$(4.8)\qquad \left|\frac{\operatorname{tr}(CP\ln S)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leq\frac{N-n}{2m}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|.$$

If we write the inequality (3.22) for the function $f:[m,M]\subset(0,\infty)\to\mathbb{R}$, $f(t)=t^{-1}$, then we have

$$\begin{aligned}(4.9)\qquad &\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leq \frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leq \frac{1}{2}\frac{M-m}{mM}\left[\frac{\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}$$

while from (3.23) we have

$$\begin{aligned}(4.10)\qquad &\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\ &\leq \frac{1}{m^2}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\leq \frac{1}{m^2}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|\left[\frac{\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\end{aligned}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]$, any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

If C is selfadjoint with $\operatorname{Sp}(C)\subseteq[n,N]$ for some real numbers $n<N$ then from (4.9) and (4.10) we have

$$(4.11)\qquad \left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leq\frac{1}{4}\frac{M-m}{mM}(N-n)$$

and

$$\left|\frac{\operatorname{tr}\left(PS^{-1}C\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\leq\frac{N-n}{2m^{2}}\left\|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right\|. \tag{4.12}$$

Now, if we take $C=S$ in (4.1), then we get

$$\begin{aligned}0&\leq\frac{\operatorname{tr}\left(PS^{r+1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PS^{r})}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\\&\leq\frac{1}{2}\left(M^{r}-m^{r}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_{H}\right)P\right|\right)\\&\leq\frac{1}{2}\left(M^{r}-m^{r}\right)\left[\frac{\operatorname{tr}\left(PS^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}\\&\leq\frac{1}{4}\left(M^{r}-m^{r}\right)(M-m),\end{aligned} \tag{4.13}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]\subset[0,\infty)$ and any $P\in\mathcal{B}_{1}^{+}(H)\setminus\{0\}$.

Also, if we take $C=S$ in (4.5), then we obtain

$$\begin{aligned}0&\leq\frac{\operatorname{tr}(PS\ln S)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P\ln S)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\\&\leq\frac{1}{2}\ln\left(\frac{M}{m}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_{H}\right)P\right|\right)\\&\leq\frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\frac{\operatorname{tr}\left(PS^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}\\&\leq\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)\end{aligned} \tag{4.14}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]\subset(0,\infty)$ and any $P\in\mathcal{B}_{1}^{+}(H)\setminus\{0\}$.

Finally, if we take $C=S$ in (4.9), then we get

$$\begin{aligned}0&\leq\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}-1\\&\leq\frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_{H}\right)P\right|\right)\\&\leq\frac{1}{2}\frac{M-m}{mM}\left[\frac{\operatorname{tr}\left(PS^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}\\&\leq\frac{1}{4}\frac{(M-m)^{2}}{mM},\end{aligned} \tag{4.15}$$

for any S a selfadjoint operator with $\operatorname{Sp}(S)\subseteq[m,M]\subset(0,\infty)$ and any $P\in\mathcal{B}_{1}^{+}(H)\setminus\{0\}$.

From the first and last terms in (4.15) we get the Kantorovich type inequality

$$1\leq\frac{\operatorname{tr}\left(PS^{-1}\right)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\leq\frac{1}{4}\frac{(M+m)^{2}}{mM}.$$

We notice that, the positivity of the first terms in (4.13), (4.14) and (4.15) follows from the Čebyšev's type trace inequality obtained in [28].

References

[1] G. A. Anastassiou, Grüss type inequalities for the Stieltjes integral. *Nonlinear Funct. Anal. Appl.* **12** (2007), no. 4, 583–593.

[2] G. A. Anastassiou, Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral. *Panamer. Math. J.* **17** (2007), no. 3, 91–109.

[3] G. A. Anastassiou, Chebyshev-Grüss type inequalities via Euler type and Fink identities. *Math. Comput. Modelling* **45** (2007), no. 9-10, 1189–1200

[4] T. Ando, Matrix Young inequalities, *Oper. Theory Adv. Appl.* **75** (1995), 33–38.

[5] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[6] E. V. Belmega, M. Jungers and S. Lasaulce, A generalization of a trace inequality for positive definite matrices. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 26, 5 pp.

[7] N. G. de Bruijn, Problem 12, *Wisk. Opgaven*, **21** (1960), 12-14.

[8] P. Cerone, On some results involving the Čebyšev functional and its generalisations. *J. Inequal. Pure Appl. Math.* **4** (2003), no. 3, Article 55, 17 pp.

[9] P. Cerone, On Chebyshev functional bounds. *Differential & difference equations and applications*, 267–277, Hindawi Publ. Corp., New York, 2006.

[10] P. Cerone, On a Čebyšev-type functional and Grüss-like bounds. *Math. Inequal. Appl.* **9** (2006), no. 1, 87–102.

[11] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications. *Tamkang J. Math.* **38** (2007), no. 1, 37–49.

[12] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional. *Appl. Math. Lett.* **18** (2005), no. 6, 603–611.

[13] P. Cerone and S. S. Dragomir, Chebychev functional bounds using Ostrowski seminorms. *Southeast Asian Bull. Math.* **28** (2004), no. 2, 219–228.

[14] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[15] L. Chen and C. Wong, Inequalities for singular values and traces, *Linear Algebra Appl.* **171** (1992), 109–120.

[16] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[17] S. S. Dragomir, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.

[18] S. S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[19] S. S. Dragomir, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 145-156.

[20] S. S. Dragomir, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397-415.

[21] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*, Nova Science Publishers Inc, New York, 2005, x+249 pp.

[22] S. S. Dragomir and G.L. Booth, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.

[23] S. S. Dragomir, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43-47.

[24] S. S. Dragomir and I. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.

[25] S. S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, *Linear Multilinear Algebra* **58** (2010), no. 7-8, 805–814. Preprint *RGMIA Res. Rep. Coll.*, **11**(2008), Supp. Art. 9. `[Online http://rgmia.org/v11(E).php]`

[26] S. S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, *Ital. J. Pure Appl. Math.* No. **28** (2011), 207–224. Preprint *RGMIA Res. Rep. Coll.* **11** (2008), Supp. Art. 11. [Online http://rgmia.org/v11(E).php]
[27] S. S. Dragomir, New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces. *Linear Algebra Appl.* **428** (2008), no. 11-12, 2750–2760.
[28] S. S. Dragomir, Some Čebyšev's type trace inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. .
[29] A. M. Fink, A treatise on Grüss' inequality, *Analytic and Geometric Inequalities,* 93-113, Math. Appl. 478, Kluwer Academic Publ., 1999.
[30] S. Furuichi and M. Lin, Refinements of the trace inequality of Belmega, Lasaulce and Debbah. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 23, 4 pp.
[31] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.
[32] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[33] H. D. Lee, On some matrix inequalities, *Korean J. Math.* **16** (2008), No. 4, pp. 565-571.
[34] L. Liu, A trace class operator inequality, *J. Math. Anal. Appl.* 328 (2007) 1484–1486.
[35] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[36] S. Manjegani, Hölder and Young inequalities for the trace of operators, *Positivity* **11** (2007), 239–250.
[37] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564.
[38] D. S. Mitrinović, J. E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[39] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.
[40] B. G. Pachpatte, A note on Grüss type inequalities via Cauchy's mean value theorem. *Math. Inequal. Appl.* **11** (2008), no. 1, 75–80.
[41] J. Pečarić, J. Mićić and Y. Seo, Inequalities between operator means based on the Mond-Pečarić method. *Houston J. Math.* **30** (2004), no. 1, 191–207
[42] M. B. Ruskai, Inequalities for traces on von Neumann algebras, *Commun. Math. Phys.* **26**(1972), 280—289.
[43] K. Shebrawi and H. Albadawi, Operator norm inequalities of Minkowski type, *J. Inequal. Pure Appl. Math.* **9**(1) (2008), 1–10, article 26.
[44] K. Shebrawi and H. Albadawi, Trace inequalities for matrices, *Bull. Aust. Math. Soc.* **87** (2013), 139–148.
[45] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.
[46] Z. Ulukök and R. Türkmen, On some matrix trace inequalities. *J. Inequal. Appl.* **2010**, Art. ID 201486, 8 pp.
[47] X. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **250** (2000) 372–374.
[48] X. M. Yang, X. Q. Yang and K. L. Teo, A matrix trace inequality, *J. Math. Anal. Appl.* **263** (2001), 327–331.
[49] Y. Yang, A matrix trace inequality, *J. Math. Anal. Appl.* **133** (1988) 573–574.
[50] C.-J. Zhao and W.-S. Cheung, On multivariate Grüss inequalities. *J. Inequal. Appl.* **2008**, Art. ID 249438, 8 pp.

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